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Local growth of solutions of linear differential equations with analytic coefficients of finite iterated order

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Abstract. In this paper, we investigate the iterated order of growth of solutions to certain homogeneous and non–homogeneous linear differential equations where the coefficients are analytic functions in the closed complex plane except a finite singular point. For that we use the Nevanlinna theory with adapted definitions.

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1 Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions on the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [8, 12, 18]). The importance of this theory has inspired many authors to find modifications and generalizations to different domains. Extensions of Nevanlinna Theory to annuli have been made by [2, 9–11, 13]. Recently in [4, 7], the authors investigated the growth of solutions of certain linear differential equations near a finite singular point. The idea began with the study of the growth of solutions near a point on the boundary of the unit disc (see [5, 6]). In this paper, we continue this investigation near a finite singular point to study other types of linear differential equations.

First, we recall the appropriate definitions. Set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and suppose that $f(z)$ is meromorphic in $\overline{\mathbb{C}} \setminus \{z_0\}$ where $z_0 \in \mathbb{C}$. Define the counting function near z_0 by

$$N_{z_0}(r, f) = -\int_{\infty}^{r} \frac{n(t, f) - n(\infty, f)}{t} dt - n(\infty, f) \log r, \tag{1.1}$$

where $n(t, f)$ counts the number of poles of $f(z)$ in the region $\{z \in \mathbb{C} : t \leq |z - z_0|\} \cup \{\infty\}$, each pole according to its multiplicity; and the proximity function by

$$m_{z_0}(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} \ln^+ \left| f\left(z_0 - re^{i\varphi}\right)\right| d\varphi. \tag{1.2}$$

The characteristic function of f is defined in the usual manner by

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f). \tag{1.3}$$

In the usual manner, we define the iterated p-order of meromorphic function $f(z)$ near z_0 by

$$\sigma_p(f,z_0) = \limsup_{r\to 0} \frac{\log_p^+ T_{z_0}(r,f)}{-\log r}, \ p \in \mathbb{N}, \tag{1.4}$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$ and $\log_{p+1}^+ x = \log^+ \log_p^+ x$ for $p \geq 1$. For an analytic function $f(z)$ in $\overline{\mathbb{C}} \setminus \{z_0\}$, we have also the definition

$$\sigma_p(f,z_0) = \limsup_{r\to 0} \frac{\log_{p+1}^+ M_{z_0}(r,f)}{-\log r}, \tag{1.5}$$

where $M_{z_0}(r,f) = \max\{|f(z)| : |z - z_0| = r\}$.

Remark 1. *It is shown in [4] that if f is a non–constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ and $g(w) = f\left(z_0 - \frac{1}{w}\right)$ then $g(w)$ is meromorphic in $\mathbb{C}$ and we have*

$$T(R,g) = T_{z_0}\left(\frac{1}{R}, f\right);$$

and so $\sigma(f,z_0) = \sigma(g)$. Also, if $f(z)$ is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, then, $g(w)$ is entire and thus $\sigma_T(f,z_0) = \sigma_M(f,z_0)$ and in general $\sigma_{n,T}(f,z_0) = \sigma_{n,M}(f,z_0)$ $n \geq 1$. So, we can use the notation $\sigma_n(f,z_0)$ without any ambiguity.

For example, the function $f(z) = \exp_p\left\{\frac{1}{(z_0 - z)^n}\right\}$, where $p, n \in \mathbb{N}\setminus\{0\}$, is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ and satisfies $M_{z_0}(r,f) = \exp_p\left\{\frac{1}{r^n}\right\}$; and then $\sigma_p(f,z_0) = n$, where $\exp_1\{x\} = \exp\{x\}$ and $\exp_{p+1}\{x\} = \exp\left\{\exp_p\{x\}\right\}$ for $p \geq 1$.

Definition 1. *The linear measure of a set $E \subset (0,\infty)$ is defined as $\int_0^\infty \chi_E(t)\,dt$ and the logarithmic measure of E is defined by $\int_0^\infty \frac{\chi_E(t)}{t} dt$ where $\chi_E(t)$ is the characteristic function of the set E.*

In 2001, Belaidi and Hamouda proved the following results.

Theorem A. *[1] Let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions such that for real constants $\alpha, \beta, \mu, \theta_1, \theta_2$ with $0 \leq \beta < \alpha$, $\mu > 0$, $\theta_1 < \theta_2$, we have*

$$|A_0(z)| \geq \exp\{\alpha r^\mu\},$$

$$|A_j(z)| \leq \exp\{\beta r^\mu\}, \quad j = 1, ..., k-1,$$

as $z \to \infty$ *with* $\theta_1 \leq \arg z \leq \theta_2$. *Then every analytic solution* $f(z) \not\equiv 0$ *in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *of the differential equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.6}$$

has infinite order.

This result has been generalized in [15] by introducing the concept of the iterated order. In 2016, the authors proved the following results.

Theorem B. *[4] Let* $A_0(z), \ldots, A_{k-1}(z)$ *be meromorphic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying*

$$|A_0(z)| \geq \exp\{\frac{\alpha}{r^\mu}\}, \tag{1.7}$$

$$|A_j(z)| \leq \exp\{\frac{\beta}{r^\mu}\}, \quad j = 1, ..., k-1, \tag{1.8}$$

where $\alpha > \beta \geq 0$, $\mu > 0$, $\arg(z_0 - z) = \theta \in (\theta_1, \theta_2) \subset [0, 2\pi)$ *and* $|z_0 - z| = r \to 0$. *Then, every analytic solution* $f(z) \not\equiv 0$ *in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *of (1.6) satisfies* $\sigma_2(f, z_0) \geq \mu$.

