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An iterative method for solving split minimization problem in Banach space with applications

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Abstract. The purpose of this paper is to study an approximation method for finding a solution of the split minimization problem which is also a fixed point of a right Bregman strongly nonexpansive mapping in p-uniformly convex real Banach spaces which are also uniformly smooth. We introduce a new iterative algorithm with a new choice of stepsize such that its implementation does not require a prior knowledge of the operator norm. Using the Bregman distance technique, we prove a strong convergence theorem for the sequence generated by our algorithm. Further, we applied our result to the approximation of solution of inverse problem arising in signal processing and give a numerical example to show how the sequence values are affected by the number of iterations. Our result in this paper extends and complements many recent results in literature.

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1 Introduction

Let E be a real Banach space and $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $dim(E) \geq 2$, the modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$, defined by

$$\delta_E := \inf\Big\{1 - ||\frac{x+y}{2}|| : ||x|| = ||y|| = 1;\ \epsilon = ||x - y||\Big\}.$$

E is said to be uniformly smooth if and only if $\delta_E(\epsilon) > 0$, for all $\epsilon \in (0, 2]$, and p-uniformly convex if there exists a $C_p > 0$, such that $\delta_E(\epsilon) \geq C_p\epsilon^p$ for any $\epsilon \in (0, 2]$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) := \sup\Big\{\frac{1}{2}(||x + y|| + ||x - y||) - 1 : ||x|| \leq 1, ||y|| \leq t\Big\}.$$

A Banach space E is said to be uniformly smooth if and only if

$$\lim_{t\to\infty} \frac{\rho_E(t)}{t} = 0,$$

and *q-uniformly smooth* if there exists a $C_q > 0$ such that $\rho_E(t) \leq C_q t^q$ for any $t > 0$. The duality mapping $J_p^E : E \to 2^{E^*}$ is defined by

$$J_p^E(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = ||x||^p, ||\bar{x}|| = ||x||^{p-1}\},$$

and is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_p^E(x_n), y\rangle \to \langle J_p^E(x), y\rangle$$

holds true for any $y \in E$. It is worth noting that the l_p $(p > 1)$ space has such property, but the L_p $(p > 2)$ space does not share this property.
It is well known that E is p-uniformly convex and uniformly smooth if and only if its dual space E^* is q-uniformly smooth and uniformly convex. Moreover, if E is reflexive and strictly convex with a strictly convex dual, then $(J_p^E)^{-1} = J_q^{E^*}$ is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E and thus $J_p^E J_q^{E^*} = I_{E^*}$ and $J_q^{E^*} J_p^E = I_E$, where I_E and I_{E^*} are the identity operators on E and E^* respectively. We note that in a real Hilbert space, the duality mappings reduce to the identity mapping. For more information on uniform convex spaces and other geometry of Banach spaces, see [4,15,39].
Let E_1 and E_2 be real Banach spaces and $A : E_1 \to E_2$ be a bounded linear operator. The Split Feasibility Problem (SFP) is to find a point

$$x \in C \quad \text{such that} \quad Ax \in Q, \tag{1}$$

where C and Q are nonempty closed and convex subsets of E_1 and E_2 respectively. The SFP has attracted the attention of many authors due to its application in signal processing and various algorithms have been developed for finding its solutions (see for example, [10,27,38,40,49] and references therein). The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [12] for modelling inverse problems which arises from phase retrieval, in medical image reconstruction and recently in modelling modulated radiation therapy [11].
For solving the SFP, Bryne [11] proposed the following CQ algorithm in real Hilbert spaces:

$$x_{n+1} = P_C(x_n - \mu_n A^*(I - P_Q)Ax_n), \quad n \geq 1, \tag{2}$$

where P_C and P_Q are metric projections onto closed convex subsets C and Q of H_1 and H_2 respectively and the stepsize $\mu_n \in \left(0, \frac{2}{||A||^2}\right)$. However, the determination of the stepsize μ_n depends on the operator norm $||A||$ (or the largest eigenvalue of A^*A) which is in general not an easy work in practice. It is found that the CQ algorithm is a special case of the Gradient-Projection Method (GPM) in convex minimization. We note that the SFP (1) can be formulated as a fixed point equation using the fact

