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Maximum nontrivial convex cover number of join and corona of graphs

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Abstract. Let G be a connected graph. We say that a set $S \subseteq X(G)$ is convex in G if, for any two vertices $x, y \in S$, all vertices of every shortest path between x and y are in S. If $3 \leq |S| \leq |X(G)| - 1$, then S is a nontrivial set. The greatest $p \geq 2$ for which there is a cover of G by p nontrivial and convex sets is the maximum nontrivial convex cover number of G. In this paper, we determine the maximum nontrivial convex cover number of join and corona of graphs.

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1 Introduction

We denote by G a graph with vertex set $X(G)$ and edge set $U(G)$. The *neighborhood* of $x \in X(G)$ is the set of all vertices $y \in X(G)$, $y \sim x$ (i.e. adjacent to x), and it is denoted by $\Gamma(x)$. A set $S \subset X(G)$ is called *nontrivial* if $3 \leq |S| \leq |X(G)| - 1$. A vertex $x \in X(G)$ is called *universal* if $\Gamma(x) = X(G) \backslash \{x\}$. We denote by $G[S]$ the subgraph of G induced by a set S. A *dominating* set of a graph G is a subset D of $X(G)$ such that every vertex not in D is adjacent to at least one member of D. A vertex $x \in X(G) \backslash S$ is an *external private neighbor* of y (with respect to S) if $\Gamma(x) \cap S = \{y\}$ [1]. A set $S \subseteq X(G)$ is said to be a *private dominating set* of a graph G if it is a dominating set and for every $y \in S$ there exists an external private neighbor $x \in X(G) \backslash S$ [2]. A *minimum dominating set* is a dominating set of the smallest size in a given graph. The *domination number* $\gamma(G)$ is the number of vertices in a minimum dominating set of G.

A set $S \subseteq X(G)$ is called *convex* in G if, for any two vertices $x, y \in S$, all vertices of every shortest path between x and y are in S [3]. By [4], a family of sets $\mathcal{P}(G)$ is called a *nontrivial convex cover* of a graph G if the following conditions hold:

1) each set of $\mathcal{P}(G)$ is nontrivial and convex in G;
2) $X(G) = \bigcup_{S \in \mathcal{P}(G)} S$;
3) $S \not\subseteq \bigcup_{C \in \mathcal{P}(G), C \neq S} C$ for each $S \in \mathcal{P}(G)$.

Also, in [4] the following notions are introduced. A vertex $x \in X(G)$ is called *resident* in $\mathcal{P}(G)$ if x belongs to only one set of $\mathcal{P}(G)$. The greatest $p \geq 2$ for which there is a cover of G by p nontrivial and convex sets is the *maximum nontrivial convex cover number* of G and it is denoted by $\varphi_{cn}^{max}(G)$. A nontrivial convex cover of a graph G that contains exactly $\varphi_{cn}^{max}(G)$ sets is called a *maximum nontrivial convex cover* of G and it is denoted by $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(G)$.

It was shown that it is NP-hard to determine the maximum nontrivial convex cover number of a given graph [5]. Later, a binary linear programming model that allows finding the maximum nontrivial convex cover number for small-sized graphs was proposed [6]. Besides, it is of interest to identify classes of graphs, including graphs resulting from some graph operation, for which the maximum nontrivial convex cover number can be easily determined. Some estimations of the maximum nontrivial convex cover number for trees are given in [7] and [8]. In this paper we establish the maximum nontrivial convex cover number of graphs resulting from join and corona of graphs.

2 Join of Graphs

The join of graphs G and H is the graph $G + H$ on vertex set $X(G + H) = X(G) \cup X(H)$ and edge set $U(G+H) = U(G) \cup U(H) \cup \{xy : x \in X(G), y \in X(H)\}$, where xy is an edge joining vertices x and y.

Before proceeding further with analysis of maximum nontrivial convex cover number of graphs resulting from join of two graphs, we state the following important result, which guarantees that the minimum cardinality of a private dominating set in a graph G always equals $\gamma(G)$.

Theorem 1. **[2]** *Every graph G without isolated vertices has a minimum dominating set which is also a private dominating set.*

Theorem 2. *Let G be a connected graph of order n and K_m the complete graph of order m. Then the following statements hold:*

1. $\varphi_{cn}^{max}(G + K_m) \geq n - \gamma(G)$ *if* $n \geq 3$ *and* $m = 1$;
2. $\varphi_{cn}^{max}(G + K_m) = n + m - 2$ *if* $n = 1$ *and* $m \geq 3$, *or* $n \geq 2$ *and* $m \geq 2$.