Theorem C. *[4] Let* $A_0(z), \ldots, A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying* $\max\{\sigma(A_j, z_0) : j \neq 0\} < \sigma(A_0, z_0)$. *Then, every analytic solution* $f(z) \not\equiv 0$ *in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *of (1.6) satisfies* $\sigma_2(f, z_0) = \sigma(A_0, z_0)$.

In this paper, we will generalize these results and others as the following.

Theorem 1. *Let* $A_0(z), \ldots, A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *such that for real constants* $\alpha, \beta, \mu, \theta_1, \theta_2$ *and a positive integer* p *with* $0 \leq \beta < \alpha$, $\mu > 0$, $\theta_1 < \theta_2$. $1 \leq p < \infty$, *satisfying*

$$|A_0(z)| \geq \exp_p\left\{\frac{\alpha}{r^\mu}\right\}, \tag{1.9}$$

$$|A_j(z)| \leq \exp_p\left\{\frac{\beta}{r^\mu}\right\}, \quad j = 1, ..., k-1 \tag{1.10}$$

where $\arg(z_0 - z) = \theta \in (\theta_1, \theta_2)$ *and* $|z_0 - z| = r \to 0$. *Then, every analytic solution* $f(z) \not\equiv 0$ *in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *of (1.6) satisfies* $\sigma_{p+1}(f, z_0) \geq \mu$.

Theorem 2. *Let* $A_0(z), \ldots, A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *and* $F \subset (0, 1)$ *be a set of infinite logarithmic measure such that*

$$|A_0(z)| \geq \exp_p\left\{\frac{\alpha}{r^\mu}\right\}, \tag{1.11}$$

$$|A_j(z)| \leq \exp_p\left\{\frac{\beta}{r^\mu}\right\}, \quad j = 1, ..., k-1, \tag{1.12}$$

with $0 \leq \beta < \alpha$, $\mu > 0$, *and* $|z_0 - z| = r \to 0$ *with* $r \in F$. *Then, every analytic solution* $f(z) \not\equiv 0$ *in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *of the differential equation (1.6) satisfies* $\sigma_{p+1}(f, z_0) \geq \mu$.

Theorem 3. *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite iterated order with* $\max\{\sigma_p(A_j, z_0) : j \neq 0\} \leq \sigma_p(A_0, z_0) = \sigma < +\infty,\ 1 < p < \infty$ *and $F \subset (0,1)$ be a set of infinite logarithmic measure such that for some constants $0 \leq \beta < \alpha$ and for any given $\epsilon > 0$, we have*

$$|A_0(z)| \geq \exp_p \left\{ \frac{\alpha}{r^{\sigma-\epsilon}} \right\}, \tag{1.13}$$

$$|A_j(z)| \leq \exp_p \left\{ \frac{\beta}{r^{\sigma-\epsilon}} \right\}, \quad j = 1, ..., k-1, \tag{1.14}$$

as $r \to 0$ with $r \in F$. Then, every analytic solution $f(z) \not\equiv 0$ in $\overline{\mathbb{C}} \setminus \{z_0\}$ of the differential equation (1.6) satisfies $\sigma_{p+1}(f, z_0) = \sigma_p(A_0, z_0) = \sigma$.

Theorem 4. *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfying* $\max\{\sigma_p(A_j, z_0) : j \neq 0\} < \sigma_p(A_0, z_0)$. *Then, every analytic solution $f(z) \not\equiv 0$ in $\overline{\mathbb{C}} \setminus \{z_0\}$ of (1.6) satisfies $\sigma_{p+1}(f, z_0) = \sigma_p(A_0, z_0)$.*

Theorem 5. *Let $A_0(z), \ldots, A_{k-1}(z)$ satisfy the hypotheses of Theorem 3, and let $H(z) \not\equiv 0$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ with $i(H) = q$.*
a) If $q < p+1$ or $q = p+1$, $\sigma_{p+1}(H, z_0) < \sigma_p(A_0, z_0)$, then every analytic solution $f(z)$ in $\overline{\mathbb{C}} \setminus \{z_0\}$ of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) f = H(z), \tag{1.15}$$

satisfies $\overline{\lambda}_{p+1}(f, z_0) = \lambda_{p+1}(f, z_0) = \sigma_{P+1}(f, z_0) = \sigma_p(A_0, z_0)$, with at most one exceptional solution f_0 satisfying $i(f_0) < p+1$ or $\sigma_{p+1}(f_0, z_0) < \sigma_p(A_0, z_0)$.
b) If $q > p+1$ or $q = p+1$, $\sigma_p(A_0, z_0) < \sigma_{p+1}(H, z_0) < +\infty$, then every analytic solution $f(z)$ in $\overline{\mathbb{C}} \setminus \{z_0\}$ of (1.15) satisfies $i(f) = q$ and $\sigma_q(f, z_0) = \sigma_q(H, z_0)$.

The analog of Theorem 5 in the complex plane case is investigated in [15].

2 Preliminaries for proving the main results

To prove these results we need the following lemmas.

