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On differentially prime subsemimodules

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Abstract. The paper is devoted to the investigation of the notion of a differentially prime subsemimodule of a differential semimodule over a commutative semiring, which generalizes the notion of differentially prime ideal of a ring. The characterization of differentially prime subsemimodules is given. The interrelation between differentially prime subsemimodules and different types of differential subsemimodules and ideals is studied.

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1 Introduction

The notion of a derivation for semirings is defined in [3] as an additive map satisfying the Leibnitz rule. Recently in [2, 13] and [11] the authors investigated different properties of semiring derivations, differential semirings, i.e. semirings considered together with a derivation, and differential ideals of such rings. Prime subsemimodules of semimodules over semirings were introduced and studied in [1]. Differentially prime ideals were introduced in [8] for differential, not necessarily commutative, rings. Differentially prime submodules of modules over associative rings were studied in [10].

The rapid development of semiring and semimodule theory in recent years motivates a further study into properties of differential semirings, differential semimodules, semiring ideals and subsemimodules defined by similar conditions. The objective of this paper is to investigate differentially prime subsemimodules of semimodules equipped with derivations over commutative semimodules, and their interrelation with other types of subsemimodules.

For the sake of completeness some definitions and properties used in the paper will be given here. For more information see [3–5, 9].

Let R be a nonempty set and let $+$ and $\cdot$ be binary operations on R. An algebraic system $(R, +, \cdot)$ is called a *semiring* if $(R, +, 0)$ is a commutative monoid, $(R, \cdot)$ is a semigroup and multiplication distributes over addition from either side. A semiring $(R, +, \cdot)$ is said to be *commutative* if $\cdot$ is commutative on R.

Zero $0_R \in R$ is called (*multiplicatively*) *absorbing* if $a \cdot 0_R = 0_R \cdot a = 0$ for all $a \in R$. An element $1_R \in R$ is called *identity* if $a \cdot 1_R = 1_R \cdot a = a$ for all $a \in R$. Suppose $1_R \neq 0_R$, otherwise $R = \{0\}$ if zero is absorbing.

Throughout the paper, we assume that all semirings are commutative with identity, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$ denotes the set of non-negative integers.

An *ideal* of a semiring R is a nonempty set $I \neq R$ which is closed under addition $+$ and satisfies the condition $ra \in I$ for all $a \in I$, $r \in R$. An ideal I of a semiring R is called *subtractive* (or *k-ideal*) if $a \in I$ and $a + b \in I$ imply $b \in I$.

Let R be a semiring with $1_R \neq 0_R$. A *semimodule over a semiring* R (or R-*semimodule*) is a nonempty set M together with two operations $+\colon M \times M \to M$ and $\cdot\colon R \times M \to M$ such that $(M, +)$ is a commutative monoid with 0_M, $(M, \cdot)$ is a semigroup, $(r+s)m = rm+sm$ for all $r, s \in R$, $m \in M$, $r(m_1+m_2) = rm_1+rm_2$ for all $r \in R$, $m_1, m_2 \in M$, $0_R \cdot m = r \cdot 0_M = 0_M$ for all $r \in R$ and $m \in M$, $1_R \cdot m = m$ for all $m \in M$.

A subset N of an R-semimodule M is called a *subsemimodule* of M if $m+n \in N$ and $rm \in N$ for any $m, n \in N$, and $r \in R$. A subsemimodule N of an R-semimodule M is called *subtractive* or *k-subsemimodule* if $m_1 \in N$ and $m_1 + m_2 \in N$ imply $m_2 \in N$. So $\{0_M\}$ is a subtractive subsemimodule of M.

Let R be a semiring. A map $\delta\colon R \to R$ is called a *derivation on* R [3] if $\delta(a+b) = \delta(a) + \delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in R$. A semiring R equipped with a derivation δ is called a *differential semiring* with respect to the derivation δ (or δ-*semiring*), and is denoted by (R, δ) [2].

