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Algebraic View over Homogeneous Linear Recurrent Processes

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Abstract. In this paper the algebraic properties of the deterministic processes with dynamic represented by a homogeneous linear recurrence over the field $\mathbb{C}$ are studied. It is started with an overview of homogeneous linear recurrent processes over $\mathbb{C}$ and its subsets. Next, it is gone deeper into homogeneous linear recurrent processes over numerical rings. After that, the recurrence criteria over sign-based ring subsets are analyzed. Also, the deterministic processes with dynamic represented by a Littlewood, Newman or Borwein homogeneous linear recurrence are considered.

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1 Introduction

The main goal of this paper is to study the algebraic properties of the deterministic processes with dynamic represented by a homogeneous linear recurrence over the field $\mathbb{C}$. We delve into the subsets of $\mathbb{C}$ to see if the dynamic of the given process is also a homogeneous linear recurrence over given subset in certain conditions. The challenge appears when we get out from comfort zone, given by field properties.

So, we start with an overview of homogeneous linear recurrent processes over $\mathbb{C}$ and its subsets. We remind the main definitions and properties, like generating vector, characteristic polynomial and minimality. Also, we formulate the minimization method based on matrix rank definition.

Next, we go deeper into homogeneous linear recurrent processes over numerical rings. We formulate and prove necessary and sufficient conditions for a homogeneous linear recurrence over $\mathbb{C}$ to be also a homogeneous linear recurrence over a subfield or subring, like $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ or an extension field of $\mathbb{Q}$.

After that, we are interested in recurrence criteria over sign-based ring subsets. We split the ring into two subsets, one containing the positive elements and the second containing the negative ones. It is shown that the recurrence criteria over these subsets are based on the number of positive real roots of the minimal characteristic polynomial over that ring and, in the most complex case when it is a single one, they are also based on the relationship of that positive real root with the rest of the roots.

The last section is dedicated to deterministic processes with dynamic represented by a Littlewood, Newman or Borwein homogeneous linear recurrence. Mainly, these are homogeneous linear recurrences over subsets of the set $\{-1, 0, 1\}$. Several results are presented, based on the properties of Littlewood, Newman and Borwein polynomials.

2 Homogeneous Linear Recurrences over subsets of $\mathbb{C}$

In this section we will remind the definitions regarding homogeneous linear recurrences from [5] and we will provide several main properties, theoretically grounded in [6].

We consider a positive integer number m and a subset K of $\mathbb{C}$. A sequence $a = \{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ is a homogeneous linear $m-$recurrence over K if there exists $q = (q_k)_{k=0}^{m-1} \in K^m$ such that

$$a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k}, \ \forall n \geq m.$$

Here q represents the generating vector and $I_m^{[a]} = (a_n)_{n=0}^{m-1}$ represents the initial state of the sequence a. The sequence a is a homogeneous linear recurrence over K if there exists a positive integer m such that the sequence a is a homogeneous linear m-recurrence over K. If $q_{m-1} \neq 0$ then the sequence a is called non-degenerate, otherwise it is called degenerate.

Denote:

$Rol[K][m]$ is the set of non-degenerate homogeneous linear m-recurrences over K;
$Rol[K]$ is the set of all non-degenerate homogeneous linear recurrences over K;
$G[K][m](a)$ is the set of all generating vectors of length m of $a \in Rol[K][m]$;
$G[K](a)$ is the set of all generating vectors of the sequence $a \in Rol[K]$.

The function $G^{[a]} : \mathbb{C} \to \mathbb{C}$, $G^{[a]}(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, represents the generating function of the sequence $a = (a_n)_{n=0}^{\infty} \subseteq \mathbb{C}$. On the other hand, the function $G_t^{[a]} : \mathbb{C} \to \mathbb{C}$, $G_t^{[a]}(z) = \sum\limits_{n=0}^{t-1} a_n z^n$, represents the partial generating function of order t of the sequence $a = (a_n)_{n=0}^{\infty} \subseteq \mathbb{C}$.

