BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Numbers 1(95)–2(96), 2021, Pages 84–92
ISSN 1024–7696

On non-discrete topologization of some countable skew fields

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Abstract. If for any finite subset M of a countable skew field R there exists an infinite subset $S \subseteq R$ such that $r \cdot m = m \cdot r$ for any $r \in S$ and for any $m \in M$, then the skew field R admits:
– A non-discrete Hausdorff skew field topology τ_0.
– Continuum of non-discrete Hausdorff skew field topologies which are stronger than the topology τ_0 and such that $\sup\{\tau_1, \tau_2\}$ is the discrete topology for any different topologies τ_1 and τ_2;
– Continuum of non-discrete Hausdorff skew field topologies which are stronger than τ_0 and such that any two of these topologies are comparable;
– Two to the power of continuum Hausdorff skew field topologies stronger than τ_0, and each of them is a coatom in the lattice of all skew field topologies of the skew fields.

Mathematics subject classification: 22A05.
Keywords and phrases: Countable skew fields, center of skew field, topological skew fields, Hausdorff topology, basis of the filter of neighborhoods, number of topologies on countable skew fields, lattice of topologies on skew fields, right Ore condition, ring of right quotients, ring of polynomials in the variable x.

1 Introduction

The study of possibility to set a non-discrete Hausdorff topology on infinite algebraic systems in which existing operations are continuous was begun in [1]. In this article for any countable group a method of constructing such group topologies was given. For countable rings the problem of the possibility to set non-discrete Hausdorff ring topologies was studied in [2, 3]. For infinite fields the problem of the possibility to set non-discrete field topologies was studied in [2]. For countable skew fields the problem of the possibility to set non-discrete Hausdorff topologies has not been yet solved. The present article is a continuation of research in this direction. The main result of this paper is Theorem 3.2, in which it is proved that if for any finite subset S of a countable skew field R there exists an infinite subset $M \subseteq R$ such that $r \cdot s = s \cdot r$ for any elements $r \in M$ and $s \in S$, then the skew field R permits non-discrete Hausdorff skew field topology. For countable groups, countable rings and countable fields similar results were obtained in [4, 5, 6, 8].

2 Notations and preliminaries

To present the main results we remind the following well-known results.
Notation 2.1. If R is a skew field, then we denote its unit by e.

Theorem 2.2. *A set Ω of subsets of a skew field R is a basis of filter of neighborhoods of zero for some Hausdorff skew field topology τ on the skew field R if and only if the following conditions are satisfied:*

1) $\bigcap_{V\in\Omega} V = \{0\}$;

2) *For any subsets V_1 and $V_2 \in \Omega$ there exists a subset $V_3 \in \Omega$ such that $V_3 \subseteq V_1 \cap V_2$;*

3) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $V_2 + V_2 \subseteq V_1$;*

4) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $-V_2 \subseteq V_1$;*

5) *For any subset $V_1 \in \Omega$ and any element $r \in R$ there exists a subset $V_2 \in \Omega$ such that $r \cdot V_2 \subseteq V_1$;*

6) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $V_2 \cdot V_2 \subseteq V_1$.*

7) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $(e+V_2)^{-1} - e \subseteq V_1$.*

Definition 2.3. If a ring R has no divisors of zero and for any non-zero elements $a, b \in R$ there are non-zero elements $a_1, b_1 \in R$ such that $a \cdot a_1 = b \cdot b_1$, then we say that the ring R satisfies the right Ore condition.

Proposition 2.4. [1] *If a ring R has no divisors of zero and the ring R satisfies the right Ore condition, then the set $\widehat{R} = \{a \cdot b^{-1} | a, b \in R$ and $0 \neq b\}$ is a skew field.*

Proposition 2.5. [2] Let R be a skew field and let $\alpha : R \longrightarrow R$ be an automorphism. If x is a variable and we will define $\alpha(r) \cdot x = x \cdot r$ for any $r \in R$, then the following statements are true:

Statement 1. $R_\alpha(x) = \{\sum_{i=0}^{n} r_i \cdot x^i | r_i \in R$ and n is a natural number $\}$ is an associative ring and the ring $R_\alpha(x)$ has no divisors of zero;

Statement 2. The ring $R_\alpha(x)$ satisfies the right Ore condition.