For an element $r \in R$ denote by $r^{(0)} = r$, $r' = \delta(r)$, $r'' = \delta(r')$, $r^{(n)} = \delta\left(r^{(n-1)}\right)$, for any $n \in \mathbb{N}_0$. An ideal I of the semiring R is called *differential* if the set I is differentially closed under δ, i.e. $\delta(r) \in I$ for any $r \in I$. The set of all derivations of an element $r \in R$ $r^{(\infty)} = \{r^{(n)} | n = 0, 1, 2, 3 \ldots\}$ is differentially closed. The ideal $[r] = \left(r^{(\infty)}\right) = (r, r', r'', \ldots)$ of R, generated by the set $r^{(\infty)}$, is differentially generated by $r \in R$; it is the smallest differential ideal containing the element $r \in R$ [11].

Let M be a semimodule over the differential semiring (R, δ). A map $d\colon M \to M$ is called a *derivation* of the semimodule M, associated with the semiring derivation $\delta\colon R \to R$ (or a δ-derivation), if $d(m+n) = d(m) + d(n)$ and $d(rm) = \delta(r)m + rd(m)$ for any $m, n \in M$, $r \in R$. A R-semimodule M together with a derivation $d\colon M \to M$ is called a *differential semimodule* (or d-δ-*semimodule*) and is denoted by (M, d).

A subsemimodule N of the R-semimodule M is called *differential* if $d(N) \subseteq N$. Any differential semimodule has two trivial differential subsemimodules: $\{0_M\}$ and itself.

For an element $m \in M$ denote by $m^{(0)} = m$, $m' = d(m)$, $m'' = d(m')$, $m^{(n)} = d\left(m^{(n-1)}\right)$, for any $n \in \mathbb{N}_0$. Moreover, let $m^{(\infty)} = \{m^{(n)} | n \in \mathbb{N}_0\}$. It is easy to see that the set $m^{(\infty)}$ is differentially closed. The subsemimodule $[m] = \left(m^{(\infty)}\right) = (m, m', m'', \ldots)$ is the smallest differential subsemimodule of M containing $m \in M$.

A subsemimodule P of a subsemimodule M is called prime if for any ideal I of R and any submodule N of M the inclusion $IN \subseteq P$ implies $N \subseteq P$ or $I \subseteq (P : M)$. Prime subsemimodules are extensively investigated in [1].

2 Differentially prime subsemimodules

Definition. Let S be a multiplicatively closed subset of R. A non-empty subset X of the semimodule M is called an *S-closed* subset of M if $sx \in X$ for every $s \in S$ and $x \in X$.

Quasi-prime ideals of differential rings were introduced and studied in [6, 7], its generalizations to differential modules, semirings and semimodules were studied by different authors, e.g.[11, 12, 14, 15].

Definition. A differential subsemimodule N of the left differential semimodule M is called *quasi-prime* if it is maximal differential subsemimodule of M disjoint from some S-closed subset of M.

For instance, every prime differential subsemimodule is quasi-prime, because the complement of the prime subsemimodule is an S-closed subset of M, where the role of S is played by the set $\{1\}$.

In the case of a regular semimodule, we obtain the notion of quasi-prime ideal of a semiring. For differential semiring ideals it is known that every maximal among differential ideals not meeting some multiplicatively closed subset of the semiring is quasi-prime. The analogue of this fact holds for differential semimodules: every maximal among differential subsemimodules of an arbitrary differential semimodule is quasi-prime.

Definition. A differential k-subsemimodule P of M is called *differentially prime* if for any $r \in R$, $m \in M$, $k \in \mathbb{N}_0$, $rm^{(k)} \in P$ implies $r \in (P : M)$ or $m \in P$.