Let $a \in Rol[K][m]$, $q \in G[K][m](a)$. For this sequence we will consider the unit characteristic polynomial $H_m^{[q]}(z) = 1 - zG_m^{[q]}(z)$ and the characteristic equation $H_m^{[q]}(z) = 0$. Every polynomial of the form $H_{m,\alpha}^{[q]}(z) = \alpha H_m^{[q]}(z)$ also represents a characteristic polynomial. We introduce the following notations:

$H[K][m](a)$ is the set of characteristic polynomials of degree m of $a \in Rol[K]$;
$H[K](a)$ is the set of characteristic polynomials of sequence $a \in Rol[K]$.

The non-zero sequence (with at least one non-zero element) $a \in Rol[K]$ is called m-minimal over K if $a \in Rol[K][m]$ and $a \notin Rol[K][t]$, $\forall t < m$. In this case, the number m represents the dimension of the sequence a over K and it is denoted $dim[K](a) = m$. The dimension of the zero sequence (with all elements equal to 0) is considered 0.

It is known from [6] that the minimal generating vector is unique, i.e.

$$|G[K][dim[K](a)](a)| = 1.$$

This unique minimal generating vector determines the unique minimal unit characteristic polynomial $P(Z) \in H[K][dim[K](a)](a)$. Next, we will omit the word "unit" and we will consider the polynomial $P(z)$ as the minimal characteristic polynomial of the sequence a.

Further, in [6], it was proved that the set of all characteristic polynomials is

$$H[K](a) = \{Q(z) \in K[z] \mid Q(z) \vdots P(z),\ Q(0) \neq 0\}.$$

This result will allow us to check if a homogeneous linear recurrence a over K is also a homogeneous linear recurrence over a subset K_2 of K. Practically, we need to determine if there exists a multiple $Q(z)$ of the minimal characteristic polynomial $P(z)$, that has the free term equal to 1 and the opposite of the rest of coefficients belonging to the set K_2.

Another important result, theoretically grounded in [6], is the following theorem that provides an efficient minimization method based on matrix rank definition.

Theorem 1. *If $a \in Rol[\mathbb{C}][m]$ is a non-zero sequence, then*

$$dim[\mathbb{C}](a) = R = rank(A_m^{[a]})$$

and

$$q = (q_0, q_1, \ldots, q_{R-1}) \in G[\mathbb{C}][R](a),$$

where the reverse vector $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ is the unique solution of the system with linear equations $A_R^{[a]} x^T = (f_R^{[a]})^T$ with free terms $f_R^{[a]} = (a_R, a_{R+1}, \ldots, a_{2R-1})$ and the matrix $A_R^{[a]} = (a_{i+j})_{i,j=\overline{0,R-1}}$ of the system.

3 Homogeneous Linear Recurrences over Numerical Rings

In this section the homogeneous linear recurrences over numerical rings will be studied. In particular, these numerical rings can be one of the well known sets $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ or an extension field of $\mathbb{Q}$.

Theorem 2. *Let K_1 a set and K_2 a subset of K_1. If $a \in Rol[K_2][m]$, then $a \in Rol[K_1][m]$ with the same generating vector.*

Proof. Let $a \in Rol[K_2][m]$ and $q \in G[K_2][m](a)$. Then $q \in K_2^m$ and, since $K_2 \subseteq K_1$, we have $q \in K_1^m$. As result, we obtain that $a \in Rol[K_1][m]$ and $q \in G[K_1][m](a)$. □

Theorem 3. *Let K_1 be a subring of $\mathbb{C}$ and K_2 a subring of K_1. Suppose that $a \in Rol[K_1]$ with the minimal characteristic polynomial $Q(z)$ over K_1. If $Q^*(z)$ is the reciprocal polynomial of $Q(z)$ and each complex root of $Q^*(z)$ is integral over K_2, then $a \in Rol[K_2]$ with reciprocal polynomial of minimal-degree monic multiple $P(z) \in K_2[z]$ of the polynomial $Q^*(z)$ as minimal characteristic polynomial. Otherwise, if $Q^*(z)$ has a complex root, which is not integral over K_2, then $a \notin Rol[K_2]$.*

Proof. Let $a \in Rol[K_1]$ with the minimal characteristic polynomial $Q(z)$ over K_1, $Q^*(z)$ the reciprocal polynomial of $Q(z)$ and each complex root of $Q^*(z)$ is integral over K_2. This means that there exists a minimal-degree monic polynomial $P(z) \in K_2[z]$, such that $P(z) \vdots Q^*(z)$. In conclusion, we obtain that $a \in Rol[K_2]$ with reciprocal polynomial of $P(z)$ as minimal characteristic polynomial.

