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Homogenization of a lubrication problem in oscillating domain by two-scale convergence method

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Abstract. In present paper we do a homogenization with respect to a small parameter of a boundary-value problem describing fluid flow between two moving in space and time rough surfaces. The two-scale convergence method was used to justify the behavior of the flow in the limit.

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1 Introduction

A general frame of this work is the boundary-value problems in domains with oscillating boundaries. In recent years the interest in this kind of problems appears in connection with the development of technologies of porous, composite and other microinhomogeneous materials, and also as a result of various physical experiments. For example, the morphology of contacting surfaces plays an important role in the frictional behavior of deformable bodies. The roughness of the contact surface and the material properties near this surface are microcharacteristics which influence the large scale behavior. The mathematical analysis of such problems based on boundary homogenization was presented e.g. in [4–6, 11, 13, 16, 17, 31] and others.

The goal of the paper is an asymptotic analysis and its rigorous mathematical justification of a problem that models fluid flow in a thin domain bounded by two moving rough surfaces. We study the asymptotic behavior of incompressible unsteady Stokes flow in narrow gap described by two small parameters ε and μ. The parameter ε is related to the distance between the surfaces whereas μ is the wavelength of the periodic roughness. Such mathematical problem has been risen by engineering applications dealing with lubrication theory. To increase the hydrodynamic performance in different lubricated machine elements, e.g. journal bearings and thrust bearings, it is important to understand the influence of surface roughness. In this connection one encounters different approaches commonly based on the equation proposed by Osborne Reynolds in 1886 [30]. The fundamental problem in lubrication theory is to describe fluid flow in a gap between two adjacent surfaces which are in relative motion. In the incompressible case the main unknown is the pressure of the fluid. Having resolved the pressure it is possible to compute other fundamental quantities such as the velocity field and the forces on the bounding surfaces, i.e. friction forces and load carrying forces.

The study of incompressible flow in a thin layer can be found, e.g. in [7,10,25–27]. An analysis of a flow in a thin layer between moving in time and space smooth surfaces have been done in [19]. The novelty of present paper is to include in consideration also the rough periodic structure of the boundary.

In order to take into account the roughness effect one needs to use some special homogenization methods. Some averaging techniques considering surfaces roughness have been introduced e.g. in [8, 18, 29]. The most natural way for mathematical justification of problems involving rough periodic geometry is the method of two-scale convergence that originally goes back to Nguetseng [28]. We refer to the paper [3] where the two-scale convergence notion was introduced and also many aspects of homogenization technique for boundary-value problems with periodically oscillating coefficients with help of two-scale convergence method are discussed. Application of two-scale convergence method to homogenization of fluid dynamic problems can be found e.g. in [12, 24].

Most previous studies have considered only the case when the stationary surface is rough. In this paper the assumptions regarding curvature and motion of the surfaces are sufficiently general to include most realistic applications and lead to a time-dependent problem with a non-cylindrical space-time domain. This causes the main difficulty compared to the stationary case.

Let $\varepsilon > 0$ be a parameter characterizing the thickness of the gap between moving surfaces while $\mu > 0$ is the size of period of the roughness.

The results presented here pertain to the asymptotic behavior of the velocity field of the fluid as both ε and μ tend to zero. The case including only the parameter ε, i.e. smooth surfaces, has been studied in [7, 10, 19]. The situation with two parameters was considered in [9, 12, 20]. The main contribution in the present work is the treatment of the unstationary problem with two small parameters. Moreover, the techniques used in the proofs differ from previous ones. In order to pass to the limit we apply the method of two-scale convergence, see e.g. [3], on extending the solution across the oscillating boundaries to a cylindrical domain. It is assumed that ε is a function of μ such that

$$\lambda = \lim_{\mu \to 0^+} \frac{\varepsilon(\mu)}{\mu} \text{ exists in } [0, \infty].$$

Three cases are distinguished: $\lambda = 0$ (Reynolds roughness), $0 < \lambda < \infty$ (Stokes roughness) and $\lambda = \infty$ (High frequency roughness). The corresponding homogenized equations are all of Reynolds type and two-dimensional. They govern the limit velocity field and have coefficients that can be calculated by solving local problems on a periodic cell, thus taking into account the surface roughness. In the High frequency roughness case we have discovered the critical value $\lim\limits_{\varepsilon,\mu \to 0}$. Depending on it two different flow behaviour are possible. In the limit as $\mu, \varepsilon \to 0$, we rigorously derive the time-dependent Reynolds equation and show how the limiting velocity field and pressure are governed by this equation. The two-scale convergence method was treated to obtain and justify the homogenized model of the flow. In particular, we have proved two-scale convergence of the original pressure and velocity field to the limit ones. Let us mention that limit equations as $\mu, \varepsilon \to 0$ for velocity and pressure for a similar problem were proposed in [20] by formal asymptotic expansion method. In some sense the results of present paper complement studies in [20].

2 Statement of the problem

We start with notations. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $x' = (x_1, x_2)$. Denote by $\Xi = [0, \Xi_1] \times [0, \Xi_2]$ a periodicity cell in $\mathbb{R}^2$, we use variables $\xi' = (\xi_1, \xi_2)$ for points from Ξ. For given positive $T, \mathbb{T}$ we write $t \in [0, T]$ and $\tau \in \mathcal{T}$, where $\mathcal{T} = [0, \mathbb{T}]$ is a periodicity cell in $\mathbb{R}$.

Let ω be an open bounded subset in $\mathbb{R}^2$, with sufficiently smooth boundary. Assume that $h^{\pm}(x', \xi', t, \tau) \in C^2(\omega \times \mathbb{R}^2 \times [0, T] \times \mathbb{R})$ are given periodic in ξ' and τ functions with

$$h_{min} \leq h = h^+ - h^- \leq h_{max},$$

where

$$h_{min} = \min_{(x', \xi', t, \tau)}(h^+ - h^-), \quad h_{max} = \max_{(x', \xi', t, \tau)}(h^+ - h^-).$$

The constants h^-_{min}, h^+_{max} are defined analogously. In our analysis we assume that the function

$$h^{\pm}(x', \xi', t, \tau) = h^{\pm}_0(x', t) + h^{\pm}_{per}(\xi' - v^{\pm}\tau)$$

describes the rough structure of surfaces moving with the velocity $v^{\pm} = (v^{\pm}_1, v^{\pm}_2, 0)$. Here $h^{\pm}_0$ describes the global film thickness whereas the Ξ-periodic functions $h^{\pm}_{per}$ represents the roughness. We assume also that v is such that $h^{\pm}$ is also periodic in τ with period $\mathbb{T}$.

For each $t \in [0, T]$ we define the domain occupied by fluid:

$$\Omega_{\varepsilon\mu}(t) = \{(x', x_3) \in \mathbb{R}^3 : \ x' \in \omega, \ \varepsilon h^-_{\mu}(x', t) < x_3 < \varepsilon h^+_{\mu}(x', t)\},$$

where

$$h^{\pm}_{\mu}(x', t) = h^{\pm}_0(x', t) + h^{\pm}_{per}\left(\frac{x' - v^{\pm}t}{\mu}\right), \quad x' \in \omega, \quad t \in [0, T].$$

In order to clarify the notations we write

$$h^{\pm}_{\mu}(x', t) = h^{\pm}(x', \xi', t, \tau)|_{\xi' = \frac{x'}{\mu}, \tau = \frac{t}{\mu}}.$$

Moreover, we define $h_{\mu} = h^+_{\mu} - h^-_{\mu}$. The boundary $\partial\Omega_{\varepsilon\mu}(t)$ can be divided in three parts:

$$\Sigma^-_{\varepsilon\mu}(t) \cup \Sigma^+_{\varepsilon\mu}(t) \cup \Sigma^w_{\varepsilon\mu}(t),$$

where

$$\Sigma^{\pm}_{\varepsilon\mu}(t) = \{(x', x_3) \in \mathbb{R}^3 : \ x' \in \omega, \ x_3 = \varepsilon h^{\pm}_{\mu}(x', t)\},$$

$$\Sigma^w_{\varepsilon\mu}(t) = \{(x', x_3) \in \mathbb{R}^3 : \ x' \in \partial\omega, \ \varepsilon h^-_\mu(x', t) \le x_3 \le \varepsilon h^+_\mu(x', t)\}.$$

We set for any $t \in [0, T]$:

$$\Omega_{\varepsilon\mu T} = \bigcup_{0\le t\le T} \Omega_{\varepsilon\mu}(t) \times \{t\}, \ \Sigma^\pm_{\varepsilon\mu T} = \bigcup_{0\le t\le T} \Sigma^\pm_{\varepsilon\mu}(t) \times \{t\}, \ \Sigma^w_{\varepsilon\mu T} = \bigcup_{0\le t\le T} \Sigma^w_{\varepsilon\mu}(t) \times \{t\}.$$

The flow is governed by the evolution Stokes equation:

$$D_t u^{\varepsilon\mu} - \nu\Delta u^{\varepsilon\mu} + \nabla p^{\varepsilon\mu} = 0 \text{ in } \Omega_{\varepsilon\mu T}, \tag{2.1}$$

$$\operatorname{div} u^{\varepsilon\mu} = 0 \text{ in } \Omega_{\varepsilon\mu T}. \tag{2.2}$$

We assume no slip boundary conditions

$$u^{\varepsilon\mu} = \left(v^\pm_1, v^\pm_2, \varepsilon\left(\frac{\partial h^\pm_\mu}{\partial t} + v^\pm \cdot \nabla h^\pm_\mu\right)\right) \qquad \text{on} \quad \Sigma^\pm_{\varepsilon\mu}(t) \tag{2.3}$$

$$u^{\varepsilon\mu} = g \equiv \begin{pmatrix} g_1\left(x_1, x_2, \frac{x_3}{\varepsilon}\right) \\ g_2\left(x_1, x_2, \frac{x_3}{\varepsilon}\right) \\ \varepsilon g_3\left(x_1, x_2, \frac{x_3}{\varepsilon}\right) \end{pmatrix} \qquad \text{on} \quad \Sigma^w_{\varepsilon\mu}(t) \tag{2.4}$$

with initial condition

$$u^{\varepsilon\mu}(x, 0) = u^{\varepsilon\mu}_0 \equiv \begin{pmatrix} u^0_1\left(x_1, x_2, \frac{x_3}{\varepsilon}\right) \\ u^0_2\left(x_1, x_2, \frac{x_3}{\varepsilon}\right) \\ \varepsilon u^0_3\left(x_1, x_2, \frac{x_3}{\varepsilon}\right) \end{pmatrix} \qquad \text{on} \quad \Omega_{\varepsilon\mu}(0) \times \{0\}, \tag{2.5}$$

where $g \in H^{\frac{1}{2}}(\Sigma^w_{\varepsilon\mu T}; \mathbb{R}^3)$ and $u^{\varepsilon\mu}_0 \in H^1(\Omega_{\varepsilon\mu}(0); \mathbb{R}^3)$ are given functions,
$\nabla h = \left(\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, 0\right)$.
Sometimes for the convenience we shall denote $v^\pm_3 = \left(\frac{\partial h^\pm_\mu}{\partial t} + v^\pm \cdot \nabla h^\pm_\mu\right)$. In addition, our assumptions on boundary and initial data are:

$$u^{\varepsilon\mu}_0 = u^{\varepsilon\mu} \text{ on } \Sigma^\pm_{\varepsilon\mu}(0) \cup \Sigma^w_{\varepsilon\mu}(0), \quad \operatorname{div} u^{\varepsilon\mu}_0 = 0 \text{ in } \Omega_{\varepsilon\mu}(0),$$
$$v^\pm \in C(\overline{\omega \times (0, T)}; \mathbb{R}^3),$$

the functions $g, u^{\varepsilon\mu}_0$ and their partial derivatives are uniformly bounded. Finally, to ensure the existence of a solution to the proposed problem, one needs to require some compatibility condition between boundary data and functions $h^\pm$. It is assumed that

$$div\ g = 0 \quad \text{ in } \Sigma^\omega_{\varepsilon\mu T}, \quad g(x_1, x_2, h^\pm) = \left(v^\pm_1, v^\pm_2, \frac{\partial h^\pm_\mu}{\partial t} + v^\pm \cdot \nabla h^\pm_\mu\right)$$

$$\int_0^T \left(\int_\omega D_t h_0 \, dx' + \int_{\partial\omega} \overline{g}^z \cdot \hat{n} \, dS(x')\right) \phi \, dt = 0 \tag{2.6}$$

for all $\phi \in L^2(0,T)$ to be valid, where

$$\overline{g}^z = \int_{h^-}^{h^+} g\,dz, \quad z = \frac{x_3}{\varepsilon}.$$

We remark that physically the condition (2.6) describes the mass conservation low. The goal of the paper is to obtain and justify the behaviour of the flow in the limit as $\varepsilon, \mu \to 0$.

3 Preliminaries to present the main result

3.1 Rescaled domain

To make an analysis of the proposed problem more simple, we use the following transformations of the domain:

$$\Omega_{\varepsilon\mu}(t) \mapsto \Omega_\mu(t) = \omega \times [h_\mu^-, h_\mu^+]$$

by the change of variables: $x' = x',\ z = \frac{x_3}{\varepsilon}$.

Observe that the domain Ω does not depend on any parameters and t. We denote

$$\omega_T = \omega \times [0,T], \quad \Omega_T = \Omega \times [0,T], \quad \Omega_{\mu T} = \bigcup_{0\le t\le T} \Omega_\mu(t).$$

Since we assume the relation $\varepsilon = \varepsilon(\mu)$, we shall write also in the sequel u^μ, p^μ instead of $u^{\varepsilon\mu}, p^{\varepsilon\mu}$.

In the rescaled domain we introduce

$$\nabla_\varepsilon = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{1}{\varepsilon}\frac{\partial}{\partial z}\right), \quad \mathrm{div}_\varepsilon = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{1}{\varepsilon}\frac{\partial}{\partial z}.$$

The following lemma (see e.g. [15], [21, Theorem 3.5] and [33]) is useful for our analysis.

Lemma 1. *Let $\partial\Omega$ be boundary of C^s class, $s \ge 2$ and $G \in H^{\frac{1}{2}}(\partial\Omega; \mathbb{R}^n)$ be a given function that satisfies $\int\limits_{\partial\Omega} G \cdot n_S\, dS = 0$ (n_S is a unit outward normal vector to the boundary). Then there exists a function $U \in H^1(\Omega; \mathbb{R}^n)$ such that*

$$\mathrm{div}\, U = 0 \ \text{in } \Omega, \quad U = G \ \text{on } \partial\Omega$$

and

$$\|U\|_{H^1(\Omega)} \le K \|G\|_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{3.1}$$

Lemma 1 implies the existence of function $G^\mu = (G_1^\mu, G_2^\mu, G_3^\mu) \in H^1(\Omega_\mu(t); \mathbb{R}^3)$ such that

$$\mathrm{div}_\varepsilon G^\mu = 0 \ \text{ in } \Omega_\mu(t), \quad G^\mu = u^\mu|_{\partial\Omega_\mu(t)} \ \text{ on } \partial\Omega_\mu(t).$$

Moreover,

$$\|G^\mu\|_{H^1(\Omega_\mu(t))} \le K\|u^\mu|_{\partial\Omega_\mu(t)}\|_{H^{\frac{1}{2}}(\partial\Omega_\mu(t))},$$

where $u^\mu|_{\partial\Omega_\mu}$ satisfies (2.3)–(2.5).

Taking into account the assumptions on boundary and initial data, the function G^μ satisfies the estimates

$$\begin{aligned}
&\|D_t G^\mu\|_{L_2(\Omega_\mu(t))} + \|G^\mu\|_{L_2(\Omega_\mu(t))} \le K\\
&\|\nabla_\varepsilon G^\mu\|_{L_2(\Omega_\mu(t))} + \|\nabla_\varepsilon D_t G^\mu\|_{L_2(\Omega_\mu(t))} \le \frac{K_1}{\varepsilon}\\
&\|D_t G^\mu\|_{L_2(\Omega_\mu(0))} + \|\nabla_\varepsilon G^\mu\|_{L_2(\Omega_\mu(0))} + \|\nabla_\varepsilon D_t G^\mu\|_{L_2(\Omega_\mu(0))} \le \frac{K_1}{\varepsilon}\\
&\|D_3 G^\mu\|_{L_2(\Omega_T)} \le \frac{K}{\varepsilon}
\end{aligned} \tag{3.2}$$

uniformly in μ. Define the extended domain

$$\Omega^{ext} = \{(x', z) : x' \in \omega,\ h^-_{min} - \delta < z < h^+_{max} + \delta\},\ \delta = \text{ const} > 0,\ \ \Omega^{ext}_T = [0,T]\times\Omega^{ext}. \tag{3.3}$$

Lemma 2. *There exists an extension $\overline{u}^\mu$ of u^μ such that*

1. $\overline{u}^\mu = u^\mu$ *in* Ω^μ_T,
2. $\overline{u}^\mu = G^\mu$ *in* $\Omega^{ext}_T \setminus \Omega^\mu_T$,
3. $\overline{u}^\mu \in L_2(0,T; H^1(\Omega^{ext};\mathbb{R}^3))$,
4. $\mathrm{div}\overline{u}^\mu = 0$ *in* Ω^{ext}_T.