Theorem 1. *Every quasi-prime k-subsemimodule N of M is differentially prime.*

Proof. Let N be a quasi-prime subsemimodule of M. Suppose that there exist $r \in R$, $m \in M$ such that $r \in R \setminus (N : M)$, $m \in M \setminus N$ and $[r] \cdot [m] \subseteq N$. Since N is maximal among the differential submodule not meeting some S-closed subset X of M, for differential ideal $(N : M) + [r]$ and differential subsemimodule $N + [m]$ the maximality of N implies $((N : P) + [r]) \cap S \neq \varnothing$ and $(N + [m]) \cap X \neq \varnothing$. As a result, there exist $s \in S$, $x \in X$ such that $s \in (N : M) + [r]$ and $x \in N + [m]$. Since X is an S-closed subset of M and $s \in S$, $x \in X$ implies that there exists $n \in \mathbb{N}_0$ such that $sx^{(n)} \in X$. Then $sx^{(n)} \in ((N : M) + [r]) \cdot (N + [m]) \subseteq N$. It follows that $sx^{(n)} \in X \bigcap N \neq \varnothing$, which contradicts the original assumption. Therefore, N is differentially prime. □

Definition. Let $S \neq \varnothing$ be a subset of R. A subset S is called *d-multiplicatively closed* if for any $a, b \in S$ there exists $n \in \mathbb{N}_0$ such that $ab^{(n)} \in S$.

Definition. Let S be a d-multiplicatively closed subset of R. A subset $X \subseteq M$ is called *Sd-multiplicatively closed* if for any $s \in S$, $x \in X$ there exists $n \in \mathbb{N}_0$ such that $sx^{(n)} \in X$.

Proposition 1. *A k-subsemimodule $N \subseteq M$ is differentially prime if and only if $M \setminus N$ is Sd-multiplicatively closed.*

Proof. Suppose $X = M \setminus N$, $S = R \setminus (N : M)$, $N \subseteq M$ is differentially prime and there exist $s \in S$ and $x \in M \setminus N$ such that for all $n \in \mathbb{N}_0$, $sx^{(n)} \notin M \setminus N$. Then $s \in (N : M)$ or $x \in N$, which contradicts $s \in S$.

Conversely, suppose $X = M \setminus N$ is Sd-multiplicatively closed, and for all $n \in \mathbb{N}_0$, $sx^{(n)} \notin X$ for some $s \in S$ and $x \in X$. Then $sx^{(n)} \in N$, and so $s \in (N : M)$, which is a contradiction. □

Theorem 2. *For a differential k-subsemimodule P of M, $P \neq M$ the following conditions are equivalent:*

1. *P is differentially prime;*
2. *For any $r \in R$, $m \in M$, $k, l \in \mathbb{N}_0$, $r^{(l)}m^{(k)} \in P$ implies $r \in (P : M)$ or $m \in P$;*
3. *For any $r \in R$, $m \in M$, $[r] \cdot [m] \subseteq P$ implies $r \in (P : M)$ or $m \in P$;*
4. *For any differential k-ideal I of R and any differential k-subsemimodule N of M, $IN \subseteq P$ implies $N \subseteq P$ or $I \subseteq (P : M)$.*

Proof. ($1 \Longrightarrow 2$) Suppose $r^{(l)}m^{(k)} \in P$ for any $k, l \in \mathbb{N}_0$. Denote $t = l + k$. For $t = 0$ we have $r^{(0)}m^{(0)} = rm \in P$. Therefore, $d(rm) = (rm)' \in P$. For a subtractive subsemimodule P, we have $(rm)' = r'm + rm' \in P$, $rm' \in P$. Hence, $r'm \in P$.

Consider $(rm^{(k)})' = r'm^{(k)} + rm^{(k+1)}$ for all $k \in \mathbb{N}_0$. As before, $(rm^{(k)})' \in P$, $rm^{(k+1)} \in P$ imply $r'm^{(k)} \in P$, by subtractiveness of P.

In a similar way, from $(r'm^{(k-1)})' = r''m^{(k-1)} + r'm^{(k)} \in P$, $r'm^{(k)} \in P$ and subtractiveness of P we obtain $r''m^{(k)} \in P$, etc.

($2 \Longrightarrow 1$) Obvious when $l = 0$.