In the case when $Q^*(z)$ has a root, which is not integral over K_2, then $Q^*(z)$ does not have any multiple in $K_2[z]$, which implies that also $Q(z)$ does not have any multiple in $K_2[z]$, i.e. we have $a \notin Rol[K_2]$. □

Corollary 1. *Let K_1 be a subfield of $\mathbb{C}$ and K_2 a subfield of K_1. If $a \in Rol[K_1]$ with the minimal characteristic polynomial $Q(z)$ over K_1 and each root of $Q(z)$ is algebraic over K_2, then $a \in Rol[K_2]$ with minimal-degree multiple $P(z) \in K_2[z]$ of the polynomial $Q(z)$ as minimal characteristic polynomial. Otherwise, if $Q(z)$ possesses a transcendental root over K_2, then $a \notin Rol[K_2]$.*

Proof. The proof follows immediately from Theorem 3. There is no need to use the monic reciprocal polynomial for the characteristic polynomial $Q(z)$, since the fields are endowed with division algebraic operation, which allows us to make the free term of the characteristic polynomial equal to 1. □

Corollary 1 allows us to decide if a general homogeneous linear recurrence (over $\mathbb{C}$) is also a homogeneous linear recurrence over a subfield, like $\mathbb{R}$, $\mathbb{Q}$ or an extension field of $\mathbb{Q}$. So, each homogeneous linear recurrence over $\mathbb{C}$ is also a homogeneous linear recurrence over $\mathbb{R}$, and, if its minimal characteristic polynomial over $\mathbb{C}$ or $\mathbb{R}$ does not have a transcendental root over $\mathbb{Q}$ (respectively over an extension of $\mathbb{Q}$), then it is also a homogeneous linear recurrence over $\mathbb{Q}$ (respectively over given extension of $\mathbb{Q}$).

Next, Theorem 3 provides a criterion to decide if a homogeneous linear recurrence over $\mathbb{C}$ is also a homogeneous linear recurrence over a subring, like $\mathbb{Z}$. However, the above remarks and the next theorem allow us to check easier if a homogeneous linear recurrence over $\mathbb{C}$ is also a homogeneous linear recurrence over $\mathbb{Z}$.

Theorem 4. *Let $a \in Rol[\mathbb{Q}]$ with minimal characteristic polynomial $Q(z)$. Then $a \in Rol[\mathbb{Z}]$ if and only if $Q(z) \in \mathbb{Z}[z]$.*

Proof. Let $a \in Rol[\mathbb{Q}]$ with minimal characteristic polynomial $Q(z)$ (over $\mathbb{Q}$). If $Q(z) \in \mathbb{Z}[z]$, then it is obvious that $a \in Rol[\mathbb{Z}]$ and $Q(z) \in H[\mathbb{Z}](a)$.

Next, we consider the converse. Let $a \in Rol[\mathbb{Z}]$ with minimal characteristic polynomial $P(z)$ (over $\mathbb{Z}$). Then $a \in Rol[\mathbb{Q}]$ and $P(z) \in H[\mathbb{Q}](a)$, which implies that $P(z) \vdots Q(z)$, i.e. there exists $T(z) \in \mathbb{Q}[z]$ such that $P(z) = Q(z)T(z)$.

Since $Q(z) \in \mathbb{Q}[z]$ and $Q(0) = 1$, there exists $q \in \mathbb{N}^*$ such that $qQ(z) \in \mathbb{Z}[z]$ and $qQ(0) = q$. Similarly, since $T(z) \in \mathbb{Q}[z]$, $P(0) = 1$ and $Q(0) = 1$, there exists $t \in \mathbb{N}^*$ such that $tT(z) \in \mathbb{Z}[z]$ and $tT(0) = t$. So, $qtP(z) = (qQ(z)) \cdot (tT(z))$.