Proof. As $x \cdot r = \alpha(r) \cdot x$ for any element $r \in R$ then

$$(\sum_{i=0}^{n} a_i \cdot x^i) \cdot (\sum_{j=0}^{m} b_j \cdot x^j) = \sum_{k=0}^{n+m} \sum_{i+j=k} a_i \cdot x^i \cdot b_j \cdot x^j = \sum_{k=0}^{n+m} \big(\sum_{i+j=k} a_i \cdot \alpha^i(b_j) \big) \cdot x^k \in R(x)$$

and hence $R(x)$ is a ring.

As $\Big((\sum_{i=0}^{n} a_i \cdot x^i) \cdot (\sum_{j=0}^{m} b_j \cdot x^j) \Big) \cdot (\sum_{t=0}^{s} c_t \cdot x^t) =$

$$(\sum_{k=0}^{n+m} \sum_{i+j=k} a_i \cdot \alpha^i(b_j) \cdot x^{i+j}) \cdot (\sum_{t=0}^{s} c_t \cdot x^t) = \sum_{p=0}^{n+m+s} \sum_{i+j+t=p} a_i \cdot \alpha^i(b_j) \cdot \alpha^{i+j}(c_t) \cdot x^{i+j+t} =$$

[1] This result is well known to specialists in algebra, and the ring $\widehat{R}$ is called the ring of skew field quotients of the ring R (see [9]).

[2] Perhaps, this result was previously obtained by other authors, but since we do not know where there exists proof of this result, then we present its proof below. However, we do not claim the authorship.

$$\Big(\sum_{i=0}^{n} a_i \cdot x^i\Big) \cdot \Big(\sum_{q=0}^{m+s} b_j \cdot \alpha^j(c_t) x^{j+t}\Big) = \Big(\sum_{i=0}^{n} a_i \cdot x^i\Big) \cdot \Big(\Big(\sum_{j=0}^{m} b_j \cdot x^j\Big) \cdot \Big(\sum_{t=0}^{s} c_t \cdot x^t\Big)\Big),$$

and hence the ring $R(x)$ is an associative ring.

As $a_n \cdot x^n \cdot b_m \cdot x^m = a_n \cdot \alpha^n(b_m) \cdot x^{n+m} \neq 0$ for any elements $a_n, b_m \in R \setminus \{0\}$ and any natural numbers n and m, then $\big(\sum\limits_{i=0}^{n} a_i \cdot x^i\big) \cdot \big(\sum\limits_{j=0}^{m} b_j \cdot x^j\big) =$

$$\sum_{k=0}^{n+m} \sum_{i+j=k} a_i \cdot x^i \cdot b_j \cdot x^j = a_n \cdot \alpha^n(b_m) \cdot x^{n+m} + \sum_{k=0}^{n+m-1} \Big(\sum_{i+j=k} a_i \cdot \alpha^i(b_j)\Big) \cdot x^k \neq 0,$$

and hence the ring $R(x)$ has no divisors of zero.

Thus Statement 1 is proved.

Let $0 \neq f(x) = \sum\limits_{i=0}^{k} a_i \cdot x^i \in R(x)$ and $0 \neq \varphi(x) = \sum\limits_{i=0}^{n} b_i \cdot x^i \in R(x)$ and let $n \leq k$.

Further we prove Statement 2 by induction on the number $k + n$.

If $k + n = k$ then $n = 0$, and hence $0 \neq f(x) = a_0 \in R$. Then

$$\varphi(x) \cdot e = a_0 \cdot (a_0^{-1} \cdot \varphi(x)) = f(x) \cdot (a_0^{-1} \cdot \varphi(x)),$$

i.e. the right Ore condition is true in this case.

Assume that Statement 2 is proved for the number $n + k = m$ and let $n + k = m + 1$. Then

$$\psi(x) = \varphi(x) - f(x) \cdot \alpha^{-n}(a_k^{-1} \cdot b_n) \cdot x^{k-n} = \sum_{i=0}^{n} b_i \cdot x^i - \Big(\sum_{i=0}^{k} a_i \cdot x^i\Big) \cdot \alpha^{-n}(a_k^{-1} \cdot b_n) \cdot x^{k-n} =$$

$$\sum_{i=0}^{n} b_i \cdot x^i - \sum_{i=0}^{k} a_i \cdot \alpha^{-n+i}(a_k^{-1} \cdot b_n) \cdot x^{k-n+i} = \sum_{i=0}^{n-1} c_i \cdot x^i.$$