Lemma 1. *[4] Let f be a non–constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$; let $\alpha > 0, \varepsilon > 0$ be given real constants and $j \in \mathbb{N}$; then*
i) there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure and a constant $A > 0$ that depends on α and j such that for all $r = |z - z_0|$ satisfying $r \in (0,1) \setminus E_1$, we have

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq A \left[\frac{1}{r^2} T_{z_0}(\alpha r, f) \log T_{z_0}(\alpha r, f) \right]^j; \tag{2.1}$$

ii) there exists a set $E_2 \subset [0, 2\pi)$ that has a linear measure zero and a constant $A > 0$ that depends on α and j such that for all $\theta \in [0, 2\pi) \setminus E_2$ there exists a constant $r_0 = r_0(\theta) > 0$ such that (2.1) holds for all z satisfying $\arg(z - z_0) \in [0, 2\pi) \setminus E_2$ *and $r = |z - z_0| < r_0$.*

Lemma 2. *[7, Theorem 8] Let f be a non–constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$. Then there exists a set $E \subset (0,1)$ that has finite logarithmic measure such that for all $j = 0, ..., k,$ we have*

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1)) \left(\frac{V(z_r)}{z_0 - z} \right)^j, \tag{2.2}$$

as $r \to 0, r \notin E$, where z_r is a point in the circle $|z_0 - z| = r$ that satisfies $|f(z_r)| = \max_{|z_0 - z| = r} |f(z)|.$

Lemma 3. *Let f be a non–constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of infinite order with $\sigma_p(f, z_0) = \sigma$, and let $V_{z_0}(r)$ be the central index of f (see [7]). Then*

$$\limsup_{r \to 0} \frac{\log_p^+ V_{z_0}(r)}{-\log r} = \sigma. \tag{2.3}$$

Proof. Set $g(w) = f\left(z_0 - \frac{1}{w}\right)$. Then $g(w)$ is entire function of infinite order with $\sigma_p(g) = \sigma_p(f, z_0) = \sigma$, and if $V(R)$ denotes the central index of g, then $V_{z_0}(r) = V\left(\frac{1}{r}\right)$. From [3, Lemma 2], we have

$$\limsup_{R \to +\infty} \frac{\log_p^+ V(R)}{\log R} = \sigma. \tag{2.4}$$

Substituting R by $\frac{1}{r}$ in (2.4), we get (2.3). □

Lemma 4. *Let $f(z)$ be a non–constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ with $i(f) = p$; Then*

$$\sigma_p(f', z_0) = \sigma_p(f, z_0). \tag{2.5}$$

Proof. By Remark 1, $g(w) = f\left(z_0 - \frac{1}{w}\right)$ is meromorphic in $\mathbb{C}$ and $\sigma_p(g) = \sigma_p(f, z_0)$. It is well known that for a meromorphic function in $\mathbb{C}$ we have $\sigma_p(g') = \sigma_p(g)$ (see [14, 17]). We have $f'(z) = \frac{1}{w^2} g'(w)$. Set $h(w) = \frac{1}{w^2} g'(w)$. Obviously, we have $\sigma_p(h) = \sigma_p(g')$. In the other hand, by Remark 1, we have $\sigma_p(h) = \sigma_p(f', z_0)$. So, we conclude that $\sigma_p(f', z_0) = \sigma_p(f, z_0)$. □

Lemma 5. *Let h be a non–constant analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of order $\sigma_p(f, z_0) > \alpha > 0$. Then, there exists a set $F \subset (0, r_0]$ of infinite logarithmic measure such that for all $r \in F$ and $|h(z)| = M_{z_0}(r, h)$, we have*

$$|h(z)| > \exp_p\left\{\frac{1}{r^\alpha}\right\}.$$

Proof. By the definition of $\sigma_p(f, z_0)$, there exists a decreasing sequence $\{r_m\} \to 0$ satisfying $\frac{m}{m+1} r_m > r_{m+1}$ and

$$\lim_{m \to \infty} \frac{\log_{p+1} M_{z_0}(r_m, f)}{-\log r_m} > \alpha.$$

Then, there exists m_0 such that for all $m > m_0$ and for a given $\varepsilon > 0$ small enough, we have

$$M_{z_0}(r_m, f) > \exp_p\left\{\frac{1}{r_m^{\alpha+\varepsilon}}\right\}. \tag{2.6}$$

There exists m_1 such that for all $m > m_1$, and for any $r \in [\frac{m}{m+1}r_m, r_m]$ and for a given $\varepsilon > 0$, we have

$$\left(\frac{m}{m+1}\right)^{\alpha+\varepsilon} > r^{\varepsilon}. \tag{2.7}$$

By (2.6) and (2.7), for all $m > m_2 = \max\{m_0, m_1\}$ and for any $r \in [\frac{m}{m+1}r_m, r_m]$, we have

$$M_{z_0}(r, f) > M_{z_0}(r_m, f) > \exp_p\left\{\frac{1}{r_m^{\alpha+\varepsilon}}\right\} > \exp_p\left\{\frac{1}{r^{\alpha+\varepsilon}}(\frac{m}{m+1})^{\alpha+\varepsilon}\right\} > \exp_p\left\{\frac{1}{r^{\alpha}}\right\}.$$