The relation $P(z) \in \mathbb{Z}[z]$ implies that the left side $qtP(z)$ is divisible by qt. Next, because $qQ(z) \in \mathbb{Z}[z]$, any integral divisor r of $qQ(z)$ satisfies the inequality

$r \leq qQ(0) = q$. Similarly, since $tT(z) \in \mathbb{Z}[z]$, any integral divisor s of $tT(z)$ satisfies the inequality $s \leq tT(0) = t$. In result, we have $rs \leq qt$. But, due to fact that qt is a divisor of $(qQ(z)) \cdot (tT(z))$, this means that $r = q$ and $s = t$, i.e. $qQ(z)$ is divisible by q and $tT(z)$ is divisible by t. In conclusion, $Q(z) \in \mathbb{Z}$.

$\square$

4 Recurrence Criteria over Sign-Based Ring Subsets

In the context of this section, we will consider K a subset of $\mathbb{R}$ and $(K, +, \cdot)$ a ring with standard addition and multiplication arithmetic operations. In particular, K can be one of the well known sets $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ or an extension field of $\mathbb{Q}$.

Next, we will see in which conditions the sequence $a \in Rol[\mathbb{R}]$ is a homogeneous linear recurrence over K or a subset of it: K_-^*, respectively K_+^*.

Lemma 1. *The sets $Rol[K_-]$ and $Rol[K_-^*]$ are equal. If $P(z) \in H[K_-][m](b)$, then $(z+1)^{m-1}P(z) \in H[K_-^*](b)$.*

Proof. We will show that the sets $Rol[K_-]$ and $Rol[K_-^*]$ are equal, i.e. $Rol[K_-] \subseteq Rol[K_-^*]$ and $Rol[K_-^*] \subseteq Rol[K_-]$.

Let $\forall a \in Rol[K_-^*]$. Since $K_-^* \subseteq K_-$, we have $a \in Rol[K_-]$. So, in consequence, we obtain the inclusion $Rol[K_-^*] \subseteq Rol[K_-]$.

Now, we consider $\forall b \in Rol[K_-]$. Let $P(z) \in H[K_-](b)$ be an arbitrary characteristic polynomial of degree m of the sequence b. Then $P(z) \in K_+[z]$, which implies that $(z+1)^{m-1}P(z) \in K_+^*[z]$. Since $(z+1)^{m-1}P(z) \vdots P(z)$, we have $b \in Rol[K_-^*]$ and $(z+1)^{m-1}P(z) \in H[K_-^*](b)$. $\square$

Lemma 2. *The relation $a \in Rol[K_-^*]$ holds if and only if $a \in Rol[K]$ and its minimal characteristic polynomial $Q(z)$ over K has no positive real roots. Additionally, the characteristic polynomial of a over K_-^* can be chosen of the form $(z+1)^n Q(z)$.*

Proof. We consider $a \in Rol[K_-^*]$. Since $K_-^* \subseteq K$, we have $a \in Rol[K]$. Let $q = (q_k)_{k=0}^{m-1} \in G[K_-^*](a)$ be a generating vector of the sequence $a \in Rol[K_-^*]$. The corresponding characteristic polynomial is $H_m^{[q]}(z) = 1 - z \sum_{k=0}^{m-1} q_k z^k \in K_+^*[z]$. According to Descartes' rule of signs (see [1]), the characteristic polynomial $H_m^{[q]}(z)$ has no positive real roots. Since the minimal characteristic polynomial $Q(z)$ over K is a divisor of the characteristic polynomial $H_m^{[q]}(z)$, it also has no positive real roots.

Now we consider that $a \in Rol[K]$ and its minimal characteristic polynomial $Q(z)$ over K has no positive real roots. According to [1] and Lemma 1, there exists $n_0 \in \mathbb{N}$ such that, $\forall n > n_0$, the polynomial $(z+1)^n Q(z)$ has all its coefficients of the same sign, i.e. $(z+1)^n Q(z) \in \mathbb{R}_+^*[z]$. Since $(z+1)^n Q(z) \in K_+^*[z]$ and $(z+1)^n Q(z) \vdots Q(z)$, we have $a \in Rol[K_-^*]$ and $H_m^{[q]}(z) = (z+1)^n Q(z) \in H[K_-^*](a)$. $\square$