As $n - 1 + k = m$, then from the induction assumption it folows that there exist $f_1(x) \neq 0$ and $\psi_1(x) \neq 0$ such that $f(x) \cdot f_1(x) = \psi(x) \cdot \psi_1(x)$. Then

$$f(x) \cdot f_1(x) = \psi(x) \cdot \psi_1(x) = \big(\varphi(x) - f(x) \cdot \alpha^{-n}(a_k^{-1} \cdot b_n \cdot x^{k-n}\big) \cdot \psi_1(x) =$$

$\varphi(x) \cdot \psi_1(x) - f(x) \cdot \alpha^{-n}(a_k^{-1} \cdot b_n \cdot x^{k-n} \cdot \psi_1(x)$, and hence

$$f(x) \cdot \big(f_1(x) + f(x) \cdot \alpha^{-n}(a_k^{-1} \cdot b_n \cdot x^{k-n}\big) \cdot \psi_1(x) = \psi(x) \cdot \psi_1(x).$$

Thus proposition is proved. □

From Proposition 2.4 and Proposition 2.5 follows

Corollary 2.6. Let R be a skew field and let $\alpha : R \longrightarrow R$ be an automorphism. If x is a variable and we assume that $\alpha(r) \cdot x = x \cdot r$ for any $r \in R$, then the set

$$\widehat{R}_\alpha(x) = \{f(x) \cdot \varphi(x)^{-1} | f(x), \varphi(x) \in R_\alpha(x) \text{ and } 0 \neq \varphi(x)\}$$

is a skew field.

Definition 2.7.

– The ring $R_\alpha(x)$ will be called *a ring of polynomials in the variable x and with the automorphism $\alpha : R \longrightarrow R$*;

– an element $a \in R$ will be called *a root of a polynomial* $f_\alpha(x) \in R_\alpha(x)$ if $f_\alpha(a) = 0$;

– any element $\hat{f}(x) \in \widehat{R}_\alpha(x)$ will be called *a rational function in the variable x*;

– an element $a \in R$ will be called *a root of a rational function* $\hat{f}(x) \in \widehat{R}_\alpha(x)(x)$ if $\hat{f}(a) = 0$.

Proposition 2.8. *Let R be a skew field and let $\alpha : R \longrightarrow R$ be an automorphism of the skew field R. If $f(x) = \sum\limits_{i=0}^{k} a_i \cdot x^i \in R_\alpha(x)$ and if $f(0) \neq 0$, then the set $\{r \in R | f(r) = 0\}$ is a finite set.*

Proof. We will prove this proposition by induction on the number k. If $k = 1$ then $f(x) = a_0 + a_1 \cdot x$. As $f(0) \neq 0$, then $a_0 \neq 0$. Then $\{r \in R | f(r) = 0\} = \emptyset$ if $a_1 = 0$ and $\{r \in R | f(r) = 0\} = \{a_1^{-1} \cdot a_0\}$ if $a_1 \neq 0$.

Assume that Proposition 2.8 is proved for the number $k = m$ and let $f(x) = \sum\limits_{i=0}^{m+1} a_i \cdot x^i \in R_\alpha(x)$.

If $\{r \in R | f(r) = 0\} = \emptyset$, then the statement of Proposition 2.8 is correct.

Let now $\{r \in R | f(r) = 0\} \neq \emptyset$ and let an element $a \in R$ be such that $f(a) = 0$. Then if $\varphi(x) = \sum\limits_{i=0}^{m} b_i \cdot x^i$ is a polynomial of $R_\alpha(x)$ such that $b_m = a_{m+1}$ and $b_{m-k} = \alpha^{-1}(a_{m-k+1} + a \cdot b_{m-k+1})$, for $1 \leq k \leq m$, and $b_0 = -a^{-1} \cdot a_0$, then $a_{m+1} = b_m$ and $a_{m-k+1} = \alpha(b_{m-k}) - a \cdot b_{m-k+1}$, for $1 \leq k \leq m$, and $a_0 = -a \cdot b_0$. Then

$$(x - a) \cdot \varphi(x) = (x - a) \cdot (\sum_{i=0}^{m} b_i \cdot x^i) = \sum_{i=0}^{m} x \cdot b_i \cdot x^i - \sum_{i=0}^{m} a \cdot b_i \cdot x^i =$$