Proof. Applying Lemma 1 we can assume that G^μ is defined on Ω^{ext}_T (as a divergence-free extension). For simplicity we drop dependence on μ, T etc. Clearly, $\overline{u}$ must be defined as

$$\overline{u} = \begin{cases} u, & \text{in } \Omega,\\ G, & \text{in } \Omega^{ext}\setminus\Omega. \end{cases}$$

Set $w = u - G$ in Ω. By assumption, u and G have the same trace on $\partial\Omega$. Hence w belongs to $H^1_0(\Omega;\mathbb{R}^3)$ and so the trivial extension

$$\overline{w} = \begin{cases} w, & \text{in } \Omega,\\ 0, & \text{in } \Omega^{ext}\setminus\Omega \end{cases}$$

belongs to $H^1_0(\Omega^{ext};\mathbb{R}^3)$. The third property is thus deduced from the relation $\overline{u} = \overline{w} + G$. To prove property 4 it suffices to show that $\mathrm{div}\overline{w}^\mu = 0$ in Ω^{ext}. The latter relation is a consequence of the identity

$$\int\limits_{\Omega^{ext}} \phi\,\mathrm{div}\overline{w}^\mu\, dx = -\int\limits_{\Omega^{ext}} \nabla\phi\overline{w}^\mu\, dx.$$

$\square$

3.2 Definition of the weak solution.

Let us define the weak solution of Stokes problem (2.1)–(2.3). Denote by

$$L_2^0(\Omega_\mu(t)) = \{p(x', z, t) : \int\limits_{\Omega_\mu(t)} p\, dx'\, dz = 0\},$$

The couple of functions $u^\mu(x', z, t) \in L_2(0, T; H^1(\Omega_\mu(t); \mathbb{R}^3))$,
$p^\mu(x', z, t) \in L_2(0, T; L_0^2(\Omega_\mu(t)))$ is the solution to (2.1)–(2.3) iff

$$\begin{aligned}\int\limits_{\Omega_\mu(t)} u^\mu \varphi\, dx'\, dz \Big|_0^T - \int\limits_0^T \int\limits_{\Omega_\mu(t)} u^\mu D_t \varphi\, dx'\, dz\, dt + \nu \int\limits_0^T \int\limits_{\Omega_\mu(t)} \nabla_\varepsilon u^\mu \nabla_\varepsilon \varphi\, dx'\, dz\, dt - \\ - \int\limits_0^T \int\limits_{\Omega_\mu(t)} p^\mu \mathrm{div}_\varepsilon \varphi\, dx'\, dz\, dt = 0\end{aligned} \tag{3.4}$$

for any $\varphi \in L_2(0, T; H_0^1(\Omega_\mu(t); \mathbb{R}^3))$,

$$\int\limits_{\Omega_\mu(t)} \mathrm{div}_\varepsilon u^\mu q\, dx'\, dz = 0 \quad \forall q \in L_2(0, T; L_2(\Omega_\mu(t))). \tag{3.5}$$

$$u^\mu - G^\mu \in L_2(0, T; H_0^1(\Omega_\mu(t); \mathbb{R}^3)). \tag{3.6}$$

For proof of the existence and uniqueness of the weak solution we refer to [19, Theorem 4.1]. Let us notice that an assumption $\int\limits_{\Omega_\mu(t)} p^\mu\, dx'\, dz = 0$ is chosen to ensure the uniqueness of p^μ satisfying (3.4). Now we shall study the asymptotic behaviour of the weak solution as $\mu \to 0$ by two-scale convergence method.

3.3 Two-scale convergence method

Before we start to formulate the main results let us remind the definition of two-scale convergence.

Definition 1. Let Ω be an open set in $\mathbb{R}^n$ and Ξ is a periodicity cell. A sequence of functions $f^\mu(x) \equiv f(x, \frac{x}{\mu})$ in $L_2(\Omega)$ is said to two-scale converge to a limit $f^0(x, \xi)$ belonging to $L_2(\Omega \times \Xi)$ ($f^\mu(x) \twoheadrightarrow f^0(x, \xi)$) if for any function $\psi(x, \xi)$ in $\mathcal{D}(\Omega; C_{per}^\infty(\Xi))$, we have

$$\lim_{\mu \to 0} \int\limits_\Omega f^\mu(x) \psi\left(x, \frac{x}{\mu}\right)\, dx = \int\limits_\Omega \int\limits_\Xi f^0(x, \xi) \psi(x, \xi)\, dx\, d\xi. \tag{3.7}$$

The class of test-functions ψ can be enlarged to $L_2(\Omega; C_{per}(\Xi))$, see [3, Remark 1.1]. Also for the reader convenience let us formulate the following result on two-scale convergence proved in [3] that will be used several times in our analysis.

Theorem 1.

- *Let* u^μ *be a bounded sequence in* $H^1(\Omega)$ *that converges weakly to a limit* u *in* $H^1(\Omega)$. *Then* $u^\mu \twoheadrightarrow u(x)$, *and there exists a function* $u_1(x,\xi)$ *in* $L_2(\Omega; H^1_{per}(\Xi)/\mathbb{R})$ *such that, up to a subsequence,* $\nabla u^\mu \twoheadrightarrow \nabla_x u(x) + \nabla_\xi u_1(x,\xi)$.

- *Let* u^μ *and* $\mu\nabla u^\mu$ *be two bounded sequences in* $L_2(\Omega)$. *Then there exists a function* $u_0(x,\xi) \in L_2(\Omega; H^1_{per}(\Xi))$ *such that, up to a subsequence,* $u^\mu \twoheadrightarrow u^0(x,\xi)$ *and* $\mu\nabla u^\mu \twoheadrightarrow \nabla_\xi u^0(x,\xi)$.

- *Let* u^μ *be a divergence-free bounded sequence in* $(L_2(\Omega))^N$ *such that* $u^\mu \twoheadrightarrow u_0(x,\xi) \in (L_2(\Omega \times \Xi))^N$.
Then the two-scale limit satisfies $\mathrm{div}_\xi u_0(x,\xi) = 0$ *and* $\int\limits_\Xi \mathrm{div}_x u_0(x,\xi)\, d\xi = 0$.

- *Let* u^μ *be a sequence of functions in* $L_2(\Omega)$ *that two-scale converges to* $u^0(x,\xi) \in L_2(\Omega; C_{per}(\Xi))$. *If*

$$\lim_{\mu\to 0} \|u^\mu\|_{L_2(\Omega)} = \|u^\mu\|_{L_2(\Omega\times\Xi)}$$

then u^μ *converges strongly to* u^0 :

$$\lim_{\mu\to 0} \|u^\mu - u^0(x, \frac{x}{\mu})\|_{L_2(\Omega)} = 0.$$

To treat two-scale convergence method for time-dependent situation we use the following theorem (see [23]).

Theorem 2.

- *Let* u^μ *be a bounded sequence in* $L_p(\Omega \times (0,T))$, *then there exists a subsequence which two-scale converges.*

- *Assume* $u^\mu \in L_p(0,T; W_0^{1,p}(\Omega))$ *such that* $u^\mu(x,t) \twoheadrightarrow u^0(x,\xi,t,\tau)$ *and* $\nabla_x u^\mu(x,t) \twoheadrightarrow z(x,t,\xi,\tau)$. *Then the two-scale limit* u^0 *is independent of* ξ *and* $u^0 \in L_p((0,T)\times\mathcal{T}; W_0^{1,p}(\Omega))$. *Moreover,*
$z(x,t,\xi,\tau) = \nabla_x u^0(x,t,\tau) + \nabla_\xi u^1(x,t,\xi,\tau)$, *where* $u^1 \in L_p((0,T)\times\mathcal{T}; W^{1,p}_{per}(\Xi))$.

4 The main results

Let us formulate now the main results of the paper. The first one is concerned with the limit behavior of the velocity field.

Theorem 3. *Let* Ω^{ext} *be defined by (3.3). There exists* $u^\lambda \in L_2((0,T)\times\mathcal{T}; L_2(\Omega^{ext}; H^1_{per}(\Xi))$, *having* $\frac{\partial u^\lambda}{\partial z} \in L_2((0,T)\times\mathcal{T}; L_2(\Omega^{ext}\times\Xi))$ *and also*

$v^\lambda \in L_2((0,T)\times \mathcal{T}; L_2(\Omega^{ext}\times \Xi; H^1_{per}([0,\Xi_3])))$ *such that*

$$\begin{aligned} &\overline{u}^\mu \twoheadrightarrow u^\lambda, \\ &\frac{\partial \overline{u}^\mu}{\partial z} \twoheadrightarrow \frac{\partial u^\lambda}{\partial z} + \frac{\partial v^\lambda}{\partial \xi_3}, \\ &\mu\frac{\partial \overline{u}^\mu}{\partial x_i} \twoheadrightarrow \frac{\partial u^\lambda}{\partial \xi_i}, \qquad i=1,2, \\ &\mu\frac{\partial \overline{u}^\mu}{\partial t} \twoheadrightarrow \frac{\partial u^\lambda}{\partial \tau}. \end{aligned} \tag{4.1}$$

In addition,

- *if* $0<\lambda<\infty$, *then* u *depends on* λ, *we denote* $u=u^\lambda$ *and*

$$\mathrm{div}_\lambda u^\lambda = \frac{\partial u_1^\lambda}{\partial \xi_1} + \frac{\partial u_2^\lambda}{\partial \xi_2} + \frac{1}{\lambda}\frac{\partial u_3^\lambda}{\partial z} = 0 \quad \text{in } \mathsf{B}(x',t,\tau),\ (x',t,\tau)\in\omega\times(0,T)\times\mathcal{T},$$
$$u^\lambda = \left(v_1^\pm, v_2^\pm, \lambda\left(\frac{\partial h^\pm}{\partial \tau} + v^\pm\cdot\nabla_\xi h^\pm\right)\right), \qquad \text{as } z=h^\pm.$$

- *if* $\lambda=0$ *then the boundary condition is*

$$u=\left(v_1^\pm, v_2^\pm, 0\right) \qquad \text{on } z=h^\pm.$$

- *if* $\lambda=\infty$ *then* $u\in L_2((0,T)\times\mathcal{T}; L_2(\Omega^*;\mathbb{R}^3))$, *where*

$$\Omega^* = \{(x',z) : x'\in\omega, h_*^-(x',t) < z < h_*^+(x',t)\}$$
$$h_*^+(x',t) = h_0^+(x'-tv) + \min_{(\xi',\tau)\in\Xi\times\mathcal{T}} h^+_{per}(\xi',\tau),$$
$$h_*^-(x',t) = h_0^-(x'-tv) + \max_{(\xi',\tau)\in\Xi\times\mathcal{T}} h^-_{per}(\xi',\tau), \quad h_* = h_*^+ - h_*^-.$$

Moreover, the boundary conditions are

$$u=\left(v_1^\pm, v_2^\pm, 0\right) \qquad \text{on } z=h_*^\pm.$$

In this case, if in addition $\lim\limits_{\mu\to 0}\frac{\mu}{\varepsilon^2(\mu)}$ *then* u *is independent of* τ.

Our second main result deals with the pressure convergence and reads as follows:

Theorem 4. *There exists the two-scale limit* $p\in L_2((0,T)\times\mathcal{T}; H^1(\omega))$ *such that*

$$\varepsilon^2\overline{p}^\mu \to p \qquad \text{strongly in } L_2((0,T)\times\mathcal{T};\Omega^{ext}) \text{ as } \mu\to 0$$

where $\overline{p}^\mu = 0$ *in* $\Omega_\mu(t)$ *and* $\overline{p}^\mu = 0$ *in* $\Omega^{ext}\setminus\Omega_\mu(t)$. *Moreover, the two-scale limit does not depend on the way of extension* p^μ *into the domain* Ω^{ext}.

The existence of pressure extension is given by Lemma 5. The result of Theorem 4 follows from Lemmas 5, 6, 7. The next theorem characterizes the two-scale limit u.

Theorem 5. *Let α^i be defined as the unique solutions of local problems (4.8) for $0<\lambda<\infty$, (4.11) for $\lambda=0$, (4.13) for $\lambda=\infty$. Then two-scale limit velocity u and pressure p are related via*

$$u^\lambda = \sum_{j=1}^{2} \frac{\partial p^\lambda}{\partial x_j}\alpha^j + \alpha^3, \tag{4.2}$$

where $p(x',t,\tau)$ is the unique solution of

$$\begin{aligned} &\sum_{i,j=1}^{2} \int\limits_{\omega_T\times\mathcal{T}} a_{ij}\frac{\partial p}{\partial x_j}\frac{\partial \varphi}{\partial x_i}\,dx'dtd\tau - \sum_{i=1}^{2}\int\limits_{\omega_T\times\mathcal{T}} b_i\frac{\partial\varphi}{\partial x_i}\,dx'dtd\tau = \\ &= \int\limits_{\omega_T\times\mathcal{T}} \overline{\frac{\partial h}{\partial t}}^{\xi'}\varphi\,dx'dtd\tau - \int\limits_{(0,T)\times\mathcal{T}}\int\limits_{\partial\omega} \overline{g}^z\cdot n\varphi\,dS(x')dtd\tau, \quad \varphi\in H^1(\omega), \end{aligned} \tag{4.3}$$

$$\int\limits_{\omega_T\times\mathcal{T}} p(x',t)\,dx'dtd\tau = 0, \qquad \overline{h}^{\xi'} = \int\limits_{\Xi} h\,d\xi'. \tag{4.4}$$

Here coefficients $A=(a_{ij})_{2\times 2}$ and $b=(b_1,b_2)$ depend on α^i through (4.8), (4.11) and (4.13) for different limiting cases.

Note that the identity (4.3) is the weak formulation to the following Reynolds equation:

$$\begin{aligned} \operatorname{div}_{x'}(A\nabla_{x'}p + b) + \frac{\partial h_0}{\partial t} &= 0 && \text{in} \quad \omega\times(0,T)\times\mathcal{T} \\ (A\nabla_{x'}p + b - \overline{g}^z)\cdot n_\omega &= 0 && \text{on} \quad \partial(\omega\times(0,T)\times\mathcal{T}), \end{aligned} \tag{4.5}$$

The function p being a solution of (4.3) is uniquely defined by (4.4). Thus, the limit solution (u,p) depends on the local problems which are different for each $\lambda\in[0,\infty]$.

4.1 Stokes roughness $(0<\lambda<\infty)$

Consider the case when $0<\lambda<\infty$, i.e. when the thickness of the layer is of the same order as the roughness wavelength. One finds that the coefficients of the homogenized Reynolds equation (4.3) are obtained by solving three-dimensional cell problems which depend on the parameter λ. More precisely,

$$\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{pmatrix} = \overline{\overline{\begin{pmatrix} \alpha_1^1 & \alpha_1^2 & \alpha_1^3 \\ \alpha_2^1 & \alpha_2^2 & \alpha_2^3 \end{pmatrix}}^{z}}^{\xi'}. \tag{4.6}$$

Here functions α^i are defined in the periodicity cell

$$B = B(x',t,\tau) = \{(\xi',z) : \xi'\in\Xi, h^-(x',\xi',t,\tau) < z < h^+(x',\xi',t,\tau)\}$$

and satisfy the system

$$\nu\Delta_\lambda \alpha^i = \nabla_\lambda q^i + f^i \qquad \text{in } \mathsf{B} \quad (i=1,2,3) \tag{4.7}$$
$$\operatorname{div}_\lambda \alpha^i = 0 \qquad \text{in } \mathsf{B} \quad (i=1,2,3), \tag{4.8}$$

where

$$f^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad f^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad f^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\Delta_\lambda = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{1}{\lambda^2}\frac{\partial^2}{\partial z^2}, \quad \nabla_\lambda = \left(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{1}{\lambda}\frac{\partial}{\partial z}\right), \quad \operatorname{div}_\lambda = \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} + \frac{1}{\lambda}\frac{\partial}{\partial z}.$$

The above systems of equations are cell problems, whose solutions α^i and q^i belong to the spaces

$$H^1_{per}(B) = \{\varphi \in H^1(B) : \varphi \text{ is } \xi' - \text{periodic}\} \quad \text{and } L^2_0(B)$$

respectively.