$$P_C(I - \mu A^*(I - P_Q)A)w = w. \tag{3}$$

This means that w is a solution of (1) if and only if w solves the fixed point problem (3), see [30,41,48] for more details.
For solving the SFP (1) in p-uniformly convex real Banach space which are also uniformly smooth, Schöpfer et.al. [34] proposed the following algorithm: For $x_1 \in E_1$ set

$$x_{n+1} = \Pi_C J_p^{E_1^*}\left[J_p^{E_1}(x_n) - \mu_n A^* J_p^{E_2}(Ax_n - \Pi_Q(Ax_n))\right], \quad n \geq 1, \tag{4}$$

where Π_C and Π_Q are the Bregman projection onto the nonempty closed convex sets $C \subseteq E_1$ and $Q \subseteq E_2$ respectively, E_1 and E_2 are p-uniformly convex real Banach spaces which are also uniformly smooth. They proved the weak convergence of algorithm (4) under the condition that the duality mapping of E_1 is sequentially weak-to-weak continuous.

We remark here that the condition that the duality mapping of E_1 is sequentially weak-to-weak continuous excludes some important Banach spaces such as the classical L_p $(2 < p < \infty)$ spaces.

In this paper, we study the more general case of Split Minimization Problem (SMP) in real Banach spaces. Let E_1 and E_2 be real Banach spaces, $A : E_1 \to E_2$ be a bounded linear operator and $f : E_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : E_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper, convex and lower semi-continuous functions. The SMP is to find a point

$$w \in \text{argmin } f \quad \text{such that} \quad Aw \in \text{argmin } g, \tag{5}$$

where argmin $f := \{\bar{x} \in E_1 : f(\bar{x}) \leq f(x), \quad \forall x \in E_1\}$
and argmin $g := \{\bar{y} \in E_2 : g(\bar{y}) \leq g(y), \quad \forall y \in E_2\}$.

We denote the set of solutions of the SMP (5) by $S(f, g)$. If $f = i_C$ [defined as $i_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise] and $g = i_Q$ are the indicator functions of nonempty, closed and convex sets $C \subseteq E_1$ and $Q \subseteq E_2$ respectively, then the SMP (5) reduces to the SFP (1).

In a real Hilbert space H, the Moreau-Yosida approximation of a proper, convex and lower semi-continuous function $f : H \to \mathbb{R} \cup \{+\infty\}$ with parameter λ also called the proximal operator of f at x is defined by

$$prox_{\lambda f} := \underset{u \in H}{\text{argmin}}\{f(u) + \frac{1}{2\lambda}||u - x||^2\}.$$

The proximal mappings have some attractive properties that make them particularly well suited for iterative algorithms. For instance, $prox_{\lambda f}$ is firmly nonexpansive, i.e $\forall x, y \in H$,

$$||prox_{\lambda f}(x) - prox_{\lambda f}(y)||^2 \leq ||x - y||^2 - ||(x - prox_{\lambda f}(x)) - (y - prox_{\lambda f}(y))||^2,$$

and its set of fixed point is precisely the set of minimizers of f.

Recenly, Moudafi and Thakur [28] studied the SMP in the case of real Hilbert spaces. They presented the following algorithm with a way of selecting the stepsize such that its implementation does not require any prior information of the operator norm:

Algorithm I:

Let $h(x_n) = \frac{1}{2}||(I - prox_{\lambda g})Ax_n||^2$, $l(x_n) = \frac{1}{2}||(I - prox_{\mu_n \lambda f})x_n||^2$
and $\theta(x_n) = \sqrt{||\nabla h(x_n)||^2 + ||\nabla l(x_n)||^2}$. For any initialization $x_0 \in H_1$, assume that a sequence $\{x_n\} \subset H_1$ has been constructed and $\theta(x_n) \neq 0$ as follows: Compute x_{n+1} via

$$x_{n+1} = prox_{\mu_n \lambda f}(x_n - \mu_n A^*(I - prox_{\lambda g})Ax_n) \quad n \geq 0, \tag{6}$$

where the stepsize $\mu_n = \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho < 4$.
If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of the problem (5) and the iterative process stops. Otherwise, we set $n := n + 1$ and go to sequence (6)
Consequently, they proved the following weak convergence theorem.