($2 \Longrightarrow 3$) Note that $[r] = \sum_{l \in \mathbb{N}_0} Rr^{(l)}$, $[m] = \sum_{k \in \mathbb{N}_0} Rm^{(k)}$, and so $[r] \cdot [m] = \sum_{k,l \in \mathbb{N}_0} Rr^{(l)}m^{(k)}$.

If $[r] \cdot [m] \subseteq P$ then $\sum_{k,l \in \mathbb{N}_0} Rr^{(l)}m^{(k)} \subseteq P$, in particular $r^{(l)}m^{(k)} \in P$. Hence, $r \in (P : M)$ or $m \in P$.

($3 \Longrightarrow 2$) Suppose for any $r \in R$, $m \in M$, $[r] \cdot [m] \subseteq P$ implies $r \in (P : M)$ or $m \in P$. Prove that for any $r \in R$, $m \in M$, $k, l \in \mathbb{N}_0$, $r^{(l)}m^{(k)} \in P$ implies $r \in (P : M)$ or $m \in P$.

If $r^{(l)}m^{(k)} \in P$, then $\sum_{k,l \in \mathbb{N}_0} Rr^{(l)}m^{(k)} \subseteq P$. Therefore, $[r] \cdot [m] \subseteq P$, which follows $r \in (P : M)$ or $m \in P$.

($3 \Longrightarrow 4$). Suppose for any $r \in R$, $m \in M$, $[r] \cdot [m] \subseteq P$ implies $r \in (P : M)$ or $m \in P$, and let $IN \subseteq P$, where I is an arbitrary differential ideal of R and N is an arbitrary differential subsemimodule of M.

Suppose $N \nsubseteq P$ or $I \nsubseteq (P : M)$. There exists $x \in N$, $x \notin P$, and $r \in I$, $r \notin (P : M)$. Clearly, $[r] \cdot [x] \subseteq IN \subseteq P$. Therefore, $r \in (P : M)$ or $m \in P$, which is a contradiction.

($4 \Longrightarrow 3$) is obvious. □

Theorem 3. *Let S be d-multiplicatively closed subset of R, X be Sd-multiplicatively closed subset of M, and let N be a differential subsemimodule of M, maximal in $M \setminus N$.*

If the ideal $(N : M)$ is differentially maximal in $R \setminus S$, then N is a differentially prime subsemimodule of M.

Proof. Suppose that there exist $r \in R$, $m \in M$ and $k \in \mathbb{N}_0$ such that $rm^{(k)}) \in N$, $r \notin (N : M)$, and $m \notin N$. It is clear that $N \subset N + [m]$ and $(N : M) \subset (N : M) + [r]$.

Since N is maximal among the differential subsemimodules not meeting some Sd-closed subset X, $(N + [m]) \cap X \neq \varnothing$. Since $(N : M)$ is maximal among the differential ideals of R not meeting some d-multiplicatively closed subset S, $((N : M) + [r]) \cap S \neq \varnothing$. Therefore there exist $a \in S$, $x \in X$ such that $a \in (N : M) + [r]$ and $x \in N + [m]$. On the other hand, since X is a Sd-multiplicatively closed subset, then $a \in S$, $x \in X$ implies the existence of $n \in \mathbb{N}_0$ such that $ax^{(n)} \in X$. Therefore $x^{(n)} \in (N + [m]) \cap X$. Then $ax^{(n)} \in ((N : M) + [r]) \cdot (N + [m]) = (N : M)N + (N : M) \cdot [m] + [r] \cdot N + [r] \cdot [m] \subseteq N$. Therefore, $ax^{(n)} \in N \bigcap X \neq \varnothing$, but it contradicts the assumption that $X \bigcap N = \varnothing$. Hence N is a differentially prime subsemimodule. □

Corollary 1. *Let P be differentially prime ideal of R, $S = R \setminus P$, X be Sd-multiplicatively closed subset of R, let a differential subsemimodule N be maximal in $M \setminus X$. If N is prime, then $(N : M) = P$.*

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