$$\sum_{i=0}^{m} \alpha(b_i) \cdot x^{i+1} - \sum_{i=0}^{m} a \cdot b_i \cdot x^i = a_{m+1} \cdot x^{m+1} + \sum_{i=0}^{m} \alpha^{-1}(b_i) \cdot x^{i+1} - \sum_{i=0}^{m} a \cdot b_i \cdot x^i =$$

$$a_{m+1} \cdot x^{m+1} + \sum_{i=0}^{m} (\alpha^{-1}(b_i) - a \cdot b_i) \cdot x^{i+1} - a \cdot b_0 = \sum_{i=0}^{m+1} a_i \cdot x^i = f_\alpha(x).$$

As $\{r \in R | f(r) = 0\} = \{r \in R | \varphi(r) = 0\} \bigcup \{a\}$ and according to the inductive assumption the set $\{r \in R | \varphi(r) = 0\}$ is a finite set, then the set $\{r \in R | f(r) = 0\}$ is a finite set.

Thus proposition is proved. □

Proposition 2.9. *Let R be a skew field and let $\alpha : R \longrightarrow R$ be an automorphism of the skew field R. If $\hat{f}(x)$ is a rational function from $\widehat{R}_\alpha(x)$ such that $\hat{f}(x) = f(x) \cdot (\varphi(x))^{-1}$ and $f(0) \neq 0$ and $\varphi(0) \neq 0$ then the set $\{r \in R | \hat{f}(r) = 0\}$ is a finite set.*

Proof. Since according to Proposition 2.8 the sets $\{r \in R | f(r) = 0\}$ and $\{r \in R | \varphi(r) = 0\}$ are finite sets and $\hat{f}(a) = f(a) \cdot (\varphi(a))^{-1} \neq 0$ for any $a \notin \{r \in R | f(r) = 0\} \bigcup \{r \in R | \varphi(r) = 0\}$ then $\{r \in R | \hat{f}(r) = 0\}$ is a finite set.

Thus proposition is proved. □

Notation 2.10. Let $V_1, V_2, \dots$ and $S_1, S_2, \dots$ be sequences of non-empty symmetric subsets of a skew field R, and let e be the unit of the skew field R. If $S_1 \subseteq S_2 \subseteq \dots$ and $e \in S_1$, then for any natural number k we define by induction the subset $F_k\big(S_1, \dots, S_k; V_1, \dots, V_k\big)$ of the skew field R by taking

$F_1(S_1; V_1) = ((e + V_1) \setminus \{0\})^{-1} \cdot V_1 \cdot S_1 + V_1 \cdot V_1 + S_1 \cdot V_1 \cdot (e + V_1) \setminus \{0\})^{-1}$, and

$$F_{k+1}(S_1, S_2, \dots, S_{k+1}; V_1, V_2, \dots, V_{k+1}) = F_1\big(S_1; V_1 + F_k(S_2, \dots, S_{k+1}; V_2, \dots, V_{k+1})\big).$$

Proposition 2.11. (see Proposition 2.4 in [8]) *Let $V_1, V_2, \dots$ and $S_1, S_2, \dots$ be some sequences of non-empty finite symmetric subsets of a skew field R. If $e \in S_1 \subseteq S_2 \subseteq \dots$, and $0 \in V_i$ for any natural number i, then the following Statements are true:*

Statement 1. *The following inclusions are true:*

1. $F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) + F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \subseteq F_k(S_1, S_2, \dots, S_k; V_1, V_2 \dots, V_k)$ *for any natural number* $k > 1$;

2. $F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \cdot F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \subseteq F_k(S_1, S_2, \dots, S_k; V_1, V_2 \dots, V_k)$ *for any natural number* $k > 1$;

3. $F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \cdot (e + F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k))^{-1} \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ *and* $(e + F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k))^{-1} \cdot F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ *for any natural number* $k > 1$;

4. $S_1 \cdot F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ *and* $F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \cdot S_1 \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ *for any natural number* $k > 1$.