Theorem 1. *Suppose $S(f,g) \neq \emptyset$. Assume that the parameters in Algorithm I satisfy the condition:*

$$\epsilon \leq \rho_n \leq \frac{4h(x_n)}{h(x_n) + l(x_n)} - \epsilon,$$

for some $\epsilon > 0$ small enough. Then the sequence $\{x_n\}$ generated by (6) weakly converges to a solution of SMP (5).

In [37], Shehu and Ogbuisi introduced the following algorithm and proved a strong convergence theorem for approximating the common solution of split minimization problem and fixed point problem of a nonlinear self mapping T in real Hilbert spaces: Given an initial point $x_1 \in H_1$, compute x_{n+1} via

$$\begin{cases} u_n = (1 - \alpha_n)x_n, \\ y_n = prox_{\lambda\mu_n f}\Big(u_n - \mu A^*(I - prox_{\lambda g})Au_n\Big), \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n T y_n, \end{cases} \tag{7}$$

where the step-size $\mu_n := \rho_n \dfrac{h(u_n) + l(u_n)}{\theta^2(u_n)}$ with $0 < \rho < 4$ and $\theta(x), h(x)$ and $l(x)$ are as defined in Algorithm I.
Also Abass et.al. [1] proved the strong convergence of the following two iterative algorithms for approximating the minimum norm solution of problem (5) in real Hilbert spaces. For any initial point $x_1 \in H_1$, assume that x_n has been constructed and $\theta(x_n) \neq 0$, then compute x_{n+1} by the following iterative schemes:

$$x_{n+1} = prox_{\lambda\mu_n f}\Big((1 - \alpha_n)x_n - \mu_n A^*(I - prox_{\lambda g})Ax_n\Big), \quad n \geq 1, \tag{8}$$

and

$$x_{n+1} = (1 - \alpha_n)prox_{\lambda\mu_n f}\Big(x_n - \mu_n A^*(I - prox_{\lambda g})Ax_n\Big), \quad n \geq 1, \tag{9}$$

where the stepsize $\mu_n := \rho_n \dfrac{h(x_n) + l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$ and $h(x_n), l(x_n)$ and $\theta(x_n)$ are as defined in Algorithm I and the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $a \leq \rho_n \leq \frac{4(1-\alpha_n)h(x_n)}{h(x_n)+l(x_n)} - a$ for some $a > 0$.

More recently, Shehu and Iyiola [36] introduced an algorithm involving an inertial extrapolation term for solving the split minimization problem in real Hilbert spaces. We note here that the initial extrapolation process has been helpful in accelerating the rate of convergence of iterative algorithms (please see [2, 3, 5, 8, 9, 20, 29, 31]). In particular, the authors in [36] presented the following algorithm: Given an initial point $x_0 = x_1 \in H_1$. Assume that x_n has been constructed and $\theta(y_n) = 0$, then compute x_{n+1} via the rule

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}), \\ z_n = y_n - \rho_n \frac{h(y_n)+l(y_n)}{\theta^2(y_n)} \Big(\nabla h(y_n) + \nabla l(y_n)\Big), \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n z_n, \quad n \geq 1, \end{cases} \tag{10}$$

where $0 < \rho_n < 4$ and $\theta(x) = \sqrt{||\nabla h(x) + \nabla l(x)||^2}$ with $h(x) = \frac{1}{2}||(I - prox_{\lambda g})Ax||^2$ and $l(x) = \frac{1}{2}||(I - prox_{\lambda f})x||^2$. They proved that under suitable conditions on β_n, α_n and ρ_n, the sequence generated by (10) converges weakly to a solution of (5). Several other modified algorithms of (6) have been presented for solving the SMP in real Hilbert spaces (see for instance [6, 50]). Then the following natural questions arise:

- *Can we obtain an algorithm which does not require a prior knowledge of the operator norm for solving the split minimization problem in higher Banach spaces than the Hilbert space?*
- *Also, can such an algorithm be strongly convergent?*

It is our goal in this paper to study the SMP (5) in a more general Banach space than the Hilbert space. Using the Bregman distance technique, we introduce a new iterative algorithm with a new choice of stepsize such that its implementation does not require a prior knowledge of the operator norm. This is very important because it is not easy to compute the norms of many linear operators as shown by the theorem of Hendrickx and Olshevsky [18]. We prove strong convergence of the sequence generated by our algorithm for solving problem (5) which is also fixed point of a right Bregman strongly nonexpansive mapping in p-uniformly convex Banach spaces which are also uniformly smooth. We further apply our result to approximation of solutions of split feasibility problems, split null point problems and the constrained least-square model to the inverse problem arising in signal processing. Our result extend and complement many important results in literature.