Proof. 1) Assume that $n \geq 3$ and $m = 1$. We define a family of nontrivial and convex sets $\boldsymbol{\mathcal{P}}(G + K_1) = \varnothing$ that will cover the graph $G + K_1$. By Theorem 1, there exists a private dominating set D of G with cardinality $\gamma(G)$. We add the set of vertices $\{x, y, z\}$ to $\boldsymbol{\mathcal{P}}(G + K_1)$ for each $x \in X(G) \backslash D$, where $y \in \Gamma(x) \cap D$ and $z \in X(K_1)$. Consequently, $|\boldsymbol{\mathcal{P}}(G + K_1)| = n - \gamma(G)$ and thus $\varphi_{cn}^{max}(G + K_1) \geq n - \gamma(G)$.

2) Assume that $n = 1$ and $m \geq 3$. We define a family of nontrivial and convex sets $\boldsymbol{\mathcal{P}}(G + K_m) = \varnothing$ that will cover the graph $G + K_m$, and choose two vertices $x, y \in X(K_m)$, $x \neq y$. Then, for each $z \in X(G+K_m) \backslash \{x, y\}$, we add the set $\{x, y, z\}$ to $\boldsymbol{\mathcal{P}}(G+K_m)$. Since every set of $\boldsymbol{\mathcal{P}}(G+K_m)$ consists of three vertices, two of which are x and y and the third vertex is resident in $\boldsymbol{\mathcal{P}}(G+K_m)$, it follows that $\boldsymbol{\mathcal{P}}(G+K_m)$ is a maximum nontrivial convex cover of $G + K_m$ and $\varphi_{cn}^{max}(G + K_m) = n + m - 2$.

3) If $n \geq 2$ and $m \geq 2$, then by following the same arguments as described above, we get $\varphi_{cn}^{max}(G + K_m) = n + m - 2$. □

For any connected non-complete graphs G and H, there is the following characterization of convex sets in $G+H$.

Theorem 3. [9] *Let G and H be connected non-complete graphs. Then a proper subset $C = S_1 \cup S_2$ of $X(G+H)$, where $S_1 \subset X(G)$ and $S_2 \subset X(H)$, is convex in $G+H$ if and only if S_1 and S_2 are cliques in G and H respectively.*

Theorem 4. *Let G and H be connected non-complete graphs of order n and m respectively. Then $\varphi_{cn}^{max}(G+H) \geq n+m-4$.*

Proof. We define a family of nontrivial and convex sets $\mathcal{P}(G+H) = \varnothing$ that will cover $G+H$. Now, we choose four vertices $g_1, g_2 \in X(G)$ and $h_1, h_2 \in X(H)$ such that $g_1 \sim g_2$, $h_1 \sim h_2$. Then, for each $h \in X(H)\backslash\{h_1, h_2\}$ and for each $g \in X(G)\backslash\{g_1, g_2\}$, we add sets $\{g_1, g_2, h\}$ and $\{h_1, h_2, g\}$ to $\mathcal{P}(G+H)$. It can be easily verified that $|\mathcal{P}(G+H)| = n+m-4$. So, $\varphi_{cn}^{max}(G+H) \geq n+m-4$. □

Theorem 5. *Let G and H be connected non-complete graphs of order n and m respectively. Then $\varphi_{cn}^{max}(G+H) = n+m-2$ if and only if there are two universal vertices in $G+H$.*

Proof. Let $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ be a maximum nontrivial convex cover of $G+H$ and $\varphi_{cn}^{max}(G+H) = n+m-2$. This entails that each set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G+K_m)$ has cardinality 3 and contains exactly one resident vertex in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$. Moreover, there are two vertices x and y of $X(G+H)$, which are common for all sets of the family $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$. By Theorem 3, it follows that $x \sim y$ and any vertex of $X(G+H)\backslash\{x, y\}$ is simultaneously adjacent to x and y. Therefore, vertices x and y are universal in $G+H$.

Assume that there are two universal vertices x and y in $G+H$, which in turn means that $x \sim y$. We define a family of nontrivial and convex sets $\mathcal{P}(G+H) = \varnothing$ that will cover $G+H$. We add the set of vertices $\{x, y, z\}$ to $\mathcal{P}(G+H)$ for each $z \in X(G+H)\backslash\{x, y\}$. It is clear that $\varphi_{cn}^{max}(G+H) = |\mathcal{P}(G+H)| = n+m-2$. □

Theorem 6. *Let G and H be connected non-complete graphs of order n and m respectively. Then $\varphi_{cn}^{max}(G+H) = n+m-3$ if and only if there are no two universal vertices in $G+H$ and there are three vertices x, y, z which induce a triangle or a path in $G+H$ such that for each vertex $v \in X(G)\backslash\{x, y, z\}$, there are two vertices $a, b \in \Gamma(v) \cap \{x, y, z\}$, $a \sim b$.*

Proof. Let $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ be a maximum nontrivial convex cover of $G+H$ such that $\varphi_{cn}^{max}(G+H) = n+m-3$. By considering Theorem 5, there are no two universal vertices in $G+H$. It follows from Theorem 3 that any convex set of $G+H$ is a clique. There is not a set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ such that $|S| \geq 5$, because in this case we get $\varphi_{cn}^{max}(G+H) \leq n+m-4$. So, each set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ contains no more than four vertices. We consider two cases.