Statement 2. *For any natural number k the set $F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ is a finite and symmetric set;*

Statement 3. $F_k(S_1, \dots, S_k; \{0\}, \dots, \{0\}) = \{0\}$ *for any natural number k;*

Statement 4. *If $0 \in U_i \subseteq V_i \subseteq R$ and $e \in T_i \subseteq S_i \subseteq R$ for any natural number i, then*

$$F_k(T_1, \dots, T_k; U_1, \dots, U_k) \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k);$$

Statement 5. *If k and p are natural numbers and $V_{k+j} = \{0\}$ for any natural number $1 \leq j \leq p$, then*

$$F_k(S_1, \dots, S_k; V_1, \dots, V_k) = F_{k+p}(S_1, \dots, S_{k+p}; V_1, \dots, V_{k+p});$$

Statement 6.

$$F_{k+1}(S_1, \dots, S_{k+1}; V_1, \dots, V_{k+1}) = F_k\big(S_1, \dots, S_k; V_1, \dots, V_{n-1}, V_k + F_1(S_{k+1}, V_{k+1})\big).$$

for any natural number n;

Statement 7. *If k and p are natural numbers then*

$$F_k(S_{p+1}, \dots, S_{k+p}; V_{p+1}, \dots, V_{k+p}) \subseteq F_{k+p}(S_1, \dots, S_{k+p}; V_1, \dots, V_{k+p}).$$

3 Basic results

Theorem 3.1. *If $R = \{0, \pm e, \pm r_1, \pm r_2, \ldots\}$ is a countable skew field and $S_k = \{\pm e, \pm r_1, \ldots, \pm r_k\}$ for any natural number k, then the skew field R permits non-discrete Hausdorff skew field topology if and only if there exists a sequence $h_1, h_2, \ldots$ of elements of the skew field R such that $h_i \neq h_j$ for any natural numbers $i \neq j$ and*

$$S_k \bigcap F_{m-k}(S_{k+1}, \ldots, S_m; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}) = \emptyset$$

for any natural numbers $0 \leq k < m$.

Proof. If a countable skew field R permits a non-discrete Hausdorff topology τ_0, then the construction of the sequence $h_1, h_2, \ldots$ is carried out similarly to the construction of correspondent sequence in the proof of Statement 1 of Theorem 3.1 in [8].

If $h_1, h_2, \ldots$ is a sequence of nonzero elements of the skew field R such that

$$S_k \bigcap F_{m-k}(S_{k+1}, \ldots, S_m; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}) = \emptyset$$

for any natural numbers $0 \leq k < m$, then the construction of the topology τ_0 is carried out similarly to the construction of the topology $\tau(A)$ in the proof of Statement 1 of Theorem 3.1 in [8].

Thus Theorem is proved. □

Theorem 3.2. *If for any finite subset S of a countable skew field $R = \{0, \pm e, \pm r_1, \pm r_2, \ldots\}$ there exists infinite subset $M \subseteq R$ such that $r \cdot s = s \cdot r$ for any elements $r \in M$ and $s \in S$, then the skew field R permits non-discrete Hausdorff skew field topology.*

Proof. If $S_0 = \emptyset$ and $S_k = \{\pm e, \pm r_1, \ldots, \pm r_k\}$ for any natulal number k then by induction we construct a sequence $h_1, h_2, \ldots$ of elements of the skew field R such that $h_i \neq h_j$ for any natural numbers $i \neq j$ and

$$S_k \bigcap F_{m-k}(S_{k+1}, \ldots, S_m; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}) = \emptyset$$

for any natural numbers $0 \leq k < m$.

We will take $h_1 = 0$. Then $F_1(S_1; \{0\}) = \{0\}$, and hence, $F_1(S_1; \{h_1\}) \bigcap S_0 = F_1(S_1; \{0\}) \bigcap \emptyset = \emptyset$.

Assume that we already defined elements $h_1, \ldots, h_m$ of the skew field R such that $h_i \neq h_j$ for any natural numbers $i \neq j$ and and such that

$$S_k \bigcap F_{m-k}(S_{k+1}, \ldots, S_m; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}) = \emptyset$$

for any natural numbers $0 \leq k < m$.

If M_m is the set all elements from R each of which is in the record of at least one element from the set

$$\bigcup_{k=0}^{m-1} \big(S_k \bigcup F_{m-k}(S_{k+1}, \ldots, S_m; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\})\big),$$

then $\{r \in R | r \cdot a = a \cdot r,\ a \in M_m\}$ is an infinite set.