Lemma 3. *If $a \in Rol[K_-^*][m]$ and $q \in G[K_-^*][m](a)$, then $a \in Rol[K_+^*][m+1]$ with characteristic polynomial $(1-\alpha z)H_m^{[q]}(z)$, $\forall \alpha \in K$, $\alpha > \max\left\{-q_0, \max_{k=\overline{0,m-2}} \frac{q_{k+1}}{q_k}\right\}$.*

Proof. Let $a \in Rol[K_-^*][m]$, $q \in G[K_-^*][m](a)$ and $\alpha > \max\left\{-q_0, \max_{k=\overline{0,m-2}} \frac{q_{k+1}}{q_k}\right\}$, where $\alpha \in K$. Then we have $H_m^{[q]}(z) = 1 - z\sum_{k=0}^{m-1} q_k z^k \in K_+^*[z]$, which implies that

$$(1-\alpha z)H_m^{[q]}(z) = 1 - \sum_{k=1}^{m} q_{k-1}z^k - \alpha z + \sum_{k=1}^{m} \alpha q_{k-1} z^{k+1} =$$

$$= 1 - (\alpha + q_0)z + \sum_{k=2}^{m} q_{k-2}\left(\alpha - \frac{q_{k-1}}{q_{k-2}}\right) z^k + \alpha q_{m-1} z^{m+1}.$$

From the inequality $\alpha > \max\left\{-q_0, \max_{k=\overline{0,m-2}} \frac{q_{k+1}}{q_k}\right\} > 0$ we obtain

$$-(\alpha + q_0) < 0,\ \ \alpha q_{m-1} < 0,\ \ q_{k-2}\left(\alpha - \frac{q_{k-1}}{q_{k-2}}\right) < 0,\ \ k = \overline{2,m}.$$

Finally, since $(1-\alpha z)H_m^{[q]}(z) \vdots H_m^{[q]}(z)$, we have $a \in Rol[K_+^*][m+1]$ with characteristic polynomial $(1-\alpha z)H_m^{[q]}(z)$. $\square$

Theorem 5. *Let $a \in Rol[K]$ with minimal characteristic polynomial $Q(z)$ over K and N_Q^+ the number of positive real roots of $Q(z)$ counted with their multiplicity.*

- *If $N_Q^+ = 0$, then $a \in Rol[K_-^*] \cap Rol[K_+^*]$. In this case, $\exists n \in \mathbb{N}$ such that*

  $$H_m^{[q]}(z) = (z+1)^n Q(z) \in H[K_-^*](a)$$

  and

  $$R(z) = (1-\alpha z)H_m^{[q]}(z) \in H[K_+^*](a),$$

  $$\forall \alpha \in K, \alpha > \max\left\{-q_0, \max_{k=\overline{0,m-2}} \frac{q_{k+1}}{q_k}\right\};$$

- *If $N_Q^+ \geq 2$, then $a \notin Rol[K_-] \cup Rol[K_+]$;*
- *If $N_Q^+ = 1$, then $a \notin Rol[K_-]$. Additionally, considering R the unique positive real root of the polynomial $Q(z)$, we have $a \notin Rol[K_+]$ in each of the following cases:*
  - *there exists a root x of the polynomial $Q(z)$ such that $|x| < R$;*

- *$x = -R$ is a simple root of the polynomial $Q(z)$ and $Q(z)$ has at least one more negative real root $y \neq -R$;*
- *$x = -R$ is a multiple root of the polynomial $Q(z)$;*
- *$R > 1$ and $K \cap (0, 1) = \varnothing$;*
- *$R = 1$, $K \cap (0, 1) = \varnothing$ and there exists a root x of the polynomial $Q(z)$ such that $|x| \neq 1$.*

Proof. Let $N_Q^+ = 0$. According to Lemma 2, we have $a \in Rol[K_-^*]$ and $\exists n \in \mathbb{N}$ such that

$$H_m^{[q]}(z) = (z+1)^n Q(z) \in H[K_-^*](a).$$

Next, applying Lemma 3, we obtain that $a \in Rol[K_+^*]$ and

$$R(z) = (1 - \alpha z) H_m^{[q]}(z) \in H[K_+^*](a),$$

$$\forall \alpha \in K, \alpha > \max \left\{ -q_0, \max_{k=\overline{0,m-2}} \frac{q_{k+1}}{q_k} \right\}.$$

So, $a \in Rol[K_-^*] \cap Rol[K_+^*]$.