1) Suppose there is a set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G+H)$, $|S| = 4$, containing two resident vertices r_1 and r_2 in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$. By replacing S with two sets $S_1 = S\backslash\{r_1\}$ and $S_2 = S\backslash\{r_2\}$ in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$, we obtain a new nontrivial convex cover of

$G+H$ that has one more set than $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$. This contradiction implies that each set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ of cardinality 4 contains exactly one resident vertex in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ and, furthermore, all vertices of $X(G+H)\backslash S$ are resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ and belong to different sets of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$.

2) Suppose that every set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ consists of three vertices. If there are two distinct sets S_1 and S_2 of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ for which $S_1 \cap S_1 = \varnothing$, then we obtain $\varphi_{cn}^{max}(G+H) \leq n+m-4$. This implies that the intersection of any two sets of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ is not empty. Now assume that there are two sets of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$, each of which has two resident vertices, or there is only one set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ containing two resident vertices and the number of non-resident vertices in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ is greater than or equal to 3. Combining this assumption with the fact that $|X(G+H)| \geq 6$, we again obtain $\varphi_{cn}^{max}(G+H) \leq n+m-4$. So, exactly one set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ has two resident vertices and the number of non-resident vertices in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ is less than 3, or each set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ contains exactly one resident vertex.

For both cases, let R be a set of resident vertices in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ that contains exactly one vertex from each set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$. It is obvious that the set $\{x,y,z\} = \bigcup_{S \in \mathcal{P}_{\varphi_{cn}^{max}}(G+H)} S\backslash R$ induces a triangle or a path in $G+H$ such that for each vertex $v \in X(G)\backslash\{x,y,z\}$, there are two vertices $a,b \in \Gamma(v) \cap \{x,y,z\}$, $a \sim b$.

Assume that there are no two universal vertices in $G+H$ and there are three vertices x, y, z which induce a triangle or a path in $G+H$ such that for each vertex $v \in X(G)\backslash\{x,y,z\}$, two different vertices $a,b \in \Gamma(v) \cap \{x,y,z\}$, $a \sim b$, exist. From Theorems 4 and 5 follows $n+m-4 \leq \varphi_{cn}^{max}(G+H) \leq n+m-3$. We define a family of nontrivial and convex sets $\mathcal{P}(G+H) = \varnothing$ that will cover $G+H$. Trivially, we add the set of vertices $\{v,a,b\}$ to $\mathcal{P}(G+H)$ for each $v \in X(H)\backslash\{x,y,z\}$, where $a,b \in \Gamma(v) \cap \{x,y,z\}$, $a \sim b$. Thus, $|\mathcal{P}(G+H)| = \varphi_{cn}^{max}(G+H) = n+m-3$. $\square$

Summarizing the above theorems, we have the following result.

Corollary 1. *Let G and H be connected non-complete graphs of order n and m respectively. Then $\varphi_{cn}^{max}(G+H) = n+m-4$ if and only if the following conditions hold:*

1. *There are no two universal vertices in $G+H$;*
2. *There are no three vertices x, y, z which induce a triangle or a path in $G+H$ such that for each vertex $v \in X(G)\backslash\{x,y,z\}$, there are two different vertices $a,b \in \Gamma(v) \cap \{x,y,z\}$, $a \sim b$.*

3 Corona of Graphs

The *corona* of graphs G and H is the graph $G \odot H$ constructed as follows:

1) $X(G \odot H) = X(G) \cup \bigcup_{x \in X(G)} X(H_x)$, where H_x is a copy of H that corresponds to a vertex $x \in X(G)$;

2) $U(G \odot H) = U(G) \cup \{xy : x \in X(G) \text{ and } y \in X(H_x)\}$.

The following theorem is important and is used in the proof of Theorem 8.