2 Preliminaries

In this section, we give some definitions and discuss some preliminary results which will be used throughout the paper. We denote the weak convergence of a sequence $\{x_n\} \subset E$ to a point $w \in E$ by $x_n \rightharpoonup w$ and the strong convergence by $x_n \to w$.

A function $\phi : E \to \mathbb{R}$ is said to be Gâteaux differentiable at $x \in E$, if there exists an element $\phi'(x) \in E^*$ such that

$$\langle \phi'(x), y\rangle = \lim_{t\to 0} \frac{\phi(x+ty)-\phi(x)}{t},$$

for every $y \in E$ and $t > 0$. We note that the function $\phi : E \to \mathbb{R}$ is Gâteaux differentiable if and only if it has a unique subgradient at x and in such case $\phi' = \partial\phi(x)$. Also in a smooth Banach space, if $\phi(x) = \frac{1}{p}||x||^p$, then the duality mapping $J_p^E(x) = \partial\phi(x)$ for any $x \in E$ and it is single-valued.
For a Gâteaux differentiable function $\phi : E \mapsto \mathbb{R}$, the function

$$D_\phi(x, y) = \phi(y) - \phi(x) - \langle \phi'(x), y - x\rangle,$$

for all $x, y \in E$ is called the Bregman distance of x to y with respect to ϕ.

Though, the Bregman distance is not a metric in the usual sense (e.g. it lacks symmetric property), but it has some distance-like properties. In smooth Banach spaces, the Bregman distance with respect to the function $\phi(x) = \frac{1}{p}||x||^p$ can be written as

$$\begin{aligned} D_p(x,y) &= \frac{1}{q}||x||^p - \langle J_p^E(x), y\rangle + \frac{1}{p}||y||^p \qquad (11)\\ &= \frac{1}{p}(||y||^p - ||x||^p) + \langle J_p^E(x), x-y\rangle \\ &= \frac{1}{q}(||x||^p - ||y||^p) - \langle J_p^E(x) - J_p^E(y), x\rangle, \quad x, y \in E. \qquad (12)\end{aligned}$$

In a Hilbert space, we have $D_2(x, y) = \frac{1}{2}||x - y||^2$.
In addition, the Bregman distance possesses the following important properties:

$$D_p(x, y) = D_p(x, z) + D_p(y, z) + \langle z - y, J_p^E(x) - J_p^E(y)\rangle, \ \ \forall x, y, z \in E,$$

and

$$D_p(x, y) + D_p(y, x) = \langle x - y, J_p^E(x) - J_p^E(y)\rangle, \ \ \forall x, y \in E.$$

The norm and Bregman distance also have the following relation

$$\tau||x - y||^p \le D_p(x, y) \le \langle x - y, J_p^E(x) - J_p^E(y)\rangle,$$

where $\tau > 0$ is some fixed number, see [34] for more details on the properties of the Bregman distance.
Let C be a nonempty closed and convex subset of a smooth Banach space E. The metric projection

$$P_C x := \underset{y\in C}{argmin}\ ||x - y||,$$

for all $x \in E$ is the unique minimizer of the norm distance which can be characterized by a variational inequality:

$$\langle J_p^E(x - P_C x), z - P_C x\rangle \le 0, \ \forall\ z \in C. \qquad (13)$$