If $\epsilon : R \longrightarrow R$ is the identical automorphism of the skew field R, then for any natural numbers $0 < k \leq n$ we defined the set

$$F_{m-k+1}(S_{k+1}, \ldots, S_m; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}, \{-x, 0, x\}) - S_k$$

of rational functions from $\widehat{R}_\epsilon(x)$. [3]

As (see Statement 5 of Proposition 2.11)

$$F_{m-k+1}(S_{k+1}, \ldots, S_m; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}, \{0\}) = F_{m-k}(S_{k+1}, \ldots, S_m; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}),$$

and

$$S_k \bigcap F_{m-k}(S_{k+1}, \ldots, S_m; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}) = \emptyset$$

for any $k < m$, then $\hat{f}(0) \neq 0$ for any rational functions from the set

$$\bigcup_{k=1}^{m} F_{m-k+1}(S_{k+1}, \ldots, S_{m+1}; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}, \{-x, 0, x\}) - S_k,$$

and hence, the set $\{r \in R | \hat{f}(r) = 0\}$ is a finite set. Then there exists an element $0 \neq h_{m+1} \in R$ such that $h_i \neq h_{m+1}$ for any $i < m+1$ and $\hat{f}(h_{m+1}) \neq 0$ for any rational functions $\hat{f}(x)$ from the set

$$\bigcup_{k=1}^{n} (F_{m-k+1}(S_{k+1}, \ldots, S_{m+1}; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}, \{-x, 0, x\}) - S_k),$$

and hence $\hat{\varphi}(h_{m+1}) \notin S_k$ for any rational functions $\hat{\varphi}(x)$ from the set

$$\bigcup_{k=1}^{n} F_{m-k+1}(S_{k+1}, \ldots, S_{m+1}; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}, \{-x, 0, x\}).$$

As $h_{m+1} \cdot a = a \cdot h_{m+1}$ for any $a \in M_m$, then

$$S_k \bigcap F_{m-k}(S_{k+1}, \ldots, S_{m+1}; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_{m+1}, 0, h_{m+1}\}) =$$

$$S_k \bigcap \{f(h_{m+1}) | f(x) \in F_{m-k}(S_{k+1}, \ldots, S_{m+1}; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\},$$
$\{-x, 0, x\})\} = \emptyset$ for any $0 \leq k < m+1$.

So, we defined the sequence $h_1, h_2, \ldots$ of nonzero elements of the skew field R such that

$$S_k \bigcap F_{n-k}(S_{k+1}, \ldots, S_n; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_n, 0, h_n\}) = \emptyset$$

for any natural numbers n and $k < n$.

Then from Theorem 3.1 it follows that there exists a non-discrete Hausdorff skew field topology on the skew field R.

Thus Theorem is proved. □

[3] As $r \cdot x = x \cdot r$ for any $r \in R$, then $F_{m-k+1}(S_{k+1}, \ldots, S_m; \{-h_{k+1}, 0, h_{k+1}\}, \ldots, \{-h_m, 0, h_m\}, \{-x, 0, x\}) - S_k \subseteq \widehat{R}(x)$.

Corollary 3.3. *If the center of a countable skew field R is an infinite set (in particular, if the characteristic of the skew field R is zero), then there exists a non-discrete Hausdorff skew field topology on the skew field R.*

From Theorem 3.2 and ([8], Theorem 3.1) follows

Corollary 3.4. *If for any finite subset S of a countable skew field R there exists infinite subset $M \subseteq R$ such that $r \cdot s = s \cdot r$ for any elements $r \in M$ and $s \in S$, then the following statements are true:*

Statement 1. *There are continuum of skew field topologies such that any two of them are comparable to each other;*

Statement 2. *There are two to the power of continuum coatoms in the lattice of all skew field topologies of the skew field R.*

Remark 3.4. There exists a countable skew field R with the finite center and such that for any finite subset $M \subseteq R$ the set $\{r \in R | r \cdot a = a \cdot r \text{ for any } a \in M\}$ is infinite. Such a skew field will be constructed in the following example. It is possible that the method that is used to build such a skew field has already been published by other authors, but since we do not know where it is published, then we present a detailed construction of such a skew field, and we not pretend on the authorship of this result.