Theorem 7. **[10]** *Let G be a connected graph and H be any graph, with $X(G) = \{x_1, \ldots, x_n\}$ and $H_{x_1}, H_{x_2}, \ldots, H_{x_n}$ being the corresponding copies of H. A nonempty set $C \subseteq X(G \odot H)$ is convex in $G \odot H$ if and only if it satisfies one of the following conditions:*

1. *C is convex in G;*
2. *C induces a complete subgraph of H_x for a vertex $x \in X(G)$;*
3. *$(G \odot H)[C] = (G[S]; \{s_1, \ldots, s_k\}) \odot (H'_{s_1}, \ldots, H'_{s_k})$, where S is a convex set in G, $\{s_1, \ldots, s_k\} \subseteq S$, and $X(s_i + H'_{s_i})$ is convex in $s_i + H_{s_i}$ for each i, $1 \leq i \leq k$, where H'_{s_i} is a subgraph of H_{s_i}.*

Theorem 8. *Let G and H be connected graphs of order n and m respectively. Then the following statements hold:*

1. *$\varphi_{cn}^{max}(G \odot H) \geq m - \gamma(H)$ if $n = 1$ and $m \geq 3$;*
2. *$\varphi_{cn}^{max}(G \odot H) = nm$ if $n \geq 2$ and $m \geq 1$.*

Proof. 1) Suppose that $n = 1$ and $m \geq 3$. Here, in fact, $G \odot H = K_1 + H$. Then by Theorem 2, we get $\varphi_{cn}^{max}(G \odot H) \geq m - \gamma(H)$.

2) Suppose that $n \geq 2$ and $m \geq 1$. Let us consider two cases:

a) Assume $m = 1$. Note that by covering each vertex $h \in \bigcup_{1 \leq i \leq n} X(H_{x_i})$ with nontrivial and convex sets, we at the same time cover all vertices of $X(G)$. So, $\varphi_{cn}^{max}(G \odot H) \leq nm$ (here, $m = 1$). We define a family of nontrivial and convex sets $\mathcal{P}(G \odot H) = \varnothing$ that will cover graph $G \odot H$. We add the set of vertices $\{h, x_i, y\}$ to $\mathcal{P}(G \odot H)$ for each $x_i \in X(G)$, where $h \in X(H_{x_i})$ and $y \in X(G)$, $y \sim x_i$. It is clear that $|\mathcal{P}(G \odot H)| = nm$ (here, also, $m = 1$). Thus, $\varphi_{cn}^{max}(G \odot H) = nm$.

b) Assume $m \geq 2$. We will show that there exists a maximum nontrivial convex cover of $G \odot H$ such that all vertices of $X(G)$ are not resident in this cover. Let $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$ be a maximum nontrivial convex cover of $G \odot H$. Let x be a vertex of $X(G)$ that is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$, $x \in S$, $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$.

If $|S \cap X(G)| \geq 2$, then by removing from $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$ all sets containing vertices of $\{x\} \cup X(H_x)$, we obtain a family of sets $\mathcal{P}(G \odot H)$. We add the set of vertices $\{y\} \cup (S \backslash X(H_x))$ to $\mathcal{P}(G \odot H)$ for each $y \in X(H_x)$. In this way, we have removed no more than $m - 1$ sets, and have added m nontrivial and convex sets such that each of them has at least one resident vertex in $\mathcal{P}(G \odot H)$. Therefore, $|\mathcal{P}(G \odot H)| > |\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)|$. This contradiction yields that $|S \cap X(G)| = 1$.

We choose an arbitrary vertex $y \in \Gamma(x) \cap X(G)$, $y \in C$, $C \in \mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$. We define a nontrivial convex cover $\mathcal{P}(G \odot H)$ consisting of sets from $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$ except for sets which contain vertices of $\{x\} \cup X(H_x)$. If y is a resident vertex in $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$ and C does not contain another resident vertex in $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$, then we also remove C from $\mathcal{P}(G \odot H)$. Finally, we add the set $\{x, y, z\}$ to $\mathcal{P}(G \odot H)$ for each $z \in X(H_x)$. It can be easily seen that $\mathcal{P}(G \odot H)$ contains at least the same number of nontrivial and convex sets as $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$, and all vertices of $X(G)$ are not resident in $\mathcal{P}(G \odot H)$. It can be concluded that any nontrivial convex cover of $G \odot H$ has at most nm sets.

Now, we define a family of sets $\mathcal{P}(G \odot H)$ in which all vertices of $X(G)$ are not resident and all vertices of each $X(H_{x_i})$, $1 \leq i \leq n$, are resident and belong to different sets of $\mathcal{P}(G \odot H)$. The family $\mathcal{P}(G \odot H)$ consists of sets $\{y, x_i, h_j\}$ for all $x_i \in X(G)$, $1 \leq i \leq n$, and for all $h_j \in X(H_{x_i})$, $1 \leq j \leq m$, where $y \in X(G)$, $y \sim x_i$. Since $|\mathcal{P}(G \odot H)| = nm$, this family is a maximum nontrivial convex cover of $G \odot H$, and furthermore $\varphi_{cn}^{max}(G \odot H) = nm$. □

4 Conclusion

We have established the maximum nontrivial convex cover number of graphs resulting from join and corona of graphs. Since it is NP-hard to determine the maximum nontrivial convex cover number for a general graph, it is important to investigate this problem for graphs resulting from any other graph operations. Further research can be focused on cartesian and lexicographic products of graphs.

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