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Numerical Implementation of Daftardar-Gejji and Jafari Method to the Quadratic Riccati Equation

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Abstract. The solution of quadratic Riccati differential equations can be found by classical numerical methods like Runge-Kutta method and the forward Euler method. Batiha *et al.* [7] applied variational iteration method (VIM) for the solution of General Riccati Equation. In the paper of El-Tawil *et al.* [19] they used the Adomian decomposition method (ADM) to solve the nonlinear Riccati equation. In [3] Abbasbandy applied Iterated He's homotopy perturbation method for solving quadratic Riccati differential equation. In [2] Abbasbandy used the Homotopy perturbation method to get an analytic solution of the quadratic Riccati differential equation, and a comparison with Adomian's decomposition method was presented. In [1] Abbasbandy employed VIM to find the solution of the quadratic Riccati equation by using Adomian's polynomials. Tan and Abbasbandy [30] employed the Homotopy Analysis Method (HAM) to find the solution of the quadratic Riccati equation. Batiha [5] used the multistage variational iteration method (MVIM) to solve the quadratic Riccati differential equation.

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1 Introduction

A strong tool to introduce real-life phenomena is differential equations but, in most cases, numerical or theoretical solutions are difficult to find, in recent years many numerical methods have been introduced to solve nonlinear differential equations,[4, 8, 31].

The solution of quadratic Riccati differential equations can be found by classical numerical methods like Runge-Kutta method and the forward Euler method. Batiha *et al.* [7] applied variational iteration method (VIM) for the solution of General Riccati Equation. In the paper [19] El-Tawil *et al.* used the Adomian decomposition method (ADM) to solve the nonlinear Riccati equation. In [3] Abbasbandy applied Iterated He's homotopy perturbation method for solving quadratic Riccati differential equation. In [2] Abbasbandy used the Homotopy perturbation method to get an analytic solution of the quadratic Riccati differential equation, and a comparison with Adomian's decomposition method was presented. In [1] Abbasbandy employed VIM to find the solution of the quadratic Riccati equation by using Adomian's polynomials. Tan and Abbasbandy [30] employed the Homotopy Analysis

Method (HAM) to find the solution of the quadratic Riccati equation. Batiha [5] used the multistage variational iteration method (MVIM) to solve the quadratic Riccati differential equation.

The purpose of this paper is to use the Daftardar-Gejji and Jafari method (DJM) to get the solution of quadratic Riccati differential equations and to present a comparison between VIM, ADM, HPM, and exact solution to prove the power of DJM to solve nonlinear differential equations.

2 The Daftardar–Gejji and Jafari Method

Daftardar-Gejji and Jafari method (DJM) was first introduced by Daftardar-Gejji and Jafari [16] in 2006, it has been proved that this method is a better technique for solving different kinds of nonlinear equations [6, 9–11, 13–15, 20–23, 29]. DJM has been used to create a new predictor-corrector method [17, 18]. Noor et al. [24–28] used DJM to create numerical methods to handle algebraic equations.

Here the Daftardar-Gejji and Jafari method will be discussed, which was successfully used to solve differential equations and nonlinear equations in the form:

$$y = f + L(y) + N(y), \tag{1}$$

where L, N are linear and non-linear operators, respectively, and f is a given function. The solution of Eq. (1) has the form:

$$y = \sum_{i=0}^{\infty} y_i. \tag{2}$$

Suppose we have

$$\begin{aligned} H_0 &= N(y_0), && (3)\\ H_m &= N\left(\sum_{i=0}^{m} y_i\right) - N\left(\sum_{i=0}^{m-1} y_i\right), && (4) \end{aligned}$$

then we get

$$\begin{aligned} H_0 &= N(y_0), && (5)\\ H_1 &= N(y_0+y_1) - N(y_0), && (6)\\ H_2 &= N(y_0+y_1+y_2) - N(y_0+y_1), && (7)\\ H_3 &= N(y_0+y_1+y_2+y_3) - N(y_0+y_1+y_2) + \cdots . && (8) \end{aligned}$$

Thus $N(y)$ is decomposed as:

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + N(y_0+y_1) - N(y_0) + N(y_0+y_1+y_2) - N(y_0+y_1)$$

$$+N(y_0+y_1+y_2+y_3)-N(y_0+y_1+y_2)+\cdots. \tag{9}$$

So, the recurrence relation is of the following form:

$$\begin{aligned} y_0 &= f \\ y_1 &= L(y_0)+H_0 \\ y_{m+1} &= L(y_m)+H_m, \quad m=1,2,\cdots. \end{aligned} \tag{10}$$

Since L is linear, then:

$$\sum_{i=0}^{m} L(y_i) = L\left(\sum_{i=0}^{m} y_i\right). \tag{11}$$

So,

$$\begin{aligned} \sum_{i=0}^{m+1} y_i &= \sum_{i=0}^{m} L(y_i) + N\left(\sum_{i=0}^{m} y_i\right) \\ &= L\left(\sum_{i=0}^{m} y_i\right) + N\left(\sum_{i=0}^{m} y_i\right), \quad m=1,2,\cdots. \end{aligned} \tag{12}$$

Thus,

$$\sum_{i=0}^{\infty} y_i = f + L\left(\sum_{i=0}^{\infty} y_i\right) + N\left(\sum_{i=0}^{\infty} y_i\right). \tag{13}$$

The $k-$ term solution is given by the following form:

$$y = \sum_{i=0}^{k-1} y_i. \tag{14}$$

3 Convergence of the DJM

In this section, we will introduce the condition of convergence of DJM.