Example 3.5. Let F be the algebraic closure of the finite simple field $\mathbb{Z}/(p \cdot \mathbb{Z})$ of characteristic p and let e be the unit of the field F. Then $F = \bigcup\limits_{n=1}^{\infty} F_n$, where $F_n = \{a \in F | a^{p^n} = a\}$ is a finite subfield of the field F and $F_n \subseteq F_{n+1}$ for any natural number n.

If $\alpha : F \longrightarrow F$ is a mapping such that $\alpha(a) = a^p$, then α is an automorphism and $\widehat{F}_\alpha(x)$ is the skew field of rational functions in variable x over field F (see Corollary 2.6) and since $a \cdot x = \alpha^{-1}(a) \neq x \cdot a$ for any $a \in F \setminus (\mathbb{Z}/(p \cdot \mathbb{Z}))$, then the center of the skew field $\widehat{F}_\alpha(x)$ is finite.

Let $M = \{\hat{f}_1(x) = f_1(x) \cdot \varphi_1(x)^{-1}, \ldots, \hat{f}_n(x) = f_n(x) \cdot \varphi_n(x)^{-1}\} \subseteq \hat{F}_\alpha(x)$ be a finite set. If $f_i(x) = \sum\limits_{j=0}^{k_i} a_{i,j} \cdot x^j$ and $\varphi_i(x) = \sum\limits_{j=0}^{k'_i} b_{i,j} \cdot x^j$ for $1 \leq i \leq n$, then there exists a natural number m such that

$$(\bigcup_{i=1}^{n} \{a_{i,1}, \ldots, a_{i,k_i}\}) \bigcup (\bigcup_{i=1}^{n} \{b_{i,1}, \ldots, b_{i,k'_i}\}) \subseteq F_m.$$

Then $x^{m \cdot s} \cdot a = a^{m \cdot s} \cdot x^{m \cdot s} = a \cdot x^{m \cdot s}$ for any natural number s and any

$$a \in (\bigcup_{i=1}^{n} \{a_{i,1}, \ldots, a_{i,k_i}\}) \bigcup (\bigcup_{i=1}^{n} \{b_{i,1}, \ldots, b_{i,k'_i}\}) \subseteq F_m,$$

and hence

$$x^{m \cdot s} \cdot f_i(x) = \sum_{j=0}^{k_i} x^{m \cdot s} \cdot a_{i,j} \cdot x^j = \sum_{j=0}^{k_i} a_{i,j} \cdot x^{m \cdot s} \cdot x^j = \sum_{j=0}^{k_i} a_{i,j} \cdot x^j \cdot x^{m \cdot s} = f_i(x) \cdot x^{m \cdot s}$$

and

$$x^{m\cdot s}\cdot\varphi_i(x)=\sum_{j=0}^{k'_i}x^{m\cdot s}\cdot b_{i,j}\cdot x^j=\sum_{j=0}^{k_i}b_{i,j}\cdot x^{m\cdot s}\cdot x^j=\sum_{j=0}^{k_i}b_{i,j}\cdot x^j\cdot x^{m\cdot s}=\varphi_i(x)\cdot x^{m\cdot s}$$

for any natural number s and any $1\leq i\leq n$. As $\varphi_i(x)\cdot x^{m\cdot s}\cdot(\varphi_i(x))^{-1}=$ $=x^{m\cdot s}\cdot\varphi_i(x)\cdot(\varphi_i(x))^{-1}=x^{m\cdot s}\cdot e=e\cdot x^{m\cdot s}=\varphi_i(x)\cdot(\varphi_i(x))^{-1}\cdot x^{m\cdot s}$ for any $1\leq i\leq n$ and $\widehat{F}_\alpha(x)$ is the skew field then $x^{m\cdot s}\cdot(\varphi_i(x))^{-1}=(\varphi_i(x))^{-1}\cdot x^{m\cdot s}$ for any $1\leq i\leq n$.

Then $x^{m\cdot s}\cdot f_i(x)\cdot(\varphi_i(x))^{-1}=f_i(x)\cdot(\varphi_i(x))^{-1}\cdot x^{m\cdot s}$ for any $1\leq i\leq n$, and hence the set $\{r\in\hat{F}_\alpha(x)|r\cdot f_i(x)\cdot(\varphi_i(x))^{-1}=f_i(x)\cdot(\varphi_i(x))^{-1}\cdot r \text{ for } 1\leq i\leq n\}$ is infinite.

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Received November 22, 2020

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