Similarly to the metric projection, we define the Bregman projection as

$$\Pi_C x := \underset{y\in C}{argmin}\ D_p(x,y),$$

for all $x \in E$, which is the unique minimizer of the Bregman distance (see [33]). The Bregman projection is also characterized by the variational inequality:

$$\langle J_p^E(x) - J_p^E(\Pi_C x), z - \Pi_C x\rangle \le 0.\ \forall\ z \in C, \tag{14}$$

which implies that

$$D_p(\Pi_C x, z) \le D_p(x,z) - D_p(x, \Pi_C x), \tag{15}$$

for all $z \in C$.
Let E be a p-uniformly convex and uniformly smooth real Banach space. Define the function $V_p : E^* \times E \to [0,\infty)$ by

$$V_p(x,y) := \frac{1}{q}||x||^q - \langle x,y\rangle + \frac{1}{p}||y||^p,\ \forall\ x \in E^*,\ y \in E. \tag{16}$$

Then V_p is nonnegative and $V_p(x,y) = D_p(J_p^{E^*}(x), y)$ for all $x \in E^*$ and $y \in E$. Moreover, by the subdifferential inequality

$$\langle \phi'(x), y - x\rangle \le \phi(y) - \phi(x),$$

with $\phi(x) = \frac{1}{q}||x||^q$ and $x \in E^*$, then $\phi'(x) = J_q^{E^*}$. Therefore we have

$$\langle J_q^{E^*}(x), y\rangle \le \frac{1}{q}||x+y||^q - \frac{1}{q}||x||^q, \tag{17}$$

and from (17), we obtain (see [35])

$$V_p(\bar{x}+\bar{y}, x) \ge V_p(\bar{x}, x) + \langle \bar{y}, J_p^{E^*}(\bar{x}) - x\rangle, \tag{18}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$. In addition, V_p is convex in the first variable. Thus, for all $z \in E$,

$$D_p(J_q^{E^*}\sum_{i=1}^N t_i J_p^E(x_i), w) \le \sum_{i=1}^N t_i D_p(x_i, w), \tag{19}$$

where $\{x_i\} \subset E$ and $\{t_i\} \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.
Let C be a convex subset of int$dom\phi_p$, where $\phi_p(x) = (\frac{1}{p})||x||^p$, $2 \le p < \infty$ and let T be a self-mapping of C. A point $\bar{x} \in C$ is said to be asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to $\bar{x}$ and $\lim_{n\to\infty}||x_n - Tx_n|| = 0$ (see [14]). The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Definition 1. A nonlinear mapping $T : C \to C$ with a nonempty asymptotic fixed point set is said to be:
(i) Right Bregman Strongly Nonexpansive (R-BSNE) mapping with respect to a nonempty $\widehat{F}(T)$ if

$$D_p(Tx, y) \leq D_p(x, y),$$

for all $x \in C$ and $y \in F(T)$ and if whenever $\{x_n\} \subset C$ is bounded, $y \in \widehat{F}(T)$ and

$$\lim_{n\to\infty} \Big(D_p(x_n, y) - D_p(Tx_n, y) \Big) = 0,$$

it follows that

$$\lim_{n\to\infty} D_p(x_n, Tx_n) = 0.$$

According to Martin-Marquez et.al. [25, 26], a R-BSNE with respect to a nonempty $\widehat{F}(T)$ is called *strict right Bregman strongly nonexpansive mapping.*
(ii) Right Bregman Firmly Nonexpansive (R-BFNE) mapping if

$$J_p^E(Tx) - J_p^E(Ty), Tx - Ty\rangle \leq \langle J_p^E(x) - J_p^E(y), Tx - Ty\rangle, \tag{20}$$

for any $x, y \in C$ or equivalently,

$$D_p(Tx, Ty) + D_p(Ty, Tx) + D_p(x, Tx) + D_p(y, Ty) \leq D_p(x, Ty) + D_p(y, Tx). \tag{21}$$

From [25, 26], we know that every right Bregman firmly nonexpansive mapping is right Bregman strongly nonexpansive if $F(T) = \widehat{F}(T)$. For more information and examples of R-BSNE and R-BFNE operators, see [25, 26].