Lemma 1. *[9] If N is $C^{(\infty)}$ in a neighborhood of u_0 and $||N^{(n)}(u_0)|| \leq L$, for any n and for some real $L > 0$ and $||u_i|| \leq M < e^{-1}, i = 1, 2, ...,$ then the series $\sum_{n=0}^{\infty} H_n$ is absolutely convergent and*

$$||H_n|| \leq LM^n e^{n-1}(e-1), \quad n = 1, 2, ...$$

Lemma 2. *[9] If N is $C^{(\infty)}$ and $||N^{(n)}(u_0)|| \leq M \leq e^{-1}, \forall n$, then the series $\sum_{n=0}^{\infty} H_n$ is absolutely convergent.*

4 Numerical Implementation

4.1 Example 1

In this example, we shall consider the quadratic Riccati equation in the form:

$$y'(t) = 2y(t) - y^2(t) + 1, \quad y(0) = 0. \tag{15}$$

The exact solution was found to be (see Fig. 1) [19]:

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \tag{16}$$

If you expand Eq. (16) by Taylor expansion about $t = 0$ we get:

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 + \frac{71}{315}t^8 + \cdots. \tag{17}$$

Bulut and Evans [12] applied the decomposition method to solve Eq. 15 and they found:

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{1}{5}t^6 + \frac{163}{315}t^7 - \frac{62}{315}t^8 + \cdots. \tag{18}$$

Abbasbandy [2] used Homotopy perturbation method (HPM) for quadratic Riccati differential equation and got:

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 - \frac{221}{1260}t^8 + \cdots. \tag{19}$$

Abbasbandy [1] applied three iterates variational iteration methods (VIM) for Eq. (15) and found the result:

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 - \frac{673}{2520}t^8 + \cdots. \tag{20}$$

To solve quadratic Riccati differential equation (15) by DJM, we integrate Eq. (15) and use initial condition $y(0) = 0$, to get:

$$y(t) = \int_0^t 2y(t) - y(t)^2 + 1dt. \tag{21}$$

By using algorithm (10) we have:

$$y_0 = 0, \quad y_1 = t, \quad y_2 = -\frac{1}{3}t^2(t-3), \;\; y_3 = -\frac{t^3\left(5\,t^4 - 35\,t^3 + 21\,t^2 + 210\,t - 210\right)}{315},$$

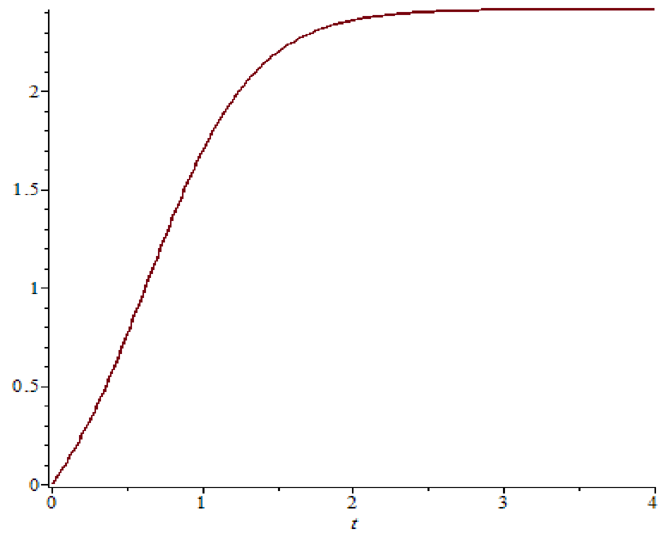

Figure 1. *The exact solution of Eq.15*

$$y_4 = -\frac{1}{170270100} t^4 (2860\,t^{11} - 42900\,t^{10} + 189420\,t^9 + 90090\,t^8 - 2388204\,t^7 + \\ +2234232\,t^6 + 11171160\,t^5 - 6891885\,t^4 - 41081040\,t^3 + 3783780\,t^2 + \\ +90810720\,t - 56756700),$$

$$\vdots$$

Thus,

$$\sum_{i=0}^{4} y_i = t - \frac{1}{3}t^2(t-3) - \frac{t^3\left(5\,t^4 - 35\,t^3 + 21\,t^2 + 210\,t - 210\right)}{315} \\ -\frac{1}{170270100} t^4 (2860\,t^{11} - 42900\,t^{10} + 189420\,t^9 + 90090\,t^8 \\ -2388204\,t^7 + 2234232\,t^6 + 11171160\,t^5 - 6891885\,t^4 \\ -41081040\,t^3 + 3783780\,t^2 + 90810720\,t - 56756700). \tag{22}$$