4.2 Reynolds roughness ($\lambda = 0$)

If $\lambda = 0$, then the limit pressure and velocity field depends on the two-dimensional local problems:

$$\begin{aligned} \operatorname{div}_{\xi'}\left(\frac{h^3}{12\nu}(\nabla_{\xi'} q^i + e_i)\right) &= 0 \qquad \text{in } \Xi, \quad (i=1,2) \\ \operatorname{div}_{\xi'}\left(-\frac{h^3}{12\nu}\nabla_{\xi'} q^3 + \frac{h}{2}(v^+ + v^-)\right) &= \frac{\partial h}{\partial \tau} \quad \text{in } \Xi, \end{aligned} \tag{4.9}$$

where $q^i \in H^1_{per}(\Xi)$ for a.e. (x', t, τ) and e_i $(i = 1, 2)$ is the canonical base in $\mathbb{R}^2$. In this case the coefficients a_{ij} and b_i are as follows:

$$\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{pmatrix} = \overline{-\frac{h^3}{12\nu}\begin{pmatrix} 1 + \frac{\partial q^1}{\partial \xi_1} & \frac{\partial q^2}{\partial \xi_1} & \frac{\partial q^3}{\partial \xi_1} \\ \frac{\partial q^1}{\partial \xi_2} & 1 + \frac{\partial q^2}{\partial \xi_2} & \frac{\partial q^3}{\partial \xi_2} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{h}{2}(v_1^+ + v_1^-) \\ 0 & 0 & \frac{h}{2}(v_2^+ + v_2^-) \end{pmatrix}}^{\xi'}. \tag{4.10}$$

The local functions α^i and q^i are linked by

$$\begin{aligned} \alpha^i &= \frac{(z-h^+)(z-h^-)}{2\nu}\left(\nabla_{\xi'} q^i + e_i\right), \quad (i=1,2) \\ \alpha^3 &= \frac{(z-h^+)(z-h^-)}{2\nu}\nabla_{\xi'} q^3 + \frac{z-h^-}{h}v^+ + \frac{h^+-z}{h}v^-, \end{aligned} \tag{4.11}$$

4.3 High frequency roughness regime ($\lambda = \infty$)

Consider the case when $\lambda = \infty$, i.e. the roughness wavelength is small as compared with the film thickness, $\varepsilon \gg \mu$. The limit pressure satisfies (4.3) with

$$\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{pmatrix} = \begin{pmatrix} -\frac{h_*^3}{12\nu} & 0 & \frac{h_*(v_1^+ + v_1^-)}{2} \\ 0 & -\frac{h_*^3}{12\nu} & \frac{h_*(v_2^+ + v_2^-)}{2}. \end{pmatrix}$$
$$\overline{g}^z = \int_{h_*^-}^{h_*^+} g\,dz. \tag{4.12}$$

Analogously to the previous two cases, u and p are related via (4.2), where

$$\alpha^i = \frac{(z - h_*^+)(z - h_*^-)}{2\nu} e_i, \quad (i = 1, 2)$$
$$\alpha^3 = \frac{(z - h_*^-)}{h_*} v^+ + \frac{(h_*^+ - z)}{h_*} v^-. \tag{4.13}$$

The proofs of main results will be given in Sections 7, 8 and 9 for each case. The crucial part for the analysis are apriori estimates that we derive in Section 5.

5 Estimates

We start with derivation of a priori estimates. For simplicity we denote the norm in $L^2(Q; \mathbb{R}^k)$, where Q is an arbitrary open set and $k = 1, 3, 3 \times 3$ is clear from the context, as $\|\cdot\|_Q$.

Lemma 3. *The following estimates are valid for* $\overline{u}^\mu$:

$$\|\overline{u}^\mu\|_{\Omega_T^{ext}} + \left\|\frac{\partial \overline{u}^\mu}{\partial z}\right\|_{\Omega_T^{ext}} \le K \tag{5.1}$$

$$\left\|\frac{\partial \overline{u}^\mu}{\partial x_1}\right\|_{\Omega_T^{ext}} + \left\|\frac{\partial \overline{u}^\mu}{\partial x_2}\right\|_{\Omega_T^{ext}} \le \frac{K}{\varepsilon} \tag{5.2}$$

$$\sup_{0 \le t \le T} \|\overline{u}^\mu\|_{\Omega^{ext}} \le \frac{K}{\varepsilon} \tag{5.3}$$

$$\tag{5.4}$$

Proof. Consider G defined in Lemma 1. According to the regularity assumptions on boundary and initial data and Lemma 3.1 $\operatorname{div}_\varepsilon G = 0$ in Ω^{ext} and

$$\|G|_{t=\tau}\|_{\Omega^{ext}} \le K_1, \tau \in [0, T], \quad \|\nabla_\varepsilon G\|_{\Omega^{ext}} \le \frac{K_2}{\varepsilon}.$$

We take $\varphi = u^\mu - G$ in the integral identity (3.4).

Then we have

$$\int_0^\tau \int_{\Omega_\mu(t)} (\overline{u}^\mu - G)_t (\overline{u}^\mu - G)\,dx'\,dz\,dt + \int_0^\tau \int_{\Omega_\mu(t)} G_t (\overline{u}^\mu - G)\,dx'\,dz\,dt +$$

$$+\nu \int_0^\tau \int_{\Omega_\mu(t)} \nabla_\varepsilon \overline{u}^\mu \nabla_\varepsilon (\overline{u}^\mu - G)\, dx'\, dz\, dt = 0.$$

The domain $\Omega_\mu(t)$ can be replaced by Ω^{ext} since the velocity can be extended by G into $\Omega^{ext} \setminus \Omega_\mu(t)$. This can be rewritten as

$$\frac{1}{2}\|(\overline{u}^\mu - G)|_{t=\tau}\|^2_{\Omega^{ext}} + \frac{1}{2}\|G|_{t=0}\|^2_{\Omega^{ext}} + \int_0^\tau \int_{\Omega^{ext}} G_t \overline{u}^\mu\, dx'\, dz\, dt + \nu \int_0^\tau \int_{\Omega^{ext}} |\nabla_\varepsilon \overline{u}^\mu|^2 \leq$$

$$\leq \frac{1}{2}\|(\overline{u}^\mu - G)|_{t=0}\|^2_{\Omega^{ext}} + \frac{1}{2}\|G|_{t=\tau}\|^2_{\Omega^{ext}} + \frac{\nu}{2} \int_0^\tau \int_{\Omega^{ext}} |\nabla_\varepsilon \overline{u}^\mu|^2\, dx'\, dz\, dt +$$

$$+\frac{\nu}{2} \int_0^\tau \int_{\Omega^{ext}} |\nabla_\varepsilon G|^2\, dx'\, dz\, dt.$$

Since

$$\int_0^\tau \int_{\Omega^{ext}} G_t \overline{u}^\mu\, dx'\, dz\, dt \geq -\frac{K}{2} \int_0^\tau \int_{\Omega^{ext}} G_t^2\, dx'\, dz\, dt - \frac{1}{2K} \int_0^\tau \int_{\Omega^{ext}} (\overline{u}^\mu)^2\, dx'\, dz\, dt$$

where K is the constant from inequality

$$\|\overline{u}^\mu\|^2_{\Omega^{ext}} \leq K \|\nabla \overline{u}^\mu\|^2_{\Omega^{ext}}, \qquad \mathrm{div}_\varepsilon \overline{u}^\mu = 0 \text{ in } \Omega^{ext}, \tag{5.5}$$

we can deduce that

$$\begin{aligned}
&\frac{1}{2}\|(\overline{u}^\mu - G)|_{t=\tau}\|^2_{\Omega^{ext}} + \frac{1}{2}\|G|_{t=0}\|^2_{\Omega^{ext}} + K_1 \nu \int_0^\tau \int_{\Omega^{ext}} |\nabla_\varepsilon \overline{u}^\mu|^2\, dx'\, dz\, dt \leq \\
&\frac{1}{2}\|(\overline{u}^\mu - G)|_{t=0}\|^2_{\Omega^{ext}} + \frac{1}{2}\|G|_{t=\tau}\|^2_{\Omega^{ext}} + \frac{K}{2} \int_0^\tau \int_{\Omega^{ext}} G_t^2\, dx'\, dz\, dt + \\
&\frac{\nu}{2} \int_0^\tau \int_{\Omega^{ext}} |\nabla_\varepsilon G|^2\, dx'\, dz\, dt.
\end{aligned} \tag{5.6}$$

The estimate (5.6), (3.2) and regularity of the initial data implies

$$\|\nabla_\varepsilon \overline{u}^\mu\|_{\Omega^{ext}} \leq \frac{K}{\varepsilon}. \tag{5.7}$$

Obviously,

$$\left\|\frac{\partial \overline{u}^\mu}{\partial x_i}\right\|_{\Omega^{ext}} \leq \frac{K}{\varepsilon}, \quad \left\|\frac{\partial \overline{u}^\mu}{\partial z}\right\|_{\Omega^{ext}} \leq K. \tag{5.8}$$

Due to regularity of G, we deduce that

$$\left\|\frac{\partial}{\partial z}(\overline{u}^{\mu}-G)\right\|^2_{\Omega^{ext}} \leq 2\left(\left\|\frac{\partial}{\partial z}\overline{u}^{\mu}\right\|^2_{\Omega^{ext}} + \left\|\frac{\partial}{\partial z}G\right\|^2_{\Omega^{ext}}\right) \leq K_1. \tag{5.9}$$

By using the Friedrichs inequality in z direction for $\overline{u}^{\mu}-G$, we conclude that

$$\|\overline{u}^{\mu}-G\|^2_{\Omega^{ext}} \leq K_2\left\|\frac{\partial}{\partial z}(\overline{u}^{\mu}-G)\right\|^2_{\Omega^{ext}} \leq K_3.$$

Hence

$$\|\overline{u}^{\mu}\|^2_{\Omega^{ext}} \leq 2(\|\overline{u}^{\mu}-G\|^2_{\Omega^{ext}} + \|G\|^2_{\Omega^{ext}}) \leq K_4.$$

Taking supremum over t from both sides of (5.6), we conclude that

$$\sup_{0\leq t\leq T}\|(\overline{u}^{\mu}-G)\|_{\Omega^{ext}} \leq \frac{K}{\varepsilon}, \text{ hence } \sup_{0\leq t\leq T}\|\overline{u}^{\mu}\|_{\Omega^{ext}} \leq \frac{K}{\varepsilon},$$

and we complete the proof of (5.1) and (5.2). □

Now we derive estimates for pressure.

Lemma 4. *There exist constants $K_1, K_2 > 0$ such that*

$$\left\|\varepsilon^2 p^{\mu}\right\|_{L_2^0(\Omega_\mu(t))} \leq K_1, \quad \left\|\varepsilon^2\nabla_\varepsilon p^{\mu}\right\|_{H^{-1}(\Omega_\mu(t))} \leq K_2 \tag{5.10}$$

Proof. Consider the integral identity:

$$\begin{aligned}&\int\limits_{\Omega_\mu(t)} u^{\mu}\varphi\,dx'\,dz\Big|_0^T - \int\limits_{\Omega_{\mu T}} u^{\mu}\varphi_t\,dx'\,dz\,dt + \nu\int\limits_{\Omega_{\mu T}}\left(\sum_{i=1}^{2}\frac{\partial u^{\mu}}{\partial x_i}\frac{\partial\varphi}{\partial x_i}+\right.\\ &\left.+\frac{1}{\varepsilon^2}\frac{\partial u^{\mu}}{\partial z}\frac{\partial\varphi}{\partial z}\right)dx'\,dz\,dt = \int\limits_{\Omega_{\mu T}} p^{\mu}\left(\frac{\partial\varphi_1}{\partial x_1}+\frac{\partial\varphi_2}{\partial x_2}+\frac{1}{\varepsilon}\frac{\partial\varphi_3}{\partial z}\right)dx'\,dz\,dt.\end{aligned} \tag{5.11}$$

We take first $(0,0,\varphi)$, $\varphi\in L_2(0,T;H_0^1(\Omega_\mu(t)))$ as a test-function.

$$\begin{aligned}&\int\limits_{\Omega_\mu(t)} u_3^{\mu}\varphi\,dx'\,dz\Big|_0^T - \int\limits_{\Omega_{\mu T}} u_3^{\mu}\varphi_t\,dx'\,dz\,dt + \nu\int\limits_{\Omega_{\mu T}}\left(\sum_{i=1}^{2}\frac{\partial u_3^{\mu}}{\partial x_i}\frac{\partial\varphi}{\partial x_i}+\right.\\ &\left.+\frac{1}{\varepsilon^2}\frac{\partial u_3^{\mu}}{\partial z}\frac{\partial\varphi}{\partial z}\right)dx'\,dz\,dt = \int\limits_{\Omega_{\mu T}} p^{\mu}\frac{1}{\varepsilon}\frac{\partial\varphi}{\partial z}\,dx'\,dz\,dt.\end{aligned} \tag{5.12}$$

Taking into account the estimates for u^{μ}, we obtain that

$$\left|\int\limits_{\Omega_{\mu T}} p^{\mu}\frac{\partial\varphi}{\partial z}\,dx'\,dz\,dt\right| \leq \frac{K}{\varepsilon}\|\varphi\|_{H_0^1(\Omega_\mu(t))}$$

Similarly, by choosing test-function as $(\varphi, 0, 0)$ we get that

$$\int\limits_{\Omega_\mu(t)} u_1^\mu \varphi \, dx' \, dz \Big|_0^\tau - \int\limits_{\Omega_{\mu T}} u_1^\mu \varphi_t \, dx' \, dz \, dt + \nu \int\limits_{\Omega_{\mu\tau}} \left(\sum_{i=1}^{2} \frac{\partial u_1^\mu}{\partial x_i} \frac{\partial \varphi}{\partial x_i} + \frac{1}{\varepsilon^2} \frac{\partial u_1^\mu}{\partial z} \frac{\partial \varphi}{\partial z} \right) dx' \, dz \, dt = \int\limits_{\Omega_{\mu\tau}} p^\mu \frac{\partial \varphi}{\partial x_1} \, dx' \, dz \, dt. \tag{5.13}$$

From this one can conclude that

$$\left| \int\limits_{\Omega_{\mu\tau}} p^\mu \frac{\partial \varphi}{\partial x_1} \, dx' \, dz \, dt \right| \leq \frac{K}{\varepsilon^2} \|\varphi\|_{H_0^1(\Omega_\mu(t))}.$$

In the similar way one can show that

$$\left| \int\limits_{\Omega_{\mu\tau}} p^\mu \frac{\partial \varphi}{\partial x_2} \, dx' \, dz \, dt \right| \leq \frac{K}{\varepsilon^2} \|\varphi\|_{H_0^1(\Omega_\mu(t))}.$$

Thus,

$$\|\nabla_\varepsilon p^\mu\|_{H^{-1}(\Omega_\mu(t))} \leq \frac{K}{\varepsilon^2}.$$

By applying the result

$$\|p^\mu\|_{L_2^0(\Omega_\mu(t))} \leq K_1 \, \|\nabla_\varepsilon p^\mu\|_{H^{-1}(\Omega_\mu(t))} \tag{5.14}$$

(see [33, Proposition 1.2 (ii)]) one concludes that

$$\|p^\mu\|_{L_2^0(\Omega_\mu(t))} \leq \frac{K_2}{\varepsilon^2}.$$

By integrating this inequality over $[0, T]$, we get the desired estimates. $\square$

6 Pressure extension and convergence

In order to apply the two-scale convergence result of passing to the limit one needs to have the unknown pressure defined in the fixed domain. A pressure extension method was introduced by L.Tartar in [32] in connection to homogenization of problems in porous media. His idea was widely used, see e.g. [1–3, 14] and [22]. We adopt this technique to extend the pressure through the oscillating boundary.