In the case $N_Q^+ \geq 2$, we have $N_P^+ \geq 2$, $\forall P(z) \vdots Q(z)$. So, according to Descartes' rule of signs, $P(z)$ has at least 2 sign changes between consecutive coefficients, which implies that $a \notin Rol[K_-] \cup Rol[K_+]$ (otherwise there would exist a polynomial $P(z) \vdots Q(z)$ with 0, respectively 1 sign changes between consecutive coefficients).

Next, we will consider $N_Q^+ = 1$ and R the unique positive real root of the polynomial $Q(z)$. Similar as above, $N_P^+ \geq 1$, $\forall P(z) \vdots Q(z)$. So, according to Descartes' rule of signs, $P(z)$ has at least 1 sign change between consecutive coefficients, which implies that $a \notin Rol[K_-]$ (otherwise there would exist a polynomial $P(z) \vdots Q(z)$ with 0 sign changes between consecutive coefficients).

Assume that $a \in Rol[K_+]$ and there exists a root x of the polynomial $Q(z)$ such that $|x| < R$. Then $\exists M \in \mathbb{N}^*$ such that $a \in Rol[K_+][M]$. Considering $p = (p_k)_{k=0}^{M-1} \in G[K_+][M](a)$, we have $H_M^{[p]}(z) = 1 - \sum\limits_{k=1}^{M} p_{k-1} z^k \in H[K_+][M]$, where $p_k \geq 0$, $k = \overline{0, M-1}$. Since $Q(z)$ is the minimal characteristic polynomial of a, we obtain that $H_M^{[p]}(z) \vdots Q(z)$, i.e. the positive real root R of $Q(z)$ is also a root of the polynomial $H_M^{[p]}(z)$. According to Descartes' rule of signs, this is the unique positive real root of $H_M^{[p]}(z)$. Then, every root z of the polynomial $H_M^{[p]}(z)$ satisfies the condition $|z| \geq r$, where r is the unique positive real root of the polynomial $T(z) = \sum\limits_{k=1}^{M} |-p_{k-1}| z^k - 1$. Since $T(z) = -H_M^{[p]}(z)$, we obtain $r = R$, which implies that every root z of the polynomial $H_M^{[p]}(z)$ satisfies the condition $|z| \geq R$. Finally, since $H_M^{[p]}(z) \vdots Q(z)$, we obtain that every root z of the polynomial $Q(z)$ satisfies the condition $|z| \geq R$, which is a contradiction with our assumption. So, if there exists a root x of the polynomial $Q(z)$ such that $|x| < R$, then $a \notin Rol[K_+]$.

Next, in the case when $x = -R$ is a root of the polynomial $Q(z)$, taking into account that $H_M^{[p]}(z) \vdots Q(z)$, we have $H_M^{[p]}(-R) = H_M^{[p]}(R) = 0$, i.e.

$$\sum_{k=1}^{M} p_{k-1} R^k = \sum_{k=1}^{M} p_{k-1} (-R)^k.$$

As result, we obtain $\sum\limits_{k-odd,\ 1 \leq k \leq M} p_{k-1} R^k = 0$, which implies that $p_k = 0$, for every even number k, $0 \leq k \leq M-1$. This means that the polynomial $H_M^{[p]}(z)$ is an even function, i.e. $H_M^{[p]}(z) = H_M^{[p]}(-z)$, $\forall z \in \mathbb{R}$. The existence of an additional negative real root $y \neq -R$ would mean that also $z = -y \neq R$ is a positive real root, contradiction with uniqueness of the positive real root R. Also, if $x = -R$ was a multiple root of the polynomial $Q(z)$, it would be a multiple root of the polynomial $H_M^{[p]}(z)$, which would imply that $\left. \frac{\partial}{\partial z} H_M^{[p]}(z) \right|_{z=-R} = 0$. But this would mean that $\sum\limits_{k=0}^{M-1} (k+1) p_k (-R)^k = 0$ and, taking into account that $p_k = 0$ for every even number k, $0 \leq k \leq M-1$, this would have as a consequence the equality $\sum\limits_{k-odd,\ 0 \leq k \leq M-1} (k+1) p_k R^k = 0$, i.e. $p_k = 0$, for every odd number k. As result, it would be $p_k = 0$, $k = \overline{0, M-1}$, which would be impossible. So, $a \notin Rol[K_+]$.