Set $F = \cup_{m=m_2}^{\infty}[\frac{m}{m+1}r_m, r_m]$; then we have

$$\sum_{m=m_2}^{\infty}\int_{\frac{m}{m+1}r_m}^{r_m}\frac{dt}{t} = \sum_{m\geq m_2}\log\frac{m+1}{m} = \infty.$$

□

Lemma 6. *Let $A_j(z)$ $(j = 0, ..., k-1)$ be analytic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$ such that z_0 is a singular point for at least one of the coefficients $A_j(z)$ and $\sigma_p(A_j, z_0) \leq \alpha < \infty$. If f is a solution, that is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$, of (1.6), then $\sigma_{p+1}(f, z_0) \leq \alpha$.*

Proof. Let $f \not\equiv 0$ be a solution of (1.6), that is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$. By Lemma 1, there exists a set $E \subset (0, 1)$ that has finite logarithmic measure, such that for all $j = 0, 1, ..., k$, we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1))\left(\frac{V_{z_0}(r)}{z_0 - z}\right)^j, \tag{2.8}$$

as $r \to 0$, $r \notin E$, where $V_{z_0}(r)$ is the central index of f near the singular point z_0, z_r is a point in the circle $|z_0 - z| = r$ that satisfies $|f(z_r)| = \max_{|z_0 - z| = r}|f(z)|$. Set

$$M_{z_0}(r) = \max_{|z_0 - z| = r}\{|A_j(z)| : j = 0, 1, ..., k-1\}. \tag{2.9}$$

Since $\sigma_p(A_j, z_0) \leq \alpha < \infty$, for any given $\varepsilon > 0$, there exists $r_0 > 0$ such that for $r_0 > r > 0$, we have

$$M_{z_0}(r) \leq \exp_{p+1}\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}.$$

From (1.6), we can write

$$\left|\frac{f^{(k)}}{f}\right| \leq |A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)|\left|\frac{f'}{f}\right| + |A_0(z)|. \tag{2.10}$$

By combining (2.8)-(2.9), we get

$$(1+o(1))\left(\frac{V_{z_0}(r)}{r}\right)^k \leq (1+o(1))\,C\left(\frac{V_{z_0}(r)}{r}\right)^{k-1} M_{z_0}(r);$$

where r near enough to z_0 and $C>0$; and then

$$V_{z_0}(r) \leq (1+o(1))\,Cr\exp_{p+1}\left\{\frac{1}{r^{\alpha+\varepsilon}}\right\}. \tag{2.11}$$

By (2.11) and Lemma 3, we obtain $\sigma_{p+1}(f,z_0) \leq \alpha$. □

Lemma 7. *Let $f(z)$ be a non–constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$, for all z satisfying* $\arg(z-z_0)=\theta$ *and* $r=|z-z_0|$, *we have*

$$M_{z_0}\left(r,\frac{f^{(k)}(z)}{f(z)}\right) = O\left(\log\left(T_{z_0}(r,f)\right)+\log\left(\frac{1}{r}\right)\right)$$

Proof. Set $g(w)=f\left(z_0-\frac{1}{w}\right)$. $g(w)$ is meromorphic in $\mathbb{C}$. We have $f(z)=g(w)$ such that $w=\frac{1}{z_0-z}$; we set $R=|w|=\frac{1}{r}$, from the logarithmic derivative lemma, we have

$$M_{z_0}\left(r,\frac{g^{(k)}(z)}{g(z)}\right) = O\left(\log\left(T(R,g)\right)+\log(R)\right), \tag{2.12}$$

then $f'(z)=\frac{1}{(z_0-z)^2}g'(w)$ and then

$$\frac{f'(z)}{f(z)} = \frac{1}{(z_0-z)^2}\frac{g'(w)}{g(w)}. \tag{2.13}$$

By (2.13) and (2.12), we get

$$\begin{aligned}
M_{z_0}\left(\frac{1}{R},\frac{f'(z)}{f(z)}\right) &= M\left(R, w^2\frac{g'(w)}{g(w)}\right)\\
&\leq M\left(R,\ w^2\right)+M\left(R,\ \frac{g'(w)}{g(w)}\right)\\
&\leq O\left(\log(R)\right)+O\left(\log\left(T(R,g)\right)+\log(R)\right)\\
&\leq O\left(\log(R)\right)+O\left(\left(\log\left(T_{z_0}\left(\frac{1}{R},f\right)\right)+\log(R)\right)\right)\\
&\leq O\left(\left(\log\left(T_{z_0}(r,f)\right)+\log\left(\frac{1}{r}\right)\right)\right).
\end{aligned}$$

Then

$$M_{z_0}\left(r,\frac{f'(z)}{f(z)}\right) = O\left(\log\left(T_{z_0}(r,f)\right)+\log\left(\frac{1}{r}\right)\right).$$

For $k=2$, we have $f''(z)=\frac{1}{(z_0-z)^4}g''(w)+\frac{2}{(z_0-z)^3}g'(w)$; and so

$$\frac{f''(z)}{f(z)}=\frac{1}{(z_0-z)^4}\frac{g''(w)}{g(w)}+\frac{2}{(z_0-z)^3}\frac{g'(w)}{g(w)}. \tag{2.14}$$