Lemma 5. *There exists an extension $\overline{p}^\mu \in L_2(0, T; L_2(\Omega^{ext}))$ of pressure p^μ defined by*

$$\overline{p}^\mu = p^\mu \quad in \ \Omega_\mu(t), \quad \overline{p}^\mu = \frac{1}{|\Omega_\mu(t)|} \int\limits_{\Omega_\mu(t)} p^\mu \, dx' \, dz \ \ in \ \Omega^{ext} \setminus \Omega_\mu(t). \tag{6.1}$$

Moreover,

$$\frac{1}{|\Omega^{ext}|}\int\limits_{\Omega^{ext}} \overline{p}^{\mu}\,dx'\,dz = \frac{1}{|\Omega_{\mu}(t)|}\int\limits_{\Omega_{\mu}(t)} p^{\mu}\,dx'\,dz. \tag{6.2}$$

Proof. The first step is to construct the operator

$$R_{\mu} : H_0^1(\Omega^{ext};\mathbb{R}^3) \mapsto H_0^1(\Omega_{\mu}(t);\mathbb{R}^3)$$

with the following properties:

$$\begin{aligned} &R_{\mu}\varphi = \varphi \;\text{ in } \Omega_{\mu}(t) \text{ if } \varphi = 0 \text{ in } \Omega^{ext}\setminus\Omega_{\mu}(t),\\ &\mathrm{div}_{\varepsilon}R_{\mu}\varphi = 0 \;\text{ in } \Omega_{\mu}(t) \text{ if } \mathrm{div}_{\varepsilon}\varphi = 0 \;\text{ in } \Omega^{ext},\\ &\|R_{\mu}\varphi\|_{H_0^1(\Omega_{\mu}(t))} \le K\|\varphi\|_{H_0^1(\Omega^{ext})} \end{aligned} \tag{6.3}$$

Fix $\delta > 0$ and constants $c^{\pm}$: $c^- = h^-_{max} + \delta$, $c^+ = h^+_{min} - \delta$. Introduce the surfaces $\gamma^{\pm}$ as $z = c^{\pm}$, $\gamma = \gamma^- \cup \gamma^+$. We denote by $\Omega^{\psi} \subset \Omega_{\mu}(t)$ the union of two domains (layers): one is between Σ^-_{μ} and γ^- and the other one is between γ^+ and Σ^+_{μ}. The result by Tartar (see [32, Lemma 3]) says that if $\varphi \in H^1(\Omega^{ext})$, then there exist $\psi \in H^1(\Omega^{\psi})$ and $q \in L_2(\Omega^{\psi})$ such that

$$\begin{aligned} &-\Delta_{\varepsilon}\psi = -\Delta_{\varepsilon}\varphi + \nabla_{\varepsilon}q \;\text{ in } \Omega^{\psi},\\ &\mathrm{div}_{\varepsilon}\psi = \mathrm{div}_{\varepsilon}\varphi + \frac{1}{|\Omega^{ext}\setminus\Omega_{\mu}(t)|}\int\limits_{\Omega^{ext}\setminus\Omega_{\mu}(t)} \mathrm{div}_{\varepsilon}\varphi\,dx'\,dz \;\text{ in } \Omega^{\psi},\\ &\psi|_{\gamma^{\pm}} = \varphi|_{\gamma^{\pm}},\; \psi|_{\Sigma^{\pm}} = 0,\; \psi|_{\Sigma^{w}} = \varphi|_{\Sigma^{w}}. \end{aligned} \tag{6.4}$$

Moreover, there exists a constant K such that

$$\|\psi\|_{H^1(\Omega_{\mu}(t))} \le \|\varphi\|_{H^1(\Omega^{ext})}.$$

Notice that $\Omega^{ext} = (\Omega_{\mu}(t)\setminus\Omega^{\psi}) \cup \Omega^{\psi} \cup (\Omega^{ext}\setminus\Omega_{\mu}(t))$. Define operator R_{μ} by

$$R_{\mu} = \begin{cases} \varphi \;\text{ in } \Omega_{\mu}(t)\setminus\Omega^{\psi},\\ \psi \;\text{ in } \Omega^{\psi},\\ 0 \;\text{ in } \Omega^{ext}\setminus\Omega_{\mu}(t). \end{cases} \tag{6.5}$$

It is easy to see that R_{μ} satisfies the properties (6.3). The next step is to define the pressure extension $\overline{p}^{\mu}$ a.e. in time by

$$(\nabla_{\varepsilon}\overline{p}^{\mu},\varphi)_{H^{-1},H_0^1(\Omega^{ext})} = (\nabla_{\varepsilon}p^{\mu}, R_{\mu}\varphi)_{H^{-1},H_0^1(\Omega_{\mu}(t))} \text{ for any } \varphi \in H_0^1(\Omega^{ext};\mathbb{R}^3). \tag{6.6}$$

This definition makes sense due to the properties of operator R_{μ}. Moreover, the equivalent definition (6.1) can be obtained from (6.6) by using an appropriate test-function φ. The equation (6.2) follows directly from the fact that

$$\overline{p}^{\mu} = \frac{1}{|\Omega_{\mu}(t)|}\int\limits_{\Omega_{\mu}(t)} p^{\mu}\,dx'\,dz \text{ in } \Omega^{ext}\setminus\Omega_{\mu}(t).$$

□

Remark 1. Since it is assumed from the beginning that $p^\mu \in L_2^0(\Omega_\mu(t))$ with respect to the space variables, the result of Lemma 5 holds with $\overline{p}^\mu = 0$ in $\Omega^{ext} \setminus \Omega_\mu(t)$.

Lemma 6. *There exists $p \in L_2((0,T)\times\mathcal{T}; H^1(\omega))$ such that*

$$\varepsilon^2 \overline{p}^\mu \twoheadrightarrow p \ \text{in } \Omega^{ext} \tag{6.7}$$

and

$$\int\limits_\omega p\, dx' = 0. \tag{6.8}$$

Moreover,

$$\varepsilon^2 \overline{p}^\mu \to p \ \text{in } L_2((0,T)\times\mathcal{T};\Omega^{ext}) \ \text{as } \mu \to 0.$$

Proof. The estimate

$$\|\varepsilon^2 \overline{p}^\mu\|_{L_2(\Omega_T^{ext})} = \|\varepsilon^2 p^\mu\|_{L_2(\Omega_{\mu T})} \le K$$

follows by the definition of $\overline{p}^\mu$ and Lemma 4. By two-scale convergence result there exists $p \in L_2(\Omega_T^{ext} \times \Xi \times [0,\Xi_3] \times \mathcal{T})$ periodic in $\xi = (\xi', \tau)$ such that $\varepsilon^2 \overline{p}^\mu \twoheadrightarrow p$ up to a subsequence of parameter μ. Now we show independence of p from ξ and z variables. Let us show first the independence of p from ξ_3. For this we take $\varphi \in C^1(0,T;\mathcal{D}(\Omega, C_{per}^\infty(\Xi\times[0,\Xi_3]\times\mathcal{T})))$, $\varphi = 0$ in $\Omega^{ext}\setminus\Omega_\mu(t)$ and choose $\left(0,0,\varphi^\mu = \varphi\left(x',z,t,\frac{x'}{\mu},\frac{z}{\mu},\frac{t}{\mu}\right)\right)$ as a test function in the integral identity. Thus, we have:

$$I^\mu = \int\limits_{\Omega^{ext}} \overline{u}_3^\mu \varphi^\mu\, dx'\, dz \Bigg|_0^T - \int\limits_{\Omega_T^{ext}} \overline{u}_3^\mu \varphi_t^\mu\, dx'\, dz\, dt +$$

$$+\nu \int\limits_{\Omega_T^{ext}} \left[\sum_{i=1}^{2} \frac{\partial \overline{u}_3^\mu}{\partial x_i}\frac{\partial \varphi^\mu}{\partial x_i} + \frac{1}{\varepsilon^2}\frac{\partial \overline{u}_3^\mu}{\partial z}\frac{\partial \varphi^\mu}{\partial z}\right] dx'\, dz\, dt = \int\limits_{\Omega_T^{ext}} \overline{p}^\mu \left[\frac{1}{\varepsilon}\frac{\partial \varphi}{\partial z} + \frac{1}{\varepsilon\mu}\frac{\partial\varphi}{\partial \xi_3}\right] dx'\, dz\, dt.$$

Multipling this equation by $\varepsilon^3\mu$ and using Lemma 3 and Lemma 6, we get that

$$\left| \int\limits_{\Omega_T^{ext}} \varepsilon^2 \overline{p}^\mu \left[\mu \frac{\partial\varphi}{\partial z} + \frac{\partial \varphi}{\partial \xi_3}\right] dx'\, dt \right| = |\varepsilon^3 \mu I^\mu| \le K\varepsilon.$$

Passing to the two-scale limit as $\mu \to 0$ ($\Rightarrow \varepsilon \to 0$), we have that

$$\int\limits_{\Omega_T^{ext}} \int\limits_{\Xi\times[0,\Xi_3]\times\mathcal{T}} p \frac{\partial \varphi}{\partial \xi_3}\, dx'\, dt\, d\xi'\, d\xi_3\, d\tau = 0.$$

This proves that p does not depend on ξ_3. In order to show independence from z variable, we take $\varphi \in C^1(0,T;\mathcal{D}(\Omega, C^\infty_{per}(\Xi \times \mathcal{T})))$ and choose $(0,0,\varphi^\mu = \varphi\Big(x', z, t, \frac{x'}{\mu}, \frac{t}{\mu}\Big))$, $\varphi^\mu = 0$ in $\Omega^{ext} \setminus \Omega_\mu(t)$ as a test function:

$$\int\limits_{\Omega^{ext}} \overline{u}_3^\mu \varphi^\mu \, dx' \, dz \Big|_0^T - \int\limits_{\Omega_T^{ext}} \overline{u}_3^\mu \varphi_t^\mu \, dx' \, dz \, dt +$$
$$+\nu \int\limits_{\Omega_T^{ext}} \left[\sum_{i=1}^2 \frac{\partial \overline{u}_3^\mu}{\partial x_i} \frac{\partial \varphi^\mu}{\partial x_i} + \frac{1}{\varepsilon^2} \frac{\partial \overline{u}_3^\mu}{\partial z} \frac{\partial \varphi^\mu}{\partial z} \right] dx' \, dz \, dt = \int\limits_{\Omega_T^{ext}} \overline{p}^\mu \frac{1}{\varepsilon} \frac{\partial \varphi}{\partial z} \, dx' \, dz \, dt.$$

Multiply this equation by ε^3 and use the estimates from Lemma 3:

$$\left| \int\limits_{\Omega_T^{ext}} \varepsilon^2 \overline{p}^\mu \left[\mu \frac{\partial \varphi}{\partial z} \right] dx' \, dt \right| \leq K\varepsilon.$$

Passing to the two-scale limit, we derive

$$\int\limits_{\Omega_T^{ext}} \int\limits_{\Xi \times \mathcal{T}} p \frac{\partial \varphi}{\partial z} \, d\xi' \, d\tau \, dx' \, dt = 0 \quad \text{for any } \varphi \in L_2(0,T;\mathcal{D}(\Omega, C^\infty_{per}(\Xi \times \mathcal{T}))), \tag{6.9}$$

what shows independence from z. Now our goal is to prove the independence of limit pressure p from slow variables ξ'. For that we take φ^μ exactly as in previous step and choose $\Big(\varphi^\mu, 0, 0\Big)$ as a test function:

$$I^\mu = \int\limits_{\Omega^{ext}} \overline{u}_1^\mu \varphi^\mu \, dx' \, dz \Big|_0^T - \int\limits_{\Omega_T^{ext}} \overline{u}_1^\mu \varphi_t^\mu \, dx' \, dz \, dt +$$
$$+\nu \int\limits_{\Omega_T^{ext}} \left[\sum_{i=1}^2 \frac{\partial \overline{u}_1^\mu}{\partial x_i} \frac{\partial \varphi^\mu}{\partial x_i} + \frac{1}{\varepsilon^2} \frac{\partial \overline{u}_1^\mu}{\partial z} \frac{\partial \varphi^\mu}{\partial z} \right] dx' \, dz \, dt = \int\limits_{\Omega_T^{ext}} \overline{p}^\mu \left[\frac{\partial \varphi}{\partial x_1} + \frac{1}{\mu} \frac{\partial \varphi}{\partial \xi_1} \right] dx' \, dz \, dt.$$

Now we multiply the obtained relation with $\mu\varepsilon^2$, pass to the two-scale limit and derive that

$$\int\limits_{\Omega_T^{ext}} \int\limits_{\Xi \times \mathcal{T}} p \frac{\partial \varphi}{\partial \xi_1} \, d\xi' \, d\tau \, dx' \, dt = 0.$$

This proves independence p from ξ_1 variable. In a similar way one can show also independence from ξ_2. We omit the details. Let us show now the strong convergence $\varepsilon^2 \overline{p}^\mu \to p$. The estimates (5.10) imply the existence of p^* such that for a.e. $t \in (0,T)$

$$\varepsilon^2 \overline{p}^\mu \rightharpoonup p^* \ \text{ weakly in } L_2(\Omega^{ext}), \ \varepsilon^2 \nabla_\varepsilon \overline{p}^\mu \rightharpoonup \nabla p^* \ \text{ weakly in } H^{-1}(\Omega^{ext}).$$

Let $\varphi^\mu \rightharpoonup \varphi^*$ weakly in $H_0^1(\Omega^{ext})$.
Then, due to the estimate $|(\varepsilon^2\nabla_\varepsilon \overline{p}^\mu, \varphi^\mu)| \le K\|\varphi^\mu\|_{H_0^1(\Omega^{ext})}$, we deduce that

$$\begin{aligned}&|(\varepsilon^2\nabla_\varepsilon \overline{p}^\mu, \varphi^\mu) - (\nabla p^*, \varphi^*)| \le |(\varepsilon^2\nabla_\varepsilon \overline{p}^\mu, \varphi^\mu - \varphi^*)| + |(\varepsilon^2\nabla_\varepsilon \overline{p}^\mu - \nabla p^*, \varphi^*)| \le \\ &\le K\|\varphi^\mu - \varphi^*\|_{H_0^1(\Omega^{ext})} + |(\varepsilon^2\nabla_\varepsilon \overline{p}^\mu - \nabla p^*, \varphi^*)| \to 0 \text{ as } \mu \to 0.\end{aligned} \tag{6.10}$$

Hence, for a.e. $t \in (0,T)$ $\varepsilon^2\nabla_\varepsilon \overline{p}^\mu \to \nabla p^*$ strongly in $H^{-1}(\Omega^{ext})$ what implies that $\varepsilon^2\overline{p}^\mu \to p^*$ strongly in $L_2^0(\Omega^{ext})$ by (5.14). Since the strong convergence implies two-scale convergence, $p^* = p$. Finally, multiplying the equation

$$\int\limits_{\Omega^{ext}} \overline{p}^\mu \, dx' \, dz = 0$$

by ε^2 and passing to the limit, we obtain $\int\limits_\omega p \, dx' = 0$ that proves (6.8). $\square$

Lemma 7. *Let $\overline{p}^\mu$ be an extension of p^μ in $z-$direction through the oscillating boundary. Then the two-scale limit p of $\varepsilon^2\overline{p}^\mu$ does not depend on the way of extension into the domain Ω^{ext}.*

Proof. Let $\overline{p}_1^\mu$ and $\overline{p}_2^\mu$ be two different extensions in $z-$direction such that $\overline{p}_i^\mu = p^\mu$ in $\Omega_\mu(t)$, $i = 1, 2$. According to Lemma 6 there exist p_i independent of z such that $\varepsilon^2\overline{p}_i^\mu \twoheadrightarrow p_i$ in Ω^{ext}. Consequently,

$$\varepsilon^2(\overline{p}_1^\mu - \overline{p}_2^\mu) \twoheadrightarrow (p_1 - p_2) \text{ in } \Omega^* \subset \Omega_\mu(t)$$

and obviously $p_1 - p_2 = 0$ in Ω^*. The function $p_1 - p_2$ does not depend on z in Ω^{ext}, therefore $p_1 \equiv p_2$ in Ω^{ext}. Thus, we have shown the uniqueness of the two-scale limit. $\square$

7 Proof of main results for Stokes roughness case

7.1 Proof of Theorem 3

Now we prove theorem on two-scale convergence of the velocity field. This result is an essential part for pressure convergence.