Now, we consider the last case, when $K \cap (0,1) = \varnothing$ and at least one of the following conditions is true: $R > 1$ or ($R = 1$ and there exists a root x of the polynomial $Q(z)$ such that $|x| \neq 1$). If $R \geq 1$, then

$$0 = H_M^{[p]}(R) = 1 - \sum_{k=1}^{M} p_{k-1} R^k \leq 1 - p_{M-1} R^M \leq 1 - R^M \leq 1 - 1 = 0,$$

which may happen only in the case when $R = 1$ and $H_M^{[p]}(z) = 1 - z^M$, contradiction with our assumption. So, $a \notin Rol[K_+]$. $\square$

As it was mentioned above, each homogeneous linear recurrence over $\mathbb{C}$ is also a homogeneous linear recurrence over $\mathbb{R}$, and, if its minimal characteristic polynomial over $\mathbb{R}$ does not have a transcendental root over $\mathbb{Q}$, then it is also a homogeneous linear recurrence over $\mathbb{Q}$. In this case, if $Q(z) \in \mathbb{Z}[z]$ is its minimal characteristic polynomial over $\mathbb{Q}$, then it is also a homogeneous linear recurrence over $\mathbb{Z}$.

Additionally, according to Theorem 5, if $Q(z)$ does not have positive real roots, then it is also a homogeneous linear recurrence over $\mathbb{N}^*$. Otherwise, if $Q(z)$ has at least two distinct positive real roots or one multiple positive real root, then it is not a homogeneous linear recurrence over $\mathbb{N}^*$.

The trickiest case is the case when $Q(z)$ has exactly one simple positive real root R. Theorem 5 analyzes several categories. According to it, the given homogeneous linear recurrence over $\mathbb{Z}$ is not homogeneous linear recurrent over $\mathbb{N}^*$ at least in the following situations:

- $R > 1$;
- $R = 1$ and $Q(z)$ has at least one root $|x| \neq 1$;
- $Q(z)$ has at least one root $|x| < R$;
- $x = -R$ is a simple root of the polynomial $Q(z)$ and $Q(z)$ has at least one more negative real root $y \neq -R$;
- $x = -R$ is a multiple root of the polynomial $Q(z)$.

It does not give necessary and sufficient conditions in this case.

5 Littlewood, Newman and Borwein Homogeneous Linear Recurrences

Littlewood, Newman and Borwein polynomials were defined in [2] in the following way. A Newman polynomial represents a polynomial with all coefficients in $\{0, 1\}$ and free term 1. Similarly, a Littlewood polynomial is a polynomial with all coefficients belonging to $\{-1, 1\}$. In the end, the polynomials with non-zero free term and all coefficients from the set $\{-1, 0, 1\}$ are called Borwein polynomials.

A Littlewood or Borwein polynomial is called alternating if the signs of every two consecutive non-zero coefficients are different. It is called even-alternating in the case when the number of non-zero coefficients is odd (there is an even number of sign changes) and odd-alternating otherwise.

Next, we will introduce similar definitions for Littlewood, Newman and Borwein homogeneous linear recurrences. A homogeneous linear $m-$recurrence is called Littlewood, Newman, respectively Borwein homogeneous linear $m-$recurrence, if it has at least one Littlewood, Newman, respectively Borwein characteristic polynomial of degree m. Similarly, a homogeneous linear recurrence is called Littlewood, Newman, respectively Borwein, if it is a Littlewood, Newman, respectively Borwein homogeneous linear m-recurrence for an arbitrary m.

Additionally, a Littlewood or Borwein homogeneous linear recurrence is called alternating, if it has at least one alternating Littlewood, respectively Borwein characteristic polynomial. It is called even-alternating in the case when the corresponding characteristic polynomial is even-alternating and odd-alternating otherwise.