By (2.13) and (2.12), we get

$$\begin{aligned} M_{z_0}\left(\frac{1}{R},\frac{f''(z)}{f(z)}\right) &= M\left(R,\frac{1}{(z_0-z)^4}\frac{g''(w)}{g(w)}+\frac{2}{(z_0-z)^3}\frac{g'(w)}{g(w)}\right)\\ &\le M\left(R,\ w^4\right)+M\left(R,\frac{g''(w)}{g(w)}\right)+M\left(R,w^3\right)+M\left(R,\frac{g'(w)}{g(w)}\right)\\ &\le O(\log(R))+O(\log(T(R,g))+\log(R))\\ &\le O(\log(R))+O\left(\left(\log\left(T_{z_0}\left(\frac{1}{R},f\right)\right)+\log(R)\right)\right)\\ &\le O\left(\left(\log(T_{z_0}(r,f))+\log\left(\frac{1}{r}\right)\right)\right). \end{aligned}$$

So

$$M_{z_0}\left(r,\frac{f''(z)}{f(z)}\right)=O\left(\log(T_{z_0}(r,f))+\log\left(\frac{1}{r}\right)\right).$$

In general, we can find that

$$f^{(k)}(z)=\frac{1}{(z_0-z)^{2k}}g^{(k)}(w)+\frac{a_{k-1}}{(z_0-z)^{2k-1}}g^{(k-1)}(w)+...+\frac{a_1}{(z_0-z)^{C}}g'(w);$$

where a_j $(j=1,2,...,k-1)$ are complex numbers; and thus

$$\frac{f^{(k)}(z)}{f(z)}=\frac{1}{(z_0-z)^{2k}}\frac{g^{(k)}(w)}{g(w)}+\frac{a_{k-1}}{(z_0-z)^{2k-1}}\frac{g^{(k-1)}(w)}{g(w)}+...+\frac{a_1}{(z_0-z)^{k+1}}\frac{g'(w)}{g(w)}. \tag{2.15}$$

By (2.15) and (2.12), we get

$$\begin{aligned} M_{z_0}\left(\frac{1}{R},\frac{f^{(k)}(z)}{f(z)}\right) &\le M\left(R,w^{2k}\right)+M\left(R,\frac{g^{(k)}(w)}{g(w)}\right)+...+M\left(R,\frac{g'(w)}{g(w)}\right)\\ &\le O(\log(R))+O(\log(T(R,g))+\log(R))\\ &\le O(\log(R))+O\left(\left(\log\left(T_{z_0}\left(\frac{1}{R},f\right)\right)+\log(R)\right)\right)\\ &\le O\left(\left(\log(T_{z_0}(r,f))+\log\left(\frac{1}{r}\right)\right)\right). \end{aligned}$$

So

$$M_{z_0}\left(r,\frac{f^{(k)}(z)}{f(z)}\right)=O\left(\log(T_{z_0}(r,f))+\log\left(\frac{1}{r}\right)\right).$$

□

Lemma 8. *Let $H(z) \not\equiv 0, A_0(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ and let f be an analytic solution in $\overline{\mathbb{C}} \setminus \{z_0\}$ of (1.15) satisfying one of the following conditions:*
i) $\max\{i(H) = q, i(A_j) : (j = 0, ..., k-1\} < i(f) = p+1 \ (1 \leq p \leq +\infty)$,
ii) $\max\{\sigma_{p+1}(H, z_0), \sigma_{p+1}(A_j, z_0) : (j = 0, ..., k-1\} < \sigma_{p+1}(f, z_0) = \sigma$.
Then $\overline{\lambda}_{p+1}(f, z_0) = \lambda_{p+1}(f, z_0) = \sigma_{p+1}(f, z_0) = \sigma_p(A_0, z_0)$.

Proof. From (1.15), we have

$$\frac{1}{f(z)} = \frac{1}{H(z)} \left(\frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z) \frac{f^{(k-1)}(z)}{f(z)} + ... + A_0(z) \right). \tag{2.16}$$

It is easy to see that if f has a zero at z_1 of order $\alpha > k$, then H must have a zero at z_1 of order $\alpha - k$, hence

$$n\left(r, \frac{1}{f}\right) \leq k\overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{H}\right), \tag{2.17}$$

$$N_{z_0}\left(r, \frac{1}{f}\right) \leq k\overline{N}_{z_0}\left(r, \frac{1}{f}\right) + N_{z_0}\left(r, \frac{1}{H}\right). \tag{2.18}$$

There exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z - z_0|$ satisfying $r \notin E_1$ and by (2.17), we have

$$m_{z_0}\left(r, \frac{1}{f}\right) \leq m_{z_0}\left(r, \frac{1}{H}\right) + \sum_{j=0}^{k-1} m_{z_0}(r, A_j) + O\left(\log T_{z_0}(r, f) + \log \frac{1}{r}\right). \tag{2.19}$$

By (2.17) and (2.18), for $r \notin E_1$, we get

$$T_{z_0}(r, f) = T_{z_0}\left(r, \frac{1}{f}\right) + O(1) \leq k\overline{N}_{z_0}\left(r, \frac{1}{f}\right) + \tag{2.20}$$

$$+T_{z_0}(r, H) + \sum_{j=0}^{k-1} T_{z_0}(r, A_j) + O\left(\log T_{z_0}(r, f) + \log \frac{1}{r}\right).$$