Proof. Lemma 3 and two-scale convergence result of Proposition 1.14 in [3] imply that there exist $u^\lambda \in L_2(\Omega^{ext}; H^1_{per}(\Xi \times [0, \Xi_3] \times \mathcal{T}))$ and
$v^\lambda \in L_2(\Omega^{ext} \times \Xi \times [0, \Xi_3] \times \mathcal{T}; H^1_{per}([0, \Xi_3]))$ satisfying (4.1). Let us prove that u^λ does not depend on ξ_3. Consider $\varphi \in L_2(0, T; \mathcal{D}(\Omega^{ext} \times \Xi \times [0, \Xi_3]))$ such that $\varphi = 0$ for $z \in [h^-_{min}, h^-) \cup (h^+, h^+_{max}]$ and choose $\varphi^\mu = \varphi(x', z, t, \frac{x'}{\mu}, \frac{t}{\mu}, \frac{z}{\mu})$. Then we derive

$$\begin{aligned}&\int\limits_{\Omega_{\mu T}} \frac{\partial u^\mu}{\partial z}\varphi^\mu \, dx' \, dz \, dt = - \int\limits_{\Omega_{\mu T}} u^\mu \left(\frac{\partial \varphi}{\partial z} + \frac{1}{\mu}\frac{\partial \varphi}{\partial \xi_3}\right) dx' \, dz \, dt = \\ &- \int\limits_{\Omega_T^{ext}} \overline{u}^\mu \left(\frac{\partial \varphi}{\partial z} + \frac{1}{\mu}\frac{\partial \varphi}{\partial \xi_3}\right) dx' \, dz \, dt.\end{aligned}$$

Integrating this equation over $[0,T]$, multiplying after that the obtained equation by μ and passing to the two-scale limit in Ω^{ext}, we obtain that

$$\int\limits_{\Omega_T^{ext}} \int\limits_{\Xi\times\mathcal{T}} u^\lambda \frac{\partial\varphi}{\partial\xi_3}\, d\xi'\, d\tau\, dx'\, dz\, dz = 0.$$

This implies that $\frac{\partial u^\lambda}{\partial\xi_3} = 0$ as $h^- \le z \le h^+$, therefore u^λ does not depend on ξ_3. Consider $\varphi^\mu = \varphi(x', z, t, \frac{x'}{\mu}, \frac{t}{\mu})$, $\varphi \in L_2(0,T;\mathcal{D}(\Omega^{ext}; C^\infty_{per}(\Xi\times\mathcal{T})))$ and $\varphi = 0$ as $z \in [h^-_{min}, h^-) \cup (h^+, h^+_{max}]$. The identity

$$\int\limits_{\Omega^{ext}} \mathrm{div}_\varepsilon \overline{u}^\mu \varphi^\mu\, dx'\, dz = 0,$$

implies

$$\int\limits_{\Omega^{ext}} \left[\sum_{i=1}^{2} \overline{u}_i^\mu \left(\frac{\partial\varphi}{\partial x_i} + \frac{1}{\mu}\frac{\partial\varphi}{\partial\xi_i}\right) + \frac{1}{\varepsilon}\overline{u}_3^\mu \frac{\partial\varphi}{\partial z}\right] dx'\, dz = 0.$$

Multiply the obtained equation by μ and pass to the two-scale limit:

$$0 = \lim_{\mu\to 0} \int\limits_{\Omega_T^{ext}\times\Xi\mathcal{T}} \left[\sum_{i=1}^{2} \overline{u}_i^\mu \frac{\partial\varphi}{\partial\xi_i} + \frac{\mu}{\varepsilon}\overline{u}_3^\mu \frac{\partial\varphi}{\partial x_3}\right] dx'\, dt\, dz\, d\xi'\, d\tau =$$

$$= \frac{1}{\Xi} \int\limits_{\Xi} \int\limits_{\Omega_T^{ext}\times\Xi\mathcal{T}} \mathrm{div}_\lambda u^\lambda \varphi\, dx'\, dz\, dt\, d\xi'\, d\tau\, d\xi'.$$

By choosing φ as $\varphi(x', z, t, \xi', \tau) = \theta(x', t, \tau)\psi(z, \xi')$ with $\theta \in L_2(0,T;\mathcal{D}(\omega))$, $\psi \in \mathcal{D}(\mathsf{B})$, $\psi = 0$ as $z \in [h^-_{min}, h^-) \cup (h^+, h^+_{max}]$, we deduce the result. Now let us investigate the boundary conditions for the limit velocity filed. By means of Green formula we derive

$$\int\limits_{\Omega_\mu(t)} \frac{\partial u^\mu}{\partial z}\varphi(x', z, t, \frac{x'}{\mu}, \frac{t}{\mu})\, dx'\, dz = -\int\limits_{\Omega_\mu(t)} u^\mu \frac{\partial\varphi}{\partial z}\, dx'\, dz +$$

$$+ \int\limits_{\omega} \left(v_1^+, v_2^+, \varepsilon\left(\frac{\partial h_\mu^+}{\partial t} + v^+\cdot\nabla h_\mu^+\right)\right)\varphi(x', h_\mu^+, t, \frac{x'}{\mu}, \frac{t}{\mu}) -$$

$$- \left(v_1^-, v_2^-, \varepsilon\left(\frac{\partial h_\mu^-}{\partial t} + v^-\cdot\nabla h_\mu^-\right)\right)\varphi(x', h_\mu^-, t, \frac{x'}{\mu}, \frac{t}{\mu})\, dx'$$

for any $\varphi \in L_2(0,T;C^\infty(\Omega^{ext}, C^\infty_{per}(\Xi\times\mathcal{T})))$.
Choose $\varphi = 0$ as $z \in [h^-_{min}, h^-) \cup (h^+, h^+_{max}]$. Extend the function u^μ to Ω^{ext},

integrate the obtained equation over $[0,T]$ and then pass to the two-scale limit as $\mu \to 0$ ($\Rightarrow \varepsilon \to 0$). Since

$$\varepsilon\left(\frac{\partial h_\mu^\pm}{\partial t} + v^\pm \cdot \nabla h_\mu^\pm\right) = \varepsilon\left(\frac{\partial h^\pm}{\partial t} + v^\pm \cdot \nabla_x h^\pm\right) + \varepsilon\left(\frac{1}{\mu}\frac{\partial h^\pm}{\partial \tau} + v^\pm \cdot \frac{1}{\mu}\nabla_\xi h^\pm\right) \twoheadrightarrow$$

$$\twoheadrightarrow \lambda\left(\frac{\partial h^\pm}{\partial \tau} + v^\pm \cdot \nabla_\xi h^\pm\right) \text{ as } \mu \to 0,$$

it holds

$$\int\limits_{\Omega_T^{ext}}\int\limits_{\Xi\times\mathcal{T}} \frac{\partial u^\lambda}{\partial z}\varphi(x',z,t,\xi',\tau)\,dx'\,dz\,dt\,d\xi'\,d\tau = -\int\limits_{\Omega_T^{ext}}\int\limits_{\Xi\times\mathcal{T}} u^\lambda\frac{\partial\varphi}{\partial z}\,dx'\,dz\,dt\,d\xi'\,d\tau +$$
$$+\int\limits_{\omega_T}\int\limits_{\Xi\times\mathcal{T}}\left(v_1^+, v_2^+, \lambda\left(\frac{\partial h^+}{\partial\tau} + v^+\cdot\nabla_\xi h^+\right)\right)\varphi(x',h^+,t,\xi',\tau) -$$
$$-\left(v_1^-, v_2^-, \lambda\left(\frac{\partial h^-}{\partial\tau} + v^-\cdot\nabla_\xi h^-\right)\right)\varphi(x',h^-,t,\xi',\tau)\,dx'\,dt\,d\xi'\,d\tau.$$

By applying again the Green formula, we conclude that

$$-\int\limits_{\Omega_T^{ext}}\int\limits_{\Xi\times\mathcal{T}} u^\lambda\frac{\partial\varphi}{\partial z}\,dx'\,dz\,dt\,d\xi'\,d\tau + \int\limits_{\omega_T}\int\limits_{\Xi\times\mathcal{T}} u^\lambda\varphi|_{z=h^-}^{z=h^+}\,dx'\,dt\,d\xi'\,d\tau =$$
$$= -\int\limits_{\Omega_T^{ext}}\int\limits_{\Xi\times\mathcal{T}} u^\lambda\frac{\partial\varphi}{\partial z}\,dx'\,dz\,dt\,d\xi'\,d\tau +$$
$$+\int\limits_{\omega_T}\int\limits_{\Xi\times\mathcal{T}}\left(\left(v_1^+, v_2^+, \lambda\left(\frac{\partial h^+}{\partial\tau} + v^+\cdot\nabla_\xi h^+\right)\right)\varphi(x',h^+,t,\xi',\tau) -\right.$$
$$\left.-\left(v_1^-, v_2^-, \lambda\left(\frac{\partial h^-}{\partial\tau} + v^-\cdot\nabla_\xi h^-\right)\right)\varphi(x',h^-,t,\xi',\tau)\right)dx'\,dt\,d\xi'\,d\tau.$$

Hence,

$$u^\lambda = \left(v_1^\pm, v_2^\pm, \lambda\left(\frac{\partial h^\pm}{\partial\tau} + v^\pm\cdot\nabla_\xi h^\pm\right)\right) \qquad \text{as } z = h^\pm$$

for almost every $(x',t,\xi',\tau) \in \omega_T \times \Xi \times \mathcal{T}$.

□

7.2 Definition of weak solution to local problems

Define the space

$$C_{0,per}^\infty(\mathbf{B}) = \{\varphi \in C^\infty(\mathbf{B}) : \varphi \text{ is } \xi',\tau - \text{ periodic, } \varphi(\xi',\tau,h^-) = \varphi(\xi',\tau,h^+) = 0\},$$

$$H^1_{per}(\mathbf{B}) = \{\varphi \in H^1(\mathbf{B}) : \varphi \text{ is } \xi', \tau - \text{ periodic}\}$$
$$H^1_{0,per}(\mathbf{B}) = \{\varphi \in H^1_{per}(\mathbf{B}) : \varphi(\xi', \tau, h^-) = \varphi(\xi', \tau, h^+) = 0\}.$$

The weak solution to local problems (4.7) satisfies the following identities in the periodicity cell $\mathbf{B}$:

$$\nu \int\limits_{\mathbf{B}} \nabla_\lambda \alpha^i \nabla_\lambda \varphi \, d\xi' \, dz = \int\limits_{\mathbf{B}} q^i \operatorname{div}_\lambda \varphi \, d\xi' \, dz - \int\limits_{\mathbf{B}} \varphi_k \, d\xi' \, dz, \ \operatorname{div}_\lambda \alpha^i = 0, \tag{7.1}$$

where $(\alpha^i, q^i) \in (H^1_{0,per}(\mathbf{B}))^3 \times L^2_0(\mathbf{B}),\ i = 1, 2.$

$$\begin{aligned} &\nu \int\limits_{\mathbf{B}} \nabla_\lambda \alpha^3 \nabla_\lambda \varphi \, d\xi' \, dz = \int\limits_{\mathbf{B}} q^3 \operatorname{div}_\lambda \varphi \, d\xi' \, dz, \ \operatorname{div}_\lambda \alpha^3 = 0, \\ &\alpha^3 = \left(v_1^\pm, v_2^\pm, \lambda \left(\frac{\partial h^\pm}{\partial \tau} + v^\pm \cdot \nabla_{\xi'} h^\pm \right) \right) \text{ as } z = h^\pm. \end{aligned} \tag{7.2}$$

7.3 The relation between velocity and pressure

In this section we prove formula (4.2) of Theorem 5 on the relation between the limit velocity and pressure.

Proof. Let us prove first the equation

$$\begin{aligned} &\nu \int\limits_{\mathcal{T}} \int\limits_{\omega_T} \int\limits_{\mathsf{B}} \left(\sum_{i=1}^{2} \frac{\partial u^\lambda}{\partial \xi_i} \frac{\partial \varphi}{\partial \xi_i} + \frac{1}{\lambda^2} \frac{\partial u^\lambda}{\partial z} \frac{\partial \varphi}{\partial z} \right) dx' \, dt \, d\xi' \, d\tau \, dz = \\ &= \nu \int\limits_{\mathcal{T}} \int\limits_{\omega_T} \int\limits_{\mathsf{B}} p^\lambda(x', t) \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} \right) dx' \, dt \, d\xi' \, d\tau \, dz \end{aligned} \tag{7.3}$$

for any $\varphi \in C^1(0, T; \mathcal{D}(\Omega^{ext}; C^\infty_{per}(\Xi \times \mathcal{T})))$ with $\varphi = 0$ for $z \in [h^-_{min}, h^-) \cup (h^+, h^+_{max}]$ and $\operatorname{div}_\lambda \varphi = 0$. Take $\varphi^\mu(x', z, t) = \varphi(x', z, t, \frac{x'}{\mu}, \frac{t}{\mu})$ as a test function in the original equation (3.4). Since $\varphi \equiv 0$ in $\Omega^{ext} \setminus \Omega_{\mu T}$, for the extended function $\overline{u}^\mu$ we have:

$$\begin{aligned} &\int\limits_{\Omega^{ext}} \overline{u}^\mu \varphi \, dy \Big|_0^T dx' \, dz - \int\limits_{\Omega^{ext}_T} \overline{u}^\mu \varphi_t \, dx' \, dz \, dt + \\ &+ \nu \int\limits_{\Omega^{ext}_T} \left(\sum_{i=1}^{2} \frac{\partial \overline{u}^\mu}{\partial x_i} \left(\frac{\partial \varphi}{\partial x_i} + \frac{1}{\mu} \frac{\partial \varphi}{\partial \xi_i} \right) + \frac{1}{\lambda^2 \mu^2} \frac{\partial \overline{u}^\mu}{\partial z} \frac{\partial \varphi}{\partial z} \right) dx' \, dz \, dt = \\ &= \int\limits_{\Omega^{ext}_T} \overline{p}^\mu(x', t) \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{1}{\mu} \frac{\partial \varphi_1}{\partial \xi_1} + \frac{\partial \varphi_2}{\partial x_2} + \frac{1}{\mu} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{1}{\lambda \mu} \frac{\partial \varphi_3}{\partial z} \right) dx' \, dz \, dt. \end{aligned} \tag{7.4}$$

Since $\operatorname{div}_\lambda \varphi = 0$, we obtain

$$\int\limits_{\Omega^{ext}} \overline{u}^{\mu}\varphi\, dy\Big|_0^T dx'\, dz - \int\limits_{\Omega^{ext}_T} \overline{u}^{\mu}\varphi_t\, dx'\, dz\, dt +$$
$$+ \int\limits_{\Omega^{ext}_T} \Big(\nu \sum_{i=1}^{2} \frac{\partial \overline{u}^{\mu}}{\partial x_i}\Big(\frac{\partial\varphi}{\partial x_i} + \frac{1}{\mu}\frac{\partial\varphi}{\partial \xi_i}\Big) + \frac{1}{\lambda^2\mu^2}\frac{\partial \overline{u}^{\mu}}{\partial z}\frac{\partial\varphi}{\partial z}\Big)\, dx'\, dz\, dt =$$
$$= \int\limits_{\Omega^{ext}_T} \overline{p}^{\mu}(x',t)\Big(\frac{\partial\varphi_1}{\partial x_1} + \frac{\partial\varphi_2}{\partial x_2}\Big)\, dx'\, dz\, dt.$$

Multiply the last equation by μ^2 and pass to the two-scale limit. Taking into account the results of Theorem 3 and Lemma 6, we deduce

$$\begin{aligned} &\nu \int\limits_{\Omega^{ext}_T}\int\limits_{\Xi\times\mathcal{T}} \Big(\sum_{i=1}^{2}\frac{\partial u^{\lambda}}{\partial \xi_i}\frac{\partial\varphi}{\partial \xi_i} + \frac{1}{\lambda^2}\Big(\frac{\partial u^{\lambda}}{\partial z} + \frac{\partial v^{\lambda}}{\partial \xi_3}\Big)\frac{\partial\varphi}{\partial z}\Big)\, d\xi'\, d\tau\, dz\, dx'\, dt = \\ &= \int\limits_{\Omega^{ext}_T}\int\limits_{\Xi\times\mathcal{T}} p^{\lambda}(x',t)\Big(\frac{\partial\varphi_1}{\partial x_1} + \frac{\partial\varphi_2}{\partial x_2}\Big)\, d\xi'\, d\tau\, dz\, dx'\, dt. \end{aligned} \tag{7.5}$$

Due to the periodicity of v^{λ}, the term with $\frac{\partial v^{\lambda}}{\partial \xi_3}$ vanishes. Finally, we use the fact that $\varphi = 0$ as $z \in [h^-_{min}, h^-) \cup (h^+, h^+_{max}]$ to deduce (7.3). Let us prove now that $p^{\lambda} \in L_2((0,T)\times\mathcal{T}; H^1(\omega))$ and u^{λ} satisfies a Stokes equation in $\mathbf{B}$. Consider $\psi(\xi_2, z) \in C^{\infty}_{0,per}([0,\Xi_2]\times[h^-,h^+])$. Suppose that $\widetilde{\psi} = \int\limits_{\mathbf{B}} \psi\, dz\, d\xi' \neq 0$.

Take $\varphi = \Big(\frac{\theta\psi}{\widetilde{\psi}}, 0, 0\Big)$, where $\theta \in \mathcal{D}(\omega)$. The function φ can be chosen as a test in (7.3) since $\mathrm{div}_{\lambda}\phi = 0$. We derive

$$\nu \int\limits_{\omega_T\times\mathcal{T}}\int\limits_{\mathbf{B}} \nabla_{\lambda} u^{\lambda}\nabla_{\lambda}\varphi\, d\xi'\, d\tau\, dz\, dx'\, dt = \int\limits_{\omega_T\times\mathcal{T}}\int\limits_{\mathbf{B}} p^{\lambda}(x',t)\frac{\partial}{\partial x_1}\Big(\frac{\theta\psi}{\widetilde{\psi}}\Big)\, d\xi'\, d\tau\, dz\, dx'\, dt.$$

The functions θ and p^{λ} do not depend on ξ', z, therefore it can be rewritten as

$$\frac{\nu}{\widetilde{\psi}} \int\limits_{\omega_T\times\mathcal{T}} \theta(x')\Big(\int\limits_{\mathbf{B}} \nabla_{\lambda} u^{\lambda}_1 \nabla_{\lambda}\psi\, d\xi'\, d\tau\, dz\Big)\, dx'\, dt =$$
$$= \int\limits_{\omega_T\times\mathcal{T}} p^{\lambda}\frac{\partial\theta}{\partial x_1}\Big(\int\limits_{\mathbf{B}} \frac{\psi}{\widetilde{\psi}}\, d\xi'\, d\tau\, dz\Big)\, dx'\, dt = \int\limits_{\omega_T\times\mathcal{T}} p^{\lambda}\frac{\partial\theta}{\partial x_1}\, dx'\, dt.$$

The function $\int\limits_{\mathbf{B}} \nabla_{\lambda} u^{\lambda}_1 \nabla_{\lambda}\overline{\psi}\, d\xi'\, d\tau\, dz$ belongs to $L_2((0,T)\times\mathcal{T}; L_2(\omega))$ since $\nabla_{\lambda} u^{\lambda}_1 \in L_2((0,T)\times\mathcal{T}; L_2(Q))$, hence $\frac{\partial p^{\lambda}}{\partial x_1} \in L_2((0,T)\times\mathcal{T}; L_2(\omega))$. Analogously, by choosing $\psi(\xi_1, x_3)$ one can show that $\frac{\partial p^{\lambda}}{\partial x_2} \in L_2((0,T)\times\mathcal{T}; L_2(\omega_T))$. This proves that

$p^\lambda \in L_2(0,T;H^1(\omega))$.