Using Taylor expansion to expand y_6 about $t = 0$ gives:

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{7}{45}t^7 - \frac{83}{315}t^8 \cdots . \tag{23}$$

4.2 Example 2

Here, we will check the following Riccati equation:

$$y'(t) = -y^2(t) + 1, \quad y(0) = 0. \tag{24}$$

The exact solution for the Riccati equation above is [19]:

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}. \tag{25}$$

When we expand Eq. (25) by Taylor expansion about $t = 0$ we get:

$$y(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2835}t^9 - \frac{1382}{155925}t^{11} + \frac{21844}{6081075}t^{13} + \cdots. \tag{26}$$

To solve quadratic Riccati differential equation (24) by DJM, we integrate Eq. (24) and use initial condition $y(0) = 0$, to get:

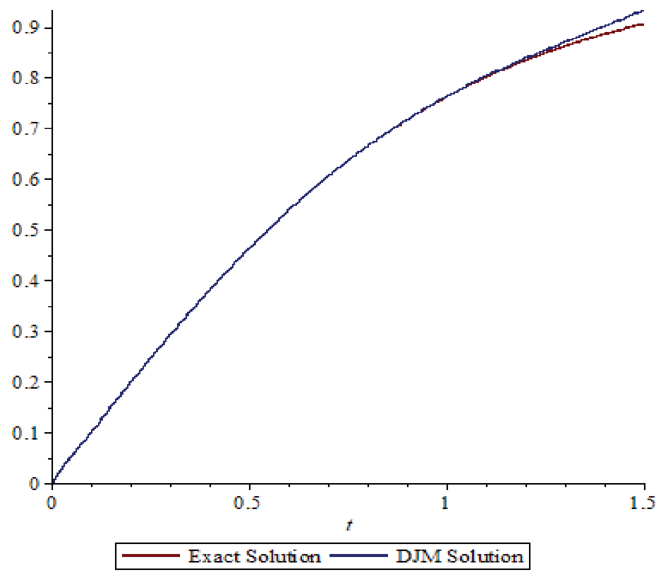

Figure 2. *The comparison between the y_4 of DJM and the exact solution*

$$y(t) = \int_0^t -y(t)^2 + 1dt. \tag{27}$$

By using algorithm (10) we have:

$$y_0 = 0, \quad y_1 = t, \quad y_2 = -\frac{1}{3}t^3, \quad y_3 = -\frac{t^5\left(5\,t^2 - 42\right)}{315},$$

$$y_4 = -\frac{t^7\left(715\,t^8 - 13860\,t^6 + 109746\,t^4 - 570570\,t^2 + 1621620\right)}{42567525}$$

$$\vdots$$

Thus,

$$\begin{aligned}\sum_{i=0}^{4} y_i &= t - 1/3\,t^3 - \frac{t^5\left(5\,t^2 - 42\right)}{315} \\ &\quad - \frac{t^7\left(715\,t^8 - 13860\,t^6 + 109746\,t^4 - 570570\,t^2 + 1621620\right)}{42567525}. \qquad (28)\end{aligned}$$

Using Taylor expansion to expand y_4 about $t = 0$ gives:

$$y(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{38}{2835}t^9 + \cdots\,. \qquad (29)$$

5 Numerical Results and Discussion

In this section, we will show the numerical solutions of quadratic Riccati differential equation.

Table 1. Numerical comparisons between exact solution and y_5 of DJM

t	Exact solution	y_5 of DJM	absolute error
0.1	0.1102951967	0.1102951631	3.360E-8
0.2	0.2419767992	0.2419752509	1.548E-6
0.3	0.3951048481	0.3950932308	1.162E-5
0.4	0.5678121656	0.5677733164	3.885E-5
0.5	0.7560143925	0.7559368137	7.758E-5
0.6	0.9535662155	0.9534634383	1.028E-4
0.7	1.1529489660	1.1528561200	9.285E-5
0.8	1.3463636550	1.3463068680	5.679E-5
0.9	1.5269113120	1.5268938270	1.748E-5
1.0	1.6894983900	1.6895510560	5.266E-5

Table 1 shows the comparison between the y_5 of DJM and the exact solution for example 1. Figure 2 shows the comparison between the y_4 of DJM and the exact solution for example 2. We can see the good accuracy of DJM compared to the exact solution, but we can note that it's accurate only for small t.

6 Conclusions

In this paper, we show a new application of the Daftardar-Gejji and Jafari method (DJM) to get the solution of the quadratic Riccati differential equation. In this paper, we use the Maple Package to calculate the series obtained from the DJM. It may be concluded that DJM is a powerful tool for finding analytical and numerical solutions for the Riccati differential equation.

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