For $r \to 0$, we have

$$O\left(\log T_{z_0}(r, f) + \log \frac{1}{r}\right) \leq \frac{1}{2} T_{z_0}(r, f), \tag{2.21}$$

$$T_{z_0}(r, A_0) + ... + T_{z_0}(r, A_{k-1}) \leq k \exp_p\left\{\frac{1}{r^{\sigma+\epsilon}}\right\}, \tag{2.22}$$

$$T_{z_0}(r, H) \leq \exp_p\left\{\frac{1}{r^{\sigma(H)+\epsilon}}\right\}. \tag{2.23}$$

From (2.21)-(2.22) and (2.23)

$$T_{z_0}(r,H)+\sum_{j=0}^{k-1}T_{z_0}(r,A_j)+O\left(\log T_{z_0}(r,f)+\log\frac{1}{r}\right)\leq\frac{1}{2}T_{z_0}(r,f)+$$

$$k\exp_p\left\{\frac{1}{r^{\sigma+\epsilon}}\right\}+\exp_p\left\{\frac{1}{r^{\sigma(H)+\epsilon}}\right\} \qquad (2.24)$$

By (2.20) and (2.24), for $r\notin E_1$,

$$T_{z_0}(r,f)\leq 2k\overline{N}_{z_0}\left(r,\frac{1}{f}\right)+2k\exp_p\left\{\frac{1}{r^{\sigma+\epsilon}}\right\}+2\exp_p\left\{\frac{1}{r^{\sigma(H)+\epsilon}}\right\}. \qquad (2.25)$$

Hence for any f with $\sigma_{p+1}(f,z_0)=\sigma_p(A_0,z_0)$, from (2.25), we have $\sigma_{p+1}(f,z_0)\leq\overline{\lambda}_{p+1}(f,z_0)$.
Therefore $\overline{\lambda}_{p+1}(f,z_0)=\lambda_{p+1}(f,z_0)=\sigma_{p+1}(f,z_0)=\sigma_p(A_0,z_0)$. □

3 Proof of theorems

Proof of Theorem 1. Assume that $f\not\equiv 0$ is analytic solution of (1.6) in $\overline{\mathbb{C}}\setminus\{z_0\}$. From (1.6), we can write

$$-A_0(z)=\frac{f^{(k)}}{f}+A_{k-1}(z)\frac{f^{(k-1)}}{f}+...+A_1(z)\frac{f'}{f}. \qquad (3.1)$$

So, we get

$$|A_0(z)|\leq\left|\frac{f^{(k)}}{f}\right|+|A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f}\right|+...+|A_1(z)|\left|\frac{f'}{f}\right|. \qquad (3.2)$$

By Lemma 1, for any given $\alpha>0$ there exists a set $E_1\subset(0,1)$ that has finite logarithmic measure and a constant $\lambda>0$ that depends only on α such that for all $r=|z-z_0|$ satisfying $r\notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leq\lambda\left[\frac{1}{r}T_{z_0}(\alpha r,f)\right]^{2j}\quad(j=1,...,k). \qquad (3.3)$$

From (3.2)-(3.3), for all z satisfying $r=|z-z_0|\notin E_1$, and the hypotheses of Theorem 1 we obtain

$$\exp_p\left\{\frac{\alpha}{r^{\mu}}\right\}\leq|A_0(z)|\leq k\lambda\left[\frac{1}{r}T_{z_0}(\alpha r,f)\right]^{2k}\exp_p\left\{\frac{\beta}{r^{\mu}}\right\};$$

and thus

$$\exp_p\left\{\frac{\alpha}{r^{\mu}}\right\}\leq k\lambda\left[\frac{1}{r}T_{z_0}(\alpha r,f)\right]^{2k}\exp_p\left\{\frac{\beta}{r^{\mu}}\right\};$$

then

$$\exp_p\left\{\frac{\alpha-\beta}{r^{\mu}}\right\}\leq k\lambda\left[\frac{1}{r}T_{z_0}(\alpha r,f)\right]^{2k}. \qquad (3.4)$$

From (3.4), it is easy to obtain that $\sigma_{p+1}(f,z_0)\geq\mu$. □

Proof of Theorem 2. From (1.6) we obtain

$$|A_0(z)| \leq \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + ... + |A_1(z)| \left|\frac{f'}{f}\right|. \tag{3.5}$$

By the hypotheses of Theorem 2, there exists a set $F_1 \subset (0,1)$ of infinite logarithmic measure such that for all $r \in F$ we have

$$|A_0(z)| \geq \exp_p\left\{\frac{\alpha}{r^\mu}\right\}, \tag{3.6}$$

$$|A_j(z)| \leq \exp_p\left\{\frac{\beta}{r^\mu}\right\}, \quad j = 1, ..., k-1, \tag{3.7}$$

as $r \to 0$ for $r \in F_1$. Hence from 3.3, 3.5, 3.6 and 3.7, it follows that for all z satisfying $r = |z - z_0| \in F \backslash E_1$, we have

$$\exp_p\left\{\frac{\alpha}{r^\mu}\right\} \leq k\lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2k} \exp_p\left\{\frac{\beta}{r^\mu}\right\};$$

then

$$\exp_p\left\{\frac{\alpha-\beta}{r^\mu}\right\} \leq k\lambda \left[\frac{1}{r} T_{z_0}(\alpha r, f)\right]^{2k}. \tag{3.8}$$

When $\alpha - \beta > 0$, then $\exp_p\left\{\frac{\alpha-\beta}{r^\mu}\right\} > 1$, as $r \to 0$. Therefore, from (3.8), it is easy to obtain that $\sigma_{p+1}(f, z_0) \geq \mu$. □