Consider now $\psi = \theta(x')\varphi(\xi', z)$, where $\theta \in \mathcal{D}(\omega)$, $\varphi \in C^\infty_{0,per}$ and $\mathrm{div}_\lambda \varphi = 0$. Taking such ψ as a test function in (7.3), we obtain

$$\nu \int\limits_{\omega_T\times\mathcal{T}} \theta(x')\Big(\int\limits_{\mathbf{B}} \nabla_\lambda u^\lambda \nabla_\lambda \varphi \, d\xi' \, d\tau \, dz\Big)\, dx' \, dt =$$
$$= \int\limits_{\omega_T\times\mathcal{T}} \int\limits_{\mathbf{B}} p^\lambda \Big(\frac{\partial}{\partial x_1}(\theta\varphi)_1 + \frac{\partial}{\partial x_2}(\theta\varphi)_2\Big)\, d\xi' \, d\tau \, dz \, dx' \, dt =$$
$$= -\int\limits_{\omega_T\times\mathcal{T}} \Big(\int\limits_{\mathbf{B}} \Big(\frac{\partial p^\lambda}{\partial x_1}\varphi_1 + \frac{\partial p^\lambda}{\partial x_2}\varphi_2\Big)\, d\xi' \, d\tau \, dz\Big)\theta \, dx' \, dt.$$

By the density, we get the following Stokes problem:

$$\begin{gathered}\nu \int\limits_{\mathbf{B}} \nabla_\lambda u^\lambda \nabla_\lambda \varphi \, d\xi' \, dz = -\int\limits_{\mathsf{B}} \Big(\frac{\partial p^\lambda}{\partial x_1}\varphi_1 + \frac{\partial p^\lambda}{\partial x_2}\varphi_2\Big)\, d\xi' \, dz \quad \text{for almost every} \\ (x', t, \tau) \in \omega \times (0,T) \times \mathcal{T}, \quad \varphi \in H^1_{0,per}(\mathsf{B}), \quad \mathrm{div}_\lambda \varphi = 0,\end{gathered} \tag{7.6}$$

where

$$\mathrm{div}_\lambda u^\lambda = 0 \quad \text{for almost every } (x', t\tau) \in \omega \times (0,T) \times \mathcal{T}.$$

Using the Green formula and the fact that p^λ does not depend on ξ', z it is possible to rewrite the last equation as follows:

$$-\nu \int\limits_{\mathbf{B}} u^\lambda \triangle_\lambda \varphi \, d\xi' \, dz + \nu \int\limits_{\partial\mathbf{B}} u^\lambda \nabla_\lambda \varphi \, dS(\xi') = -\sum_{i=1}^{2} \frac{\partial p^\lambda}{\partial x_i} \int\limits_{\mathbf{B}} \varphi_i \, d\xi' \, dz. \tag{7.7}$$

According to the integral identities (7.1) and (7.2), we can substitute the integral $-\int\limits_{\mathsf{B}} \varphi_i \, d\xi' \, dz$ with

$$\nu \int\limits_{\mathbf{B}} \nabla_\lambda \alpha^i \nabla_\lambda \varphi \, d\xi' \, dz - \int\limits_{\mathbf{B}} q^i \mathrm{div}_\lambda \varphi \, d\xi' \, dz = \nu \int\limits_{\mathbf{B}} \nabla_\lambda \alpha^i \nabla_\lambda \varphi \, d\xi' \, dz$$

since $\mathrm{div}_\lambda \varphi = 0$. In addition, we add zero term

$$0 = \nu \int\limits_{\mathbf{B}} \nabla_\lambda \alpha^3 \nabla_\lambda \varphi \, d\xi' \, dz - \int\limits_{\mathbf{B}} q^3 \mathrm{div}_\lambda \varphi \, d\xi' \, dz = \nu \int\limits_{\mathbf{B}} \nabla_\lambda \alpha^3 \nabla_\lambda \varphi \, d\xi' \, dz$$

to the right-hand side of (7.7). Thus, we derive

$$-\nu\int\limits_{\mathbf{B}} u^\lambda \triangle_\lambda \varphi\, d\xi'\, dz + \nu \int\limits_{\partial\mathbf{B}} u^\lambda \nabla_\lambda\varphi\, dS(\xi') = \nu\sum_{i=1}^{2}\frac{\partial p^\lambda}{\partial x_i}\int\limits_{\mathbf{B}} \nabla_\lambda\alpha^i\nabla_\lambda\varphi\, d\xi'\, dz +$$
$$+\nu\int\limits_{\mathbf{B}} \nabla_\lambda\alpha^3\nabla_\lambda\varphi\, d\xi'\, dz = -\nu\sum_{i=1}^{2}\frac{\partial p^\lambda}{\partial x_1}\Bigg(\int\limits_{\mathsf{B}} \alpha^i\triangle_\lambda\varphi\, d\xi'\, dz + \int\limits_{\partial\mathbf{B}} \alpha^i\nabla_\lambda\varphi\, dS(\xi')\Bigg) -$$
$$-\nu\int\limits_{\mathbf{B}} \alpha^3\triangle_\lambda\varphi\, d\xi'\, dz + \nu\int\limits_{\partial\mathsf{B}} \alpha^3\nabla_\lambda\varphi\, dS(\xi').$$

This obviously implies

$$\int\limits_{\mathsf{B}}\Bigg(u^\lambda - \sum_{i=1}^{2}\frac{\partial p^\lambda}{\partial x_i}\alpha^i - \alpha^3\Bigg)\triangle_\lambda\varphi\, d\xi'\, dz = 0 \quad \text{for almost every } (x', t, \tau) \in \omega\times(0,T)\times\mathcal{T},$$

$$u^\lambda = \alpha^3 = \left(v_1^\pm, v_2^\pm, \lambda\left(\frac{\partial h^\pm}{\partial \tau} + v^\pm\cdot\nabla_{\xi'}h^\pm\right)\right) \qquad \text{on } z = h^\pm.$$

Therefore (4.2) is proved.

□

7.4 Proof of Reynolds equation (4.3)

Multiply the relation $\mathrm{div}_\varepsilon\, u^\mu = 0$ with $\varphi(x') \in C^1(\overline{\omega}) \in H^1(\omega)$, integrate it over the domain $\Omega_\mu(t)$ and use the Green formula, we get

$$-\int\limits_{\Omega\mu(t)} u_1^\mu\frac{\partial\varphi}{\partial x_1}\, dx'\, dz + \int\limits_{\partial\Omega\mu(t)} u_1^\mu n_1\varphi\, dS - \int\limits_{\Omega\mu(t)} u_2^\mu\frac{\partial\varphi}{\partial x_2}\, dx'\, dz +$$
$$+\int\limits_{\partial\Omega\mu(t)} u_2^\mu n_2\varphi\, dS + \frac{1}{\varepsilon}\int\limits_{\partial\Omega\mu(t)} u_3^\mu n_3\varphi\, dS = 0.$$

Taking into account the boundary conditions for u^μ, we have for every $t \in (0, T]$:

$$\int\limits_{\Omega\mu(t)} u_1^\mu\frac{\partial\varphi}{\partial x_1}\, dx'\, dz + \int\limits_{\Omega\mu(t)} u_2^\mu\frac{\partial\varphi}{\partial x_2}\, dx'\, dz - \int\limits_{\omega}\frac{\partial h_\mu}{\partial t}\varphi\, dx' - \int\limits_{\partial\omega}\int\limits_{h_\mu^-}^{h_\mu^+} g\cdot n\varphi\, dS(x', z) = 0.$$

Extend the function u^μ to the fixed domain Ω^{ext}. Then we obtain:

$$\int\limits_{\Omega^{ext}} \overline{u}_1^\mu\frac{\partial\varphi}{\partial x_1}\, dx'\, dz + \int\limits_{\Omega^{ext}} \overline{u}_2^\mu\frac{\partial\varphi}{\partial x_2}\, dx'\, dz - \int\limits_{\omega}\frac{\partial h_\mu}{\partial t}\varphi\, dx' - \int\limits_{\partial\omega}\int\limits_{h_{min}^- -\delta}^{h_{max}^+ +\delta} g\cdot n\varphi\, dS(x', z) =$$

$$= \int\limits_{\Omega^{ext}\setminus\Omega_\mu(t)} \left(G_1 \frac{\partial\varphi}{\partial x_1} + G_2 \frac{\partial\varphi}{\partial x_2} \right) dz\, dx' - \int\limits_{\partial\omega} \varphi \int\limits_{[h^-_{min}-\delta, h^-_\mu]\cup[h^+_\mu, h^+_{max}+\delta]} g\cdot n\, dS(x', z).$$

The right-hand side equals 0 by $\mathrm{div}_\varepsilon G = 0$ in $\Omega^{ext}\setminus\Omega_\mu(t)$ and since G takes the same values on boundaries $z = h^+_\mu$, $z = h^+_{max} + \delta$ (analogously on $z = h^-_\mu$, $z = h^-_{min} - \delta$). Integrate this equation over $(0, T] \times \mathcal{T}$ and pass to the two-scale limit.

$$\begin{aligned} &\int\limits_{\mathcal{T}} \int\limits_{\omega_T} \int\limits_{\mathsf{B}} \left(u_1^\lambda \frac{\partial\varphi}{\partial x_1} + u_2^\lambda \frac{\partial\varphi}{\partial x_2} \right) d\xi'\, dx'\, dz\, dt\, d\tau - \int\limits_{\mathcal{T}} \int\limits_{\omega_T} \int\limits_{\Xi\times\mathcal{T}} \frac{\partial h}{\partial t} \varphi\, d\xi'\, dx'\, dt\, d\tau = \\ &= \int\limits_{(0,T)\times\mathcal{T}} \int\limits_{\partial\omega_T} \overline{g}^z \cdot n\varphi\, dS(x')\, dt, \quad \varphi \in H^1(\omega), \end{aligned} \tag{7.8}$$

where

$$\overline{g}^z = \int\limits_{h^-}^{h^+} g\, dz.$$

Since φ is arbitrary it holds that

$$\begin{aligned} &\mathrm{div}_{x'} \overline{\overline{u^\lambda}^z}^{\xi'} + \frac{\partial \overline{h}^{\xi'}}{\partial t} = 0, && \text{in} \quad \omega_T \times \mathcal{T} \\ &\left(\overline{\overline{u^\lambda}^z}^{\xi'} - \overline{g}^z \right) \cdot \hat{n} = 0, && \text{on} \quad \partial\omega \times (0, T] \times \mathcal{T}. \end{aligned} \tag{7.9}$$

Now we substitute u^λ with (4.2) in (7.8):

$$\begin{aligned} &\int\limits_{\omega_T\times\mathcal{T}} \int\limits_{\mathbf{B}} \sum_{i=1}^{2} \left(\sum_{j=1}^{2} \frac{\partial p^\lambda}{\partial x_j} \alpha_i^j + \alpha_i^3 \right) \frac{\partial\varphi}{\partial x_i}\, d\xi'\, d\tau\, dx'\, dz\, dt - \\ &- \int\limits_{\omega_T\times\mathcal{T}} \int\limits_{\Xi\times\mathcal{T}} \frac{\partial h}{\partial t} \varphi\, d\xi'\, d\tau\, dx' = \int\limits_{(0,T)\times\mathcal{T}} \int\limits_{\partial\omega} \overline{g}^z \cdot n\varphi\, dS(x')\, dt\, d\tau. \end{aligned}$$

Since p^λ and φ do not depend on ξ' and z, the last equation can be rewritten as

$$\begin{aligned} &\sum_{i,j=1}^{2} \int\limits_{\omega_T\times\mathcal{T}} a_{ij}^\lambda \frac{\partial p^\lambda}{\partial x_j} \frac{\partial\varphi}{\partial x_i}\, dx'\, d\tau = \sum_{i=1}^{2} \int\limits_{\omega_T\times\mathcal{T}} b_i^\lambda \frac{\partial\varphi}{\partial x_i}\, dx'\, d\tau - \\ &- \int\limits_{\omega_T\times\mathcal{T}} \overline{\frac{\partial h}{\partial t}}^{\xi',\tau} \varphi\, dx'\, d\tau - \int\limits_{(0,T)\times\mathcal{T}} \int\limits_{\partial\omega} \overline{g}^z \cdot n\varphi\, dS(x')\, dt\, d\tau, \quad \varphi \in H^1(\omega), \end{aligned} \tag{7.10}$$

where

$$a_{ij}^\lambda(x, t, \tau) = -\int\limits_{\mathbf{B}} \alpha_i^j\, d\xi'\, dz, \quad b_i^\lambda(x, t, \tau) = \int\limits_{\mathbf{B}} \alpha_i^3\, d\xi'\, dz, \quad \overline{\frac{\partial h}{\partial t}}^{\xi'} = \int\limits_{\Xi\times\mathcal{T}} \frac{\partial h}{\partial t}\, d\xi'.$$

By integrating (4.2) we obtain

$$\overline{\overline{u^\lambda}}^{z\xi'} = \sum_{i=1}^{2} \frac{\partial p^\lambda}{\partial x_i} \overline{\overline{\alpha^i}}^{z\xi'} + \overline{\overline{\alpha^0}}^{z\xi'} = A^\lambda \nabla_{x'} p^\lambda + b^\lambda, \tag{7.11}$$

where

$$A^\lambda = \overline{\overline{\begin{pmatrix} \alpha_1^1 & \alpha_1^2 & 0 \\ \alpha_2^1 & \alpha_2^2 & 0 \\ \alpha_3^1 & \alpha_3^2 & 0 \end{pmatrix}}}^{z\xi',\tau} \tag{7.12}$$

and

$$b^\lambda = \overline{\overline{\begin{pmatrix} \alpha_1^3 \\ \alpha_2^3 \\ \alpha_3^3 \end{pmatrix}}}^{z\xi'} \tag{7.13}$$

This proves (4.5) and (4.6).