Taking into account these definitions, we can conclude that $Rol[\{-1, 0\}][m]$ is the set of all Newman homogeneous linear m-recurrences, $Rol[\{-1, 1\}][m]$ is the set of all Littlewood homogeneous linear m-recurrences and $Rol[\{-1, 0, 1\}][m]$ is the set of all Borwein homogeneous linear m-recurrences. Similarly, $Rol[\{-1, 0\}]$ represents the set of all Newman homogeneous linear recurrences, $Rol[\{-1, 1\}]$ is the set of all Littlewood homogeneous linear recurrences and $Rol[\{-1, 0, 1\}]$ represents the set of all Borwein homogeneous linear recurrences.

Newman polynomials were studied in [3]. It is known that all real roots of Newman polynomials must lie in the interval $\left(-\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$. Additionally, in [7] several bounds for the location of complex roots of Newman polynomials were obtained. These results can be considered as necessary conditions for a homogeneous linear recurrence over K to be Newman, where $K \supseteq \{-1, 0\}$. For instance, if the minimal characteristic polynomial over K has one real root outside of the interval $\left(-\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$, then the recurrence is not Newman (since the roots of minimal characteristic polynomial are also roots of any characteristic polynomial, which cannot be Newman in this case).

Additionally, the even-alternating Borwein polynomials were mentioned in [3]. More exactly, given any positive integers $d_1 < d_2 < \ldots < d_t$ with t even, Borwein polynomial defined as $Q(z) = 1 - z^{d_1} + z^{d_2} - \ldots + z^{d_t}$ has as multiple the Newman polynomial $Q(z)(1 + z + z^2 + \ldots + z^{d_t})$. In consequence, every even-alternating Borwein homogeneous linear m-recurrence is a Newman homogeneous linear $2m$-recurrence.

The main result from [3] is referred to cyclotomic polynomials not vanishing at 1. It claims that every such polynomial is a divisor of a Newman polynomial. Also, a constructive way for determining it is given. As a consequence, every homogeneous linear recurrence, with cyclotomic polynomial not vanishing at 1 as characteristic polynomial, is a Newman homogeneous linear recurrence.

According to [4], every Newman polynomial of degree at most 8 divides some Littlewood polynomials. This means that each Newman homogeneous linear m-recurrence, with $m \leq 8$, is also a Littlewood homogeneous linear recurrence.

The next Theorem summarizes all above results regarding Littlewood, Newman and Borwein homogeneous linear recurrences.

Theorem 6. *The following statements are true:*

- *If a is a homogeneous linear recurrence over K, where $K \supseteq \{-1, 0\}$, and the minimal characteristic polynomial of a over K has one real root outside of the interval $\left(-\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$, then the recurrence is not Newman;*
- *Every even-alternating Borwein homogeneous linear m-recurrence with characteristic polynomial of the form $Q(z) = 1 - z^{d_1} + z^{d_2} - \ldots + z^{d_t}$ is a Newman homogeneous linear $2m$-recurrence with $Q(z)(1 + z + z^2 + \ldots + z^{d_t})$ as characteristic polynomial;*
- *Each homogeneous linear recurrence, with cyclotomic polynomial not vanishing at 1 as characteristic polynomial, is a Newman homogeneous linear recurrence;*
- *Every Newman homogeneous linear m-recurrence, with $m \leq 8$, is also a Littlewood homogeneous linear recurrence.*

The following Theorem is a direct consequence of the corresponding theorem from [4]. It describes necessary condition for a homogeneous linear recurrence over $\mathbb{Z}$ to not be a Littlewood homogeneous linear recurrence.

Theorem 7. *Consider the sequence $a \in Rol[\mathbb{Z}]$ with minimal characteristic polynomial $Q(z) \in H[\mathbb{Z}](a)$. Let $Q^*(z)$ be the reciprocal polynomial of $Q(z)$, with roots of modulus strictly greater than 1 labelled as $\alpha_1, \ldots, \alpha_k$, where $k \geq 1$. Suppose that there exist a positive integer N and a real number $\delta \geq 0$ with the property that, for each of the 2^N vectors $b = (b_1, \ldots, b_N)$, where $b_1, \ldots, b_N \in \{-1, 1\}$, there are two positive integers $n = n(b) \leq N$ and $i = i(b) \leq k$ such that*

$$(|\alpha_i| - 1)|\alpha_i^n + b_1\alpha_i^{n-1} + \ldots + b_n| \geq 1 + \delta.$$

Then a is not a Littlewood homogeneous linear recurrence.

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