Proof of Theorem 3. From Theorem (2), we have $\sigma_{p+1}(f, z_0) \geq \sigma - \epsilon$, since ϵ is arbitrary, we get

$$\sigma_{P+1}(f, z_0) \geq \sigma_p(A_0, z_0) = \sigma. \tag{3.9}$$

On the other hand, from Lemma (2), for any given $\varepsilon > 0$, there exists $r_0 > 0$ such that for $0 < r = |z_0 - z| < r_0$, we have

$$|A_j(z)| \leq \exp\left\{\frac{1}{r^{\sigma+\varepsilon}}\right\}, \ j = 0, ..., k-1. \tag{3.10}$$

By the Wiman-Valiron theory near a finite singular point (see [7]), we have

$$\frac{f^{(j)}(z_r)}{f(z_r)} = (1 + o(1)) \left(\frac{V_{z_0}(r)}{z_0 - z_r}\right)^j, \ (j = 0, ..., k-1), \tag{3.11}$$

where $V_{z_0}(r)$ is the central index of f and $|f(z_r)| = M(r, f) = \max_{|z_0 - z| = r} |f(z)|$. From (1.6), we can write

$$-\frac{f^{(k)}}{f} = A_{k-1}(z) \frac{f^{(k-1)}}{f} + ... + A_1(z) \frac{f'}{f} + A_0(z). \tag{3.12}$$

Substituting (3.10) and (3.11) into (3.12), we obtain

$$(1+o(1))\frac{(V_{z_0}(r))^k}{r^k}\le k\exp_p\left\{\frac{1}{r^{\sigma+\varepsilon}}\right\}\frac{(V_{z_0}(r))^{k-1}}{r^{k-1}}((1+o(1))),$$

and so

$$V_{z_0}(r)\le kr\exp_p\left\{\frac{1}{r^{\sigma+\varepsilon}}\right\}(1+o(1)). \tag{3.13}$$

By (3.13), we get

$$\limsup_{r\to 0}\frac{\log^+_{p+1}V_{z_0}(r)}{-\log r}\le\sigma+\epsilon.$$

Since ϵ is arbitrary, we get

$$\sigma_{p+1}(f,z_0)\le\sigma_p(A_0,z_0)=\sigma. \tag{3.14}$$

From (3.9) and (3.14), we get

$$\sigma_{p+1}(f,z_0)=\sigma_p(A_0,z_0).$$

□

Proof of Theorem 4. From (1.6), we can write

$$|A_0(z)|\le|\frac{f^{(k)}}{f}|+|A_{k-1}(z)||\frac{f^{(k-1)}}{f}|+\cdots+|A_1(z)||\frac{f'}{f}|. \tag{3.15}$$

Set $\max\{\sigma_p(A_j,z_0):j\neq 0\}<\beta<\alpha<\sigma_p(A_0,z_0)$. For any given $\varepsilon>0$, there exists $r_0>0$ such that for all r satisfying $r_0\ge r>0$, we have

$$|A_j(z)|\le\exp_p\{\frac{1}{r^{\beta+\varepsilon}}\},\quad j=1,2,\dots,k-1. \tag{3.16}$$

By taking $\beta+\varepsilon<\alpha<\sigma_p(A_0,z_0)$, and by Lemma 5, there exists a set $F\subset(0,r_0]$ of infinite logarithmic measure such that for all $r\in F$ and $|A_0(z)|=M_{z_0}(r,A_0)$, we have

$$|A_0(z)|>\exp_p\{\frac{1}{r^\alpha}\}. \tag{3.17}$$

Using (3.16)–(3.17) with (3.3) in (3.15), we obtain

$$\exp_p\{\frac{1}{r^\alpha}\}\le\frac{\lambda}{r^{2k}}\left[T_{z_0}(\frac{r}{\gamma},f)\right]^{2k}\exp_p\{\frac{1}{r^{\beta+\varepsilon}}\}. \tag{3.18}$$

From (3.18), we obtain that $\sigma_{p+1}(f,z_0)\ge\alpha$.

On the other hand, applying Lemma 6, we obtain that $\sigma_{p+1}(f,z_0)\le\sigma_p(A_0,z_0)$. Since $\alpha\le\sigma_{p+1}(f,z_0)\le\sigma_p(A_0,z_0)$ holds for every $\alpha<\sigma_p(A_0,z_0)$, then we conclude that $\sigma_{p+1}(f,z_0)=\sigma_p(A_0,z_0)$. □