Let us prove now the uniqueness of the solution of (4.3) (or, equivalently, (4.5)). It is sufficient to show that $A^\lambda_{2\times 2} = {a^\lambda_{ij}}_{i,j=1,2}$ is symmetric and positive definite matrix. We choose α^1 as a test function in (7.2) for $i=2$. The function α^1 belongs to $H^1_{0,per}(\mathbf{B})$ due to its boundary conditions. Moreover, the fact that $\operatorname{div}_\lambda \alpha^1 = 0$ leads to

$$\nu \int\limits_{\mathbf{B}} \nabla_\lambda \alpha^1 \nabla_\lambda \alpha^2 \, d\xi' \, dz = -\int\limits_{\mathbf{B}} \alpha_1^2 \, d\xi' \, dz = a^\lambda_{12}.$$

Analogously, taking α^2 as a test function in (7.1) for $i=1$, we deduce that

$$\nu \int\limits_{\mathbf{B}} \nabla_\lambda \alpha^1 \nabla_\lambda \alpha^2 \, d\xi' \, dz = -\int\limits_{\mathbf{B}} \alpha_2^1 \, d\xi' \, dz = a^\lambda_{21}.$$

Hence, $a^\lambda_{12} = a^\lambda_{21}$, what proves the symmetry. Let (η_1, η_2) be an arbitrary vector in $\mathbb{R}^2$. Choose now $\eta_i \eta_j \alpha^{j\lambda}$ as a test function in equation with number i $(i=1,2)$ in (7.1). Then we get:

$$\nu \eta_i \eta_j \int\limits_{\mathbf{B}} \nabla_\lambda \alpha^i \nabla_\lambda \alpha^j \, d\xi' \, dz = -\int\limits_{\mathbf{B}} \eta_i \eta_j \alpha_i^j \, d\xi' \, dz = \eta_i \eta_j a^\lambda_{ij}.$$

Summing up, this implies

$$\sum_{i,j=1}^{2} \eta_i \eta_j a^\lambda_{ij} = \nu \int\limits_{\mathbf{B}} (\eta_1 \nabla_\lambda \alpha^1 + \eta_2 \nabla_\lambda \alpha^2)^2 \, d\xi' \, dz \geq 0.$$

Suppose that $\sum\limits_{i,j=1}^{2} \eta_i \eta_j a^\lambda_{ij} = 0$. This means that

$$\int\limits_{\mathbf{B}} \nabla_\lambda (\eta_1 \alpha^1 + \eta_2 \alpha^2)^2 \, d\xi' \, dz = 0.$$

Consequently,

$$\int_{\mathbf{B}} \frac{\partial}{\partial z_3}(\eta_1\alpha^1 + \eta_2\alpha^2)^2 \, d\xi' \, dz = 0.$$

Taking into account the fact that $\eta_1\alpha^1 + \eta_2\alpha^2 \in H^1_{0,per}(\mathbf{B})$ and using the Friedrichs inequality in $z-$direction, we deduce that

$$\eta_1\alpha^1 + \eta_2\alpha^2 = 0.$$

By choosing $\eta_1\alpha^1 + \eta_2\alpha^2$ as a test function in (7.1), we have that

$$\int_{\mathbf{B}} (\eta_1\varphi_1 + \eta_2\varphi_2) \, d\xi' \, dz = 0,$$

for any $(\varphi_1, \varphi_2, \varphi_3) = \varphi \in H^1_{0,per}(\mathbf{B})$ with $\mathrm{div}_\lambda \varphi = 0$.
Now choose $\varphi = (\varphi_1(\xi', \tau, z), 0, 0)$ such that $\varphi_1 \in H^1_{0,per}(\mathbf{B})$, $\int_{\mathbf{B}} \varphi_1 \, d\xi' \, dz \neq 0$. Then we have $\eta_1 = 0$. Analogously, one can prove that $\eta_2 = 0$. Thus, the assumption $\sum_{i,j=1}^{2} \eta_i\eta_j a^\lambda_{ij} = 0$ implies that $\eta_1 = \eta_2 = 0$, and we have proved the positiveness of the matrix.

8 Proof of Theorems 3 and 5 in the case $\lambda = 0$

The proof of Theorem 3 is similar to the previous case. We use the uniform estimates for the case $\frac{\varepsilon}{\mu} \to 0$ and two-scale convergence result. One can verify the boundary conditions

$$u = (v_1^\pm, v_2^\pm, 0) \quad \text{ on } z = h^\pm$$

exactly in the same way by using the fact that $\lambda = 0$ in the considered case.
Moreover,

$$\int_{\Xi} \sum_{i=1}^{2} \overline{u}_i^z \frac{\partial \psi}{\partial \xi} \, d\xi' = 0 \quad \text{ for any } \psi \in H^1(\Xi). \tag{8.1}$$

The last equation follows from the integral identity for $\mathrm{div}_\varepsilon \, \overline{u}^\mu = 0$ with test-function $\varphi = \varphi_1(x')\varphi_2(\xi')$, $\varphi_1 \in \mathcal{D}(\omega)$ and $\varphi_2 \in H^1_{per}(\Xi)$.

8.1 Proof of (4.11)

Following [10] introduce the spaces:

$$V_0 = \{\varphi = (\varphi_1, \varphi_2) : \varphi_i \in L_2(\mathbf{B}),\ \frac{\partial \varphi_i}{\partial z} \in L_2(\mathbf{B}),\ \varphi(x', h^-, t) = \varphi(x', h^+, t) = 0\},$$

$$V_{00} = \{\varphi \in V_0 : \int_{\Xi} \sum_{i=1}^{2} \overline{\varphi}^z \nabla \psi \, d\xi' = 0 \ \text{ for any } \psi \in H^1_{per}(\Xi)\}$$

and

$$V = \{\varphi = (\varphi_1, \varphi_2) \in C^\infty_{0,per}(\mathbf{B}) : \exists \varphi_3 \in C^\infty_{0,per}(\mathbf{B}) \text{ such that } \frac{\partial \varphi_1}{\partial \xi_1} + \frac{\partial \varphi_2}{\partial \xi_2} + \frac{\partial \varphi_3}{\partial z} = 0\}.$$

The spaces V_0 and V_00 are Hilbert ones. In the same way as [10, Lemma 4.8], it can be proved that

$$\overline{V} = V_{00}, \tag{8.2}$$

where the closure is taken with respect to the norm

$$\|v\|^2 \equiv \|v\|^2_{L_2} + \|\frac{\partial v}{\partial z}\|^2_{L_2}.$$

Let us prove that

$$u = \alpha^1 \frac{\partial p}{\partial x_1} + \alpha^2 \frac{\partial p}{\partial x_2} + \alpha^3, \tag{8.3}$$

where α^i are the unique solutions to auxiliary problems:

$$\nu \sum_{i=1}^{2} \int_{\mathsf{B}} \frac{\partial \alpha_i^k}{\partial z} \frac{\partial \varphi_i}{\partial z} \, d\xi' \, dz = - \int_{\mathsf{B}} \varphi_k \, d\xi' \, dz \quad \text{for all } \varphi \in V_{00}, \ \alpha^k \in V_{00} \tag{8.4}$$

and

$$\nu \sum_{i=1}^{2} \int_{\mathsf{B}} \frac{\partial \alpha_i^3}{\partial z} \frac{\partial \varphi_i}{\partial z} \, d\xi' \, dz = 0 \text{ for all } \varphi \in V_{00}, \ \alpha^3 \in R + V_{00} \tag{8.5}$$

with $R \in L_2(\mathsf{B})$, $\frac{\partial R}{\partial z} \in L_2(\mathsf{B})$ such that $R = (v_1^\pm, v_2^\pm, 0)$ on $\Sigma^\pm$.

Proof. Take $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in (\mathcal{D}(\Omega^{ext}, C^\infty_{per}(\mathsf{B})))^3$ such that $\varphi \equiv 0$ in $\Omega^{ext} \setminus \Omega_{\mu T}$ and $\mathrm{div}_\varepsilon \varphi = 0$. Define $\varphi^\mu(x', z, t) = \varphi(x', z, t, \frac{x'}{\mu}, \frac{t}{\mu})$ and take $(\varphi_1^\mu, \varphi_2^\mu, \frac{\varepsilon}{\mu}\varphi_3^\mu)$ as a test function in the original equation (3.4). Since $\varphi \equiv 0$ in $\Omega^{ext} \setminus \Omega_{\mu T}$, for the extended function $\overline{u}^\mu$ we have:

$$\begin{aligned}
&\int_{\Omega^{ext}} \overline{u}^\mu \varphi \Big|_0^T dx' \, dz - \int_0^T \int_{\Omega^{ext}} \Bigg(\overline{u}^\mu \Big(\frac{\partial \varphi}{\partial t} + \frac{1}{\mu} \frac{\partial \varphi}{\partial t} \Big) + \nu \sum_{i,j=1}^{2} \frac{\partial \overline{u}_i^\mu}{\partial x_j} \Big(\frac{\partial \varphi_i}{\partial x_j} + \frac{1}{\mu} \frac{\partial \varphi_i}{\partial \xi_j} \Big) + \\
&+ \frac{\varepsilon}{\mu} \frac{\partial \overline{u}_3^\mu}{\partial z} \Big(\frac{\partial \varphi_3}{\partial x_j} + \frac{1}{\mu} \frac{\partial \varphi_3}{\partial \xi_j} \Big) + \frac{1}{\varepsilon^2} \sum_{i=1}^{2} \frac{\partial \overline{u}_i^\mu}{\partial z} \frac{\partial \varphi_i}{\partial z} + \frac{1}{\varepsilon\mu} \frac{\partial \overline{u}_3^\mu}{\partial z} \frac{\partial \varphi_3}{\partial z} \Bigg) dx' \, dz \, dt = \\
&= \int_0^T \int_{\Omega^{ext}} \overline{p}^\mu(x', t) \Big(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} \Big) dx' \, dz \, dt.
\end{aligned} \tag{8.6}$$

Multiply the last equation by ε^2 and pass to the two-scale limit. Taking into account the result of Lemma 4.1, we deduce

$$\nu \int_{\omega_T} \int_{\mathsf{B}} \sum_{i=1}^{2} \frac{\partial u_i}{\partial z} \frac{\partial \varphi_i}{\partial z} \, d\xi' \, d\tau \, dz \, dx' \, dt = \int_{\omega_T} \int_{\mathsf{B}} p(x', t) \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} \right) d\xi' \, d\tau \, dz \, dx' \, dt. \tag{8.7}$$

Exactly as in Theorem 5 one can show that $p \in L_2((0,T) \times \mathcal{T}; H^1(\omega))$ and that

$$\nu \int_{\mathsf{B}} \sum_{i=1}^{2} \frac{\partial u_i}{\partial z} \frac{\partial \varphi_i}{\partial z} \, dz \, dx' = - \int_{\mathsf{B}} \sum_{i=1}^{2} \frac{\partial p(x', t)}{\partial x_i} \varphi_i \, d\xi' \, dz, \tag{8.8}$$

for almost every $(x', t, \tau) \in \omega \times (0,T) \times \mathcal{T}$. By the density result (8.2), the equation (8.8) holds also for any $\varphi \in V_{00}$. The uniqueness of $(u_1, u_2) \in R + V_{00}$ satisfying (8.8) for all $\varphi \in V_{00}$ follows from the Lax-Milgram theorem. The same motivates the uniqueness of the solutions to (8.4), (8.5). In the same way as in the proof of Theorem 5, by using linearity and fact that p does not depend on z, we deduce (8.3) from (8.4), (8.5) and (8.8). □

8.2 Proof of (4.11)

The weak solution to local problems (8.9) satisfy the following identities by the definition:

$$\int_{\Xi} \frac{h^3}{6} (e_i + \nabla_\xi q^i) \nabla_\xi \varphi \, d\xi' = 0, \;\; \varphi \in H^1_{per}(\Xi \times \mathcal{T}) \tag{8.9}$$

where $q^i \in H^1_{per}(\Xi) \times L^2_0(\Xi)$, $\quad i = 1, 2$ and e_i is the canonical base in $\mathbb{R}^2$.

$$\int_{\Xi} \frac{h^3}{12\nu} \nabla_\xi q^3 \nabla_\xi \varphi \, d\xi' = \int_{\Xi} \frac{h(v^+ + v^-)}{2} \nabla_\xi \varphi \, d\xi' + \int_{\Xi} \frac{\partial h}{\partial \tau} \varphi \, d\xi', \tag{8.10}$$

where $q^3 \in H^1_{per}(\Xi) \times L^2_0(\Xi \times \mathcal{T})$, $\varphi \in H^1_{per}(\Xi \times \mathcal{T})$.

Lemma 8. *The solutions to auxiliary problems* (8.4), (8.5) are related to q^i by

$$\begin{aligned} \alpha^i &= \frac{(z - h^+)(z - h^-)}{2\nu} \left(\nabla_{\xi'} q^i + e_i \right), \quad (i = 1, 2) \\ \alpha^3 &= \frac{(z - h^+)(z - h^-)}{2\nu} \nabla_{\xi'} q^3 + \frac{z - h^-}{h} v^+ + \frac{h^+ - z}{h} v^-. \end{aligned} \tag{8.11}$$

Proof. By using De Rham theorem we can conclude from (8.4) the existence of $\widetilde{q}^k \in L^2_0(\mathsf{B})$ such that

$$\nu \sum_{i=1}^{2} \int_{\mathsf{B}} \frac{\partial \alpha_i^k}{\partial z} \frac{\partial \varphi_i}{\partial z} \, d\xi' \, dz = - \int_{\mathsf{B}} \varphi_k \, d\xi' \, dz + \int_{\mathsf{B}} \widetilde{q}^k \mathrm{div} \varphi \, d\xi' \, dz \forall \varphi \in H^1_0(\mathsf{B}). \tag{8.12}$$

Take φ as $(\varphi, 0, 0)$, $(0, \varphi, 0)$ and $(0, 0, \varphi)$ in (8.13). This implies

$$\nu \frac{\partial^2 \alpha_i^k}{\partial z^2} = \delta_{ik} + \frac{\partial \widetilde{q}^k}{\partial \xi_i}, \ i, k = 1, 2, \tag{8.13}$$

$$\frac{\partial \widetilde{q}^k}{\partial z} = 0, \ k = 1, 2. \tag{8.14}$$

Integrating twice with respect to z and taking into account the boundary conditions, we derive

$$\alpha_i^k = \frac{1}{2\nu} \left(\frac{\partial \widetilde{q}^k}{\partial \xi_i} + \delta_{ki} \right) (z - h^+)(z - h^-), \ i, k = 1, 2. \tag{8.15}$$

Since $\alpha_i^k \in V_{00}$,

$$\int\limits_{\Xi} \left(\int\limits_{h^-}^{h^+} \left(\frac{\partial \widetilde{q}^k}{\partial \xi_i} + \delta_{ki} \right) (z^2 - z(h^+ + h^-) + h^+ h^-) \, dz \right) \nabla \varphi \, d\xi' = 0.$$

This can be rewritten as

$$\int\limits_{\Xi \times \mathcal{T}} \frac{h^3}{6} (e_i + \nabla_\xi \widetilde{q}^i) \nabla_\xi \varphi \, d\xi' = 0, \ i = 1, 2, \ \varphi \in H^1_{per}(\Xi).$$

Due to the uniqueness, $\widetilde{q}^i = q^i$. In a similar way one can verify the validity of (8.11) by using (8.5) and uniqueness of the solution to (8.10). $\square$

8.3 Proof of (4.3)

Proof. Similarly to the proof of Theorem 3 one can show the validity of equation (4.3) for limit pressure with

$$a_{ij}(x', t) = \int\limits_{\Xi} \frac{h^3}{12\nu} \left(\delta_{ij} + \frac{\partial q^i}{\partial \xi_j} \right) d\xi', \ i = 1, 2,$$

$$b_i(x', t) = \int\limits_{\Xi} \left(-\frac{h^3}{12\nu} \frac{\partial q^3}{\partial \xi_j} + \frac{h(v^+ + v^-)_i}{2} \right) d\xi'.$$

The different formulae for a_{ij} and b_i in this case (Reynolds roughness) come from the formula for a_i. Now let us prove the uniqueness of the solution. For this goal we need to show that matrix $A^\lambda_{2\times 2} = {a^\lambda_{ij}}_{i,j=1,2}$ is symmetric and positive definite. We take $\varphi = q^j$ in (8.9). Then

$$\int\limits_{\Xi} \frac{h^3}{6} (e_i + \nabla_\xi q^i) \nabla_\xi q^j \, d\xi' = 0$$

and a_{ij} can be written in the form

$$a_{ij} = \int\limits_{\Xi} \frac{h^3}{6}(e_i + \nabla_\xi q^i)(e_j + \nabla_\xi q^j)\, d\xi' = a_{ji}$$

and the symmetry is proved. Let $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. According to the definition,

$$\eta^t A \eta = \int\limits_{\Xi} \frac{h^3}{12\nu} \left(\sum_{i=1}^{2} \eta_i (e_i + \nabla_\xi q^i) \right)^2 d\xi \geq C \int\limits_{\Xi} \| \sum_{i=1}^{2} \eta_i (e_i + \nabla_\xi q^i) \|_2^2 \, d\xi =$$

$$C \int\limits_{\Xi} \sum_{i=1}^{2} \eta_i^2 \, d\xi + C \int\limits_{\Xi} \| \eta_1 \nabla_\xi q^1 + \eta_2 \nabla_\xi q^2 \|_2^2 \, d\xi \geq C \|\eta\|_2^2.$$

Here

$$C = \frac{\min_\Xi h^3}{2\nu} > 0.$$

Thus, matrix A is positive definite, and the uniqueness is proved. $\square$

9 Analysis of case $\lambda = \infty$

Now we are going to prove the results of Theorem 3 for the case when $\lambda = \infty$. The principal difference in this case as compared with the previously considered is that we shall work in "cut-off" domain $\Omega_T^* = \{(x', z) : x' \in \omega, h_*^-(x', t) < z < h_*^+(x', t)\} \times (0, T)$, where

$$h_*^+(x', t) = h_0^+(x' - tv) + \min_{(\xi', \tau) \in \Xi \times \mathcal{T}} h_{per}^+(\xi', \tau),$$

$$h_*^-(x', t) = h_0^-(x' - tv) + \max_{(\xi', \tau) \in \Xi \times \mathcal{T}} h_{per}^-(\xi', \tau), \quad h_* = h_*^+ - h_*^-.$$

Lemma 4 implies the estimate for the pressure in Ω_T^* :

Lemma 9.

$$\|\varepsilon^2 p^\mu\|_{\Omega_T^*} \leq K$$

Lemmas 5 and 6 give the two-scale convergence to a limit pressure.

Lemma 10. *There exists a function $p(x', t, \tau) \in L_2(\omega_T \times \mathcal{T})$ such that*

$$\varepsilon^2 p^\mu \twoheadrightarrow p(x', t, \tau) \quad \text{in } \Omega_T^* \times \mathcal{T} \tag{9.1}$$

and

$$\int\limits_{\omega} p \, dx' = 0. \tag{9.2}$$

Remark 2. It will be shown that p does not depend on τ when $\lim\limits_{\mu \to 0} \frac{\mu}{\varepsilon^2(\mu)} < \infty$. The proof of this fact is based on the properties of the limit velocity field and is given in Section 9.1

9.1 Proof of Theorem 3

The estimates from Lemma 3 are obviously valid in Ω^*, and we conclude the existence of $u \in L_2(\Omega_T^*; H^1_{per}(\Xi \times [0, \Xi_3] \times \mathcal{T}))$ and
$v \in L_2(\Omega_T^* \times \Xi \times [0, \Xi_3] \times \mathcal{T}; H^1_{per}([0, \Xi_3]))$ satisfying (4.1). Let us verify the boundary conditions for two-scale limit u on $z = h_*^\pm$. By means of Green formula we derive

$$\int\limits_{\Omega_T^*} \frac{\partial u^\mu}{\partial z} \varphi(x', z, t, \frac{x'}{\mu})\, dx'\, dz\, dt = -\int\limits_{\Omega_T^*} u^\mu \frac{\partial \varphi}{\partial z}\, dx'\, dz\, dt +$$
$$+ \int\limits_{\omega_T} \Bigg(\left(v_1^+, v_2^+, \varepsilon \left(\frac{\partial h_*^+}{\partial t} + v^+ \cdot \nabla h_*^+ \right) \right) \varphi(x', h_*^+, t, \frac{x'}{\mu}) -$$
$$- \left(v_1^-, v_2^-, \varepsilon \left(\frac{\partial h_*^-}{\partial t} + v^- \cdot \nabla h_*^- \right) \right) \varphi(x', h_*^-, t, \frac{x'}{\mu}) \Bigg)\, dx'\, dt$$

for any $\varphi \in L_2(0, T; C^\infty(\Omega^*, C^\infty_{per}(\Xi)))$. Passing to the two-scale limit, one obtains that

$$\int\limits_{\Omega_T^*} \int\limits_{\Xi} \frac{\partial u}{\partial z} \varphi(x', z, t, \xi')\, dx'\, dz\, dt\, d\xi' = -\int\limits_{\Omega_T^*} \int\limits_{\Xi} u \frac{\partial \varphi}{\partial z}\, dx'\, dz\, dt\, d\xi' +$$
$$+ \int\limits_{\omega_T} \int\limits_{\Xi} \Big(\left(v_1^+, v_2^+, 0 \right) \varphi(x', h_*^+, t,\ d\xi') - \left(v_1^-, v_2^-, 0 \right) \varphi(x', h_*^-, t, \xi') \Big)\, dx'\, dt\, d\xi'$$

Applying again the Green formula, we conclude that

$$-\int\limits_{\Omega_T^*} \int\limits_{\Xi} u \frac{\partial \varphi}{\partial z}\, dx'\, dz\, dt\, d\xi' + \int\limits_{\omega_T} \int\limits_{\Xi} u\varphi\Big|_{z=h_*^-}^{z=h_*^+}\, dx'\, dt\, d\xi' = -\int\limits_{\Omega_T^*} \int\limits_{\Xi} u \frac{\partial \varphi}{\partial z}\, dx'\, dz\, dt\, d\xi' +$$
$$+ \int\limits_{\omega_T} \int\limits_{\Xi} \Big(\left(v_1^+, v_2^+, 0 \right) \varphi(x', h_*^+, t,\ d\xi') - \left(v_1^-, v_2^-, 0 \right) \varphi(x', h_*^-, t, \xi') \Big)\, dx'\, dt\, d\xi'.$$

Hence,

$$u = (v_1^\pm, v_2^\pm, 0) \quad \text{ as } z = h_*^\pm$$

for almost every $(x', t, \xi') \in \omega_T \times \Xi$. Independence of u from ξ_3 can be proved analogously to the proofs in previous cases. In additional, one can show that u is independent of ξ' and τ. First we derive that $\operatorname{div}_{\xi'} u = 0$. Indeed, we use first the fact that $diver_\varepsilon u^\mu = 0$ in Ω^*. Multiply this equation with
$\varphi^\mu(x', z, t) = \varphi\left(x', z, t, \frac{x'}{\mu}, \frac{t}{\mu}\right) \in L_2(0, T; \mathcal{D}(\Omega^*; C^\infty_{per}(\Xi \times \mathcal{T})))$, then

$$0 = \int\limits_{\Omega_T^*} \left(\frac{\partial u_1^\mu}{\partial x_1} + \frac{\partial u_2^\mu}{\partial x_2} + \frac{1}{\varepsilon} \frac{\partial u_3^\mu}{\partial z} \right) \varphi^\mu\, dx'\, dz\, dt =$$

$$= -\int\limits_{\Omega_T^*} \left(\sum_{i=1}^{2} u_i^\mu \left(\frac{\partial \varphi^i}{\partial x_i} + \frac{1}{\mu} \frac{\partial \varphi^i}{\partial \xi_i} \right) + \frac{u_3^\mu}{\varepsilon} \frac{\partial \varphi_3^\mu}{\partial z} \right) dx'\, dz\, dt.$$

Multiply the last equation by μ and pass to the two-scale limit, one gets that

$$0 = \int\limits_{\mathcal{T}} \int\limits_{\omega_T} \int\limits_{\mathbf{B}^*} \sum_{i=1}^{2} u_i \frac{\partial \varphi_i}{\partial \xi_i}\, dx'\, dt\, d\xi'\, d\tau\, dz = -\int\limits_{\mathcal{T}} \int\limits_{\omega_T} \int\limits_{\mathbf{B}^*} \varphi \operatorname{div}_{\xi'} u\, dx'\, dt\, d\xi'\, d\tau\, dz$$

what proves the result. Let us derive now the equation

$$\nu \int\limits_{\mathcal{T}} \int\limits_{\omega_T} \int\limits_{\mathbf{B}^*} \sum_{i=1}^{2} \frac{\partial u}{\partial \xi_i} \frac{\partial \varphi}{\partial \xi_i}\, dx'\, dt\, d\xi'\, d\tau\, dz = 0 \tag{9.3}$$

for any $\varphi \in H^1(0,T;\mathcal{D}(\Omega^*; C^\infty_{per}(\Xi \times \mathcal{T})))$ with $\operatorname{div}_{\xi'} \varphi = 0$, $\frac{\partial \varphi_3}{\partial z} = 0$, where

$$B^*(x',t) = \{(\xi', z) : \xi' \in \Xi, h_*^-(x',t) < z < h_*^+(x',t)\}.$$

Take $\varphi^\mu(x',z,t) = \varphi\left(x', z, t, \xi' = \frac{x'}{\mu}, \tau = \frac{t}{\mu}\right)$ such that $\operatorname{div}_{\xi'} \varphi = 0$, $\frac{\partial \varphi_3}{\partial z} = 0$, as a test-function in the original equation (3.4):

$$\begin{aligned}
&\int\limits_{\Omega^*} u^\mu \varphi\, dy \Big|_0^T dx'\, dz - \int\limits_{\Omega_T^*} u^\mu \varphi_t\, dx'\, dz\, dt + \\
&+ \nu \int\limits_{\Omega_T^*} \left(\sum_{i=1}^{2} \frac{\partial u^\mu}{\partial x_i} \left(\frac{\partial \varphi}{\partial x_i} + \frac{1}{\mu} \frac{\partial \varphi}{\partial \xi_i} \right) + \frac{1}{\varepsilon^2} \frac{\partial u^\mu}{\partial z} \frac{\partial \varphi}{\partial z} \right) dx'\, dz\, dt = \\
&= \int\limits_{\Omega_T^*} p^\mu(x',t) \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{1}{\mu} \frac{\partial \varphi_1}{\partial \xi_1} + \frac{\partial \varphi_2}{\partial x_2} + \frac{1}{\mu} \frac{\partial \varphi_2}{\partial \xi_2} + \frac{1}{\varepsilon} \frac{\partial \varphi_3}{\partial z} \right) dx'\, dz\, dt = \\
&= \int\limits_{\Omega_T^*} p^\mu(x',t) \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} \right) dx'\, dz\, dt.
\end{aligned} \tag{9.4}$$

Multiply this equation with $\varepsilon\mu$ and pass to the two-scale limit. Since $\frac{\mu}{\varepsilon} \to 0$ and $\varepsilon^2 p^\mu \twoheadrightarrow p$ as $\mu \to 0$, then $\varepsilon\mu p^\mu \hookrightarrow 0$ and we deduce exactly (9.3). Take now $\varphi = (u_1 - v_1, u_2 - v_2, 0)$ in (9.3). Then

$$\nu \int\limits_{\mathcal{T}} \int\limits_{\omega_T} \int\limits_{\mathbf{B}^*} \sum_{i=1}^{2} \left| \frac{\partial u}{\partial \xi_i} \right|^2 dx'\, dt\, d\xi'\, d\tau\, dz = 0, \tag{9.5}$$

hence, u is independent of ξ'. Now we want to show that $u_3 = 0$. For this we pass to the two-scale limit in the identity

$$0 = \int\limits_{\Omega^*} \varepsilon \left(\frac{\partial u_1^\mu}{\partial x_1} + \frac{\partial u_2^\mu}{\partial x_2} + \frac{1}{\varepsilon} \frac{\partial u_3^\mu}{\partial z} \right) \varphi^\mu\, dx'\, dz$$

and conclude that

$$0 = \int\limits_{\mathcal{T}} \int\limits_{\omega_T} \int\limits_{\mathbf{B}^*} \left(\frac{\partial u_1}{\partial \xi_1} + \frac{\partial u_2}{\partial \xi_2} + \frac{\partial u_3}{\partial z} + \frac{\partial v_3}{\partial z} \right) \varphi \, dx' \, dt \, d\xi' \, d\tau \, dz.$$

Taking into account independence of u_1, u_2 from ξ' and periodicity of v_3, we obtain that

$$\frac{\partial u_3}{\partial z} = 0 \quad \text{ in } \mathbf{B}^*.$$

This together with the fact that $u_3 = 0$ on $z = h_*^{\pm}$ implies that $u_3 = 0$ in $\mathbf{B}^*$ by the Friedrich's inequality. Let us show that

$$\nu \int\limits_{\Omega_T^*} \int\limits_{\mathcal{T}} \frac{\partial u}{\partial z} \frac{\partial \varphi}{\partial z} \, dx' \, dt \, d\tau \, dz = \int\limits_{\Omega_T^*} \int\limits_{\mathcal{T}} p \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} \right) dx' \, dt \, d\tau \, dz, \tag{9.6}$$

for φ vanishing on $z = h_*^{\pm}$ and $\frac{\partial \varphi_3}{\partial z} = 0$. Indeed, it is derived by multiplying the integral identity with ε^2, where $\varphi^\mu(x', z, t) = \varphi\left(x', z, t, \tau = \frac{t}{\mu}\right)$ and passing to the two-scale limit.

Choose the test function $(\varphi_1^\mu, \varphi_2^\mu, 0)$ with $\varphi_i^\mu = \varphi\left(x', z, t, \tau = \frac{t}{\mu}\right)$ in the integral identity (3.4), multiplied by μ :

$$\begin{aligned} &\mu \int\limits_{\Omega^*} u^\mu \varphi \, dy \bigg|_0^T dx' \, dz - \int\limits_{\Omega_T^*} u^\mu \left(\mu \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial \tau} \right) dx' \, dz \, dt + \\ &+ \nu \int\limits_{\Omega_T^*} \left(\sum_{i=1}^{2} \mu \frac{\partial u^\mu}{\partial x_i} \frac{\partial \varphi}{\partial x_i} + \frac{\mu}{\varepsilon^2} \frac{\partial u^\mu}{\partial z} \frac{\partial \varphi}{\partial z} \right) dx' \, dz \, dt = \\ &= \int\limits_{\Omega_T^*} \mu p^\mu(x', t) \left(\frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2} \right) dx' \, dz \, dt. \end{aligned} \tag{9.7}$$

Let us notice that in the considered case (when $\lim\limits_{\mu \to 0} \frac{\varepsilon}{\mu} = \infty$)

$$\mu \frac{\partial u^\mu}{\partial x_i} = \varepsilon \frac{\partial u^\mu}{\partial x_i} \frac{\mu}{\varepsilon} \twoheadrightarrow 0 \quad \text{ and } \mu p^\mu = \frac{\mu}{\varepsilon^2} \varepsilon^2 p^\mu \twoheadrightarrow \gamma p \quad \text{ as } \mu \to 0, \text{ where } \gamma := \lim_{\mu \to 0} \frac{\mu}{\varepsilon^2}.$$

Three cases are possible depending on either $\gamma = 0, \infty$ or $0 < \gamma < \infty$. Now we prove that in the case $0 \le \gamma < \infty$ u_1, u_2 and p does not depend on τ. If $\gamma = 0$, then passing to the two-scale limit in (9.7), one derives

$$\nu \int\limits_{\Omega_T^*} \int\limits_{\mathcal{T}} u \frac{\partial \varphi}{\partial \tau} \, dx' \, dt \, d\tau \, dz = 0. \tag{9.8}$$

This implies that u is independent of τ, hence, due to (9.6), p does not depend on τ as well. Consider now the situation when $0 < \gamma < \infty$. Then the two-scale limit of (9.7) is the following identity:

$$\nu \int\limits_{\Omega_T^*} \int\limits_{\mathcal{T}} \left(u \frac{\partial \varphi}{\partial \tau} - \nu\gamma \frac{\partial u}{\partial z}\frac{\partial \varphi}{\partial z} + \gamma p \operatorname{div}_{x'} \varphi \right) dx'\, dt\, d\tau\, dz = 0. \tag{9.9}$$

This equation can be simplified due to (9.6). It becomes exactly (9.8) and we complete the proof as in the case $\gamma = 0$. Finally, in the case $\gamma = \infty$, dividing equation (9.9) by γ one gets (9.6) since the term $\int\limits_{\Omega_T^*}\int\limits_{\mathcal{T}} u \frac{\partial \varphi}{\partial \tau}\, dx'\, dt\, d\tau\, dz$ vanishes.

9.2 Proof of Theorem 5

Proof. Let us show first the validity of (4.2) with

$$\begin{aligned} \alpha^i &= \frac{(z-h_*^+)(z-h_*^-)}{2\nu} e_i, \quad (i=1,2) \\ \alpha^3 &= \frac{(z-h^+)(z-h^-)}{2\nu} \nabla_{\xi'} q^3 + \frac{z-h_*^-}{h_*} v^+ + \frac{h_*^+ - z}{h_*} v^-. \end{aligned} \tag{9.10}$$

Choose $\varphi = (u_1 - v_1, u_2 - v_2, 0)$ in (9.6). Since u is independent of $\xi^{'}$ and p is independent of $\xi^{'}$ and τ, it follows that

$$\nu \int\limits_{\mathcal{T}\times\omega_T} \int\limits_{h_*^-}^{h_*^+} \left(\nu \frac{\partial^2 u}{\partial z^2} - \nabla_{x'} p \right) \varphi\, dx'\, dt\, d\tau\, dz = 0.$$

If we take into account also the boundary conditions for u on $z = h_*^\pm$, this equation reduces to

$$\nu \int\limits_{\mathcal{T}\times\omega_T} \left(u - \sum_{i=1}^{2} \frac{\partial p}{\partial x_i} \frac{(z-h_*^+)(z-h_*^-)}{2\nu} e_i - \frac{z-h_*^-}{h_*} v^+ + \frac{h_*^+ - z}{h_*} v^- \right) \varphi\, dx'\, dt\, d\tau\, dz = 0.$$

Thus, (4.2) and (9.10) are proved. Then, similarly to previous two cases, we write the integral identity for $\operatorname{div}_\varepsilon u^\mu = 0$ in Ω^*, integrate by parts taking into account boundary conditions and pass to the two-scale limit. After that we substitute u with (5.8), where α^i satisfy (9.10). Since p does not depend on z, and $\alpha^i, h_*^\pm$ do not depend on $\xi^{'}$, we derive (4.3) with

$$\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{pmatrix} = \begin{pmatrix} -\frac{h_*^3}{12\nu} & 0 & \frac{h_*(v_1^+ + v_1^-)}{2} \\ 0 & -\frac{h_*^3}{12\nu} & \frac{h_*(v_2^+ + v_2^-)}{2}. \end{pmatrix}, \quad \overline{g}^z = \int_{h_*^-}^{h_*^+} g\, dz.$$

$\square$

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