Proof of Theorem 5. a) First, we show that (1.15) can possess at most one exceptional solution f_0 satisfying $\sigma_{p+1}(f_0, z_0) \leq \sigma_p(A_0, z_0)$ or $i(f_0) < p+1$. In fact, if f_0^* is a second solution with $\sigma_{p+1}(f_0^*, z_0) \leq \sigma_p(A_0, z_0)$ or $i(f_0^*) < p+1$, then $\sigma_{p+1}(f_0 - f_0^*, z_0) \leq \sigma_p(A_0, z_0)$ or $i(f_0 - f_0^*) < p+1$. But $f_0 - f_0^*$ is a solution of the corresponding homogeneous equation (1.6) of (1.15), this contradicts Theorem (3). We assume that f is a solution with $\sigma_{p+1}(f,\ z_0) \geq \sigma_p(A_0, z_0)$, and $f_{1,}\ f_2, ...,$ f_k is solutions base of the corresponding homogeneous equation (1.6). Then f can be expressed in the form

$$f(z) = B_1(z)f_1(z) + B_2(z)f_2(z) + ... + B_k(z)f_k(z), \tag{3.19}$$

where $B_1(z), B_2(z), ..., B_k(z)$ are determined by

$$\begin{aligned} &B_1'(z)_1(z)f_1(z) + B_2'(z)f_2(z) + ... + B_k'(z)f_k(z) = 0, \\ &B_1'(z)f_1'(z) + B_2'(z)f_2'(z) + ... + B_k'(z)f_k'(z) = 0, \\ &\qquad . \\ &\qquad . \\ &B_1'(z)f_1^{(k-1)}(z) + B_2'(z)f_2^{(k-1)}(z) + ... + B_k'(z)f_k^{(k-1)}(z) = H. \end{aligned} \tag{3.20}$$

Since the Wronskian $W(f_1,\ f_2, ...,\ f_k) = \begin{vmatrix} f_1(z) & f_2(z) & . & . & f_k(z) \\ f_1'(z) & f_2'(z) & . & . & f_k'(z) \\ . & . & . & . & . \\ . & . & . & . & . \\ f_1^{(k-1)} & f_2^{(k-1)} & . & . & f_k^{(k-1)} \end{vmatrix}$ is a differential polynomial in $f_1,\ f_2, ...,\ f_k$ with constant coefficients, it is easy to deduce that

$$\sigma_{p+1}(W,\ z_0) \leq \sigma_{p+1}(f_j,\ z_0) = \sigma_p(A_0,\ z_0).$$

From (3.20),

$$B_J'(Z) = \frac{\begin{vmatrix} f_1(z) & . & 0 & f_k(z) \\ f_1'(z) & . & 0 & f_k'(z) \\ . & . & . & . \\ . & . & . & . \\ f_1^{(k-1)} & . & H & f_k^{(k-1)} \end{vmatrix}}{W(f_1,\ f_2, ...,\ f_k)} = \frac{H.\begin{vmatrix} f_1(z) & . & . & f_k(z) \\ f_1'(z) & . & . & f_k'(z) \\ . & . & . & . \\ . & . & . & . \\ f_1^{(k-1)} & . & . & f_k^{(k-1)} \end{vmatrix}}{W(f_1,\ f_2, ...,\ f_k)}$$

$$B_j' = H.G_j(f_1,\ f_2, ...,\ f_k).W(f_1,\ f_2, ...,\ f_k)^{-1},\ j = 1, ..., k, \tag{3.21}$$

where $G_j(f_1,\ f_2, ...,\ f_k)$ are differential polynomial in $f_1,\ f_2, ...,\ f_k$ with constant coefficients, thus

$$\sigma_{p+1}(G_j, z_0) \leq \sigma_{p+1}(f_j, z_0) = \sigma_p(A_0, z_0). \tag{3.22}$$

Since $i\left(H\right)<p+1$ or $i\left(H\right)=p+1,\sigma_{p+1}\left(H,z_{0}\right)<\sigma_{p}\left(A_{0},z_{0}\right)$ from Lemma (4) and (3.22), we have

$$\sigma_{p+1}\left(B_{j},\ z_{0}\right)=\sigma_{p+1}\left(B_{j}^{\prime},\ z_{0}\right)\leq\max\left\{\sigma_{p+1}\left(H,\ z_{0}\right),\ \sigma_{p}\left(A_{0},\ z_{0}\right)\right\}=\sigma_{p}\left(A_{0},\ z_{0}\right) \tag{3.23}$$

for $j=1,...,k.$ By (3.19) and (3.23), we obtain

$$\sigma_{p+1}(f,\ z_{0})\leq\max\left\{\sigma_{p+1}\left(f_{j},z_{0}\right),\sigma_{p+1}\left(B_{j},z_{0}\right)\right\}=\sigma_{p}\left(A_{0},z_{0}\right). \tag{3.24}$$

From this and the assumption $\sigma_{p+1}(f,z_{0})\geq\sigma_{p}\left(A_{0},z_{0}\right)$, we conclude that

$$\sigma_{p+1}(f,z_{0})=\sigma_{p}\left(A_{0},z_{0}\right).$$

If f is a solution of equation (1.15) satisfying $\sigma_{p+1}(f,z_{0})=\sigma_{p}\left(A_{0},z_{0}\right)$, from Lemma 8, we have

$$\overline{\lambda}_{p+1}\left(f,z_{0}\right)=\lambda_{p+1}\left(f,z_{0}\right)=\sigma_{P+1}\left(f,z_{0}\right)=\sigma_{p}\left(A_{0},z_{0}\right).$$

b) From the hypotheses of Theorem (5) and (3.19)-(3.24), we have

$$\sigma_{q}(f,z_{0})\leq\sigma_{q}\left(H,z_{0}\right). \tag{3.25}$$

From (1.15), a simple consideration of order implies

$$\sigma_{q}(f,z_{0})\geq\sigma_{q}\left(H,z_{0}\right). \tag{3.26}$$

By (3.25)-(3.26), we have $\sigma_{q}(f,z_{0})=\sigma_{q}\left(H,z_{0}\right).$ □

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