Let E be a p-uniformly convex and uniformly smooth real Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontiuous function, the proximal mapping associated with f with respect to the Bregman distance is defined as

$$prox_{\lambda f}(x) = \underset{w\in E}{argmin}\Big\{ f(w) + \frac{1}{\lambda} D_p(w, x)\Big\}.$$

Bauschke et.al. [7] explored some important properties of the operator $prox_{\lambda f}$. We note from [7] that

$$\text{dom } prox_{\lambda f} \subset \text{intdom } \phi \ \text{ and } \ \text{ran } prox_{\lambda f} \subset \text{dom } \phi \cap \text{dom } f,$$

where $\phi(x) = \frac{1}{p}||x||^p$ and it is convex and Gâteaux differentiable. In addition, if ran $prox_{\lambda f} \subset$ intdom ϕ, then $prox_{\lambda f}$ is R-BFNE and single-valued on its domain if $\phi|_{\text{intdom } \phi}$ is strictly convex. The set of fixed points of $prox_{\lambda f}$ are indeed the set of minimizers of f (see [7] for more details). Throughtout this paper, we shall assume that ran $prox_{\lambda f} \subset$ intdom ϕ.
We now state the following lemmas which will be used in the sequel.

Lemma 1. *(Xu [46]): Let $x, y \in E$ and $q > 1$. If a Banach space E is q-uniformly smooth, then there is a $C_q > 0$ so that*

$$||x - y||^q \leq ||x||^q - q\langle y, J_q^E(x)\rangle + C_q||y||^q. \tag{22}$$

Lemma 2. *[15] If $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for arbitrary constants $a > 0$ and $b > 0$, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{23}$$

Lemma 3. *[24] Let $\{a_n\}$ be a sequence of real numbers such that there exists a nondecreasing subsequence $\{n_i\}$ of $\{n\}$, that is, $a_{n_i} \leq a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied for all (sufficiently large) numbers $k \in \mathbb{N}$: $a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$, $m_k = \max\{j \leq k : a_j \leq a_{j+1}\}$.*

Lemma 4. [47] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + t_n\delta_n \quad \forall n \geq 0,$$

where $\{t_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that:

i. $\sum_{n=o}^{\infty} t_n = \infty$,

ii. $\limsup_{n\to\infty} \delta_n \leq 0$.

Then, $\lim_{n\to\infty} a_n = 0$.

3 Main Result

In this section, we introduce an iterative algorithm which does not require a prior knowledge of the operator norm $||A||$ for approximating a solution of SMP (5) which is also a fixed point of a R-BSNE mapping and then prove the strong convergence of the sequence generated by the algorithm in p-uniformly convex real Banach spaces which are also uniformly smooth. Before we establish our main theorem in this paper, let us prove the following lemma which will be used in proving the main theorem.

Lemma 5. *Let E be a p-uniformly convex Banach space which is uniformly smooth. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and let $prox_{\lambda f} : E \to E$ be the proximal operator associated with f for $\lambda > 0$, then the following inequalities hold:*

(i) for all $x \in E$ and $z \in F(prox_{\lambda f})$, we have

$$D_p(prox_{\lambda f}(x), z) + D_p(x, prox_{\lambda f}(x)) \leq D_p(x, z), \tag{24}$$

(ii) for all $x, z \in E$, we have

$$\langle J_p^E(x) - J_p^E(prox_{\lambda f}(x)), prox_{\lambda f}(x) - z\rangle \geq 0. \tag{25}$$

Proof. (i) By the firm nonexpansivity of $prox_{\lambda f}$, it follows from Definition 1 and (21) that for any $x, y \in E$, we have

$$\begin{aligned} D_p(prox_{\lambda f}(x), prox_{\lambda f}(y)) + D_p(prox_{\lambda f}(y), prox_{\lambda f}(x)) \\ + D_p(x, prox_{\lambda f}(x)) + D_p(y, prox_{\lambda f}(y)) \\ \leq D_p(x, prox_{\lambda f}(y)) + D_p(y, prox_{\lambda f}(x)). \end{aligned} \tag{26}$$

Putting $y = z \in F(prox_{\lambda f})$, then (26) becomes

$$\begin{aligned} D_p(prox_{\lambda f}(x), z) + D_p(z, prox_{\lambda f}(x)) + D_p(x, prox_{\lambda f}(x)) + D_p(z, z) \\ \leq D_p(x, z) + D_p(z, prox_{\lambda f}(x)), \end{aligned}$$

which implies that

$$D_p(prox_{\lambda f}(x), z) \leq D_p(x, z) - D_p(x, prox_{\lambda f}(x)). \tag{27}$$

(ii) It follows from (11) and (24) that

$$\begin{aligned} \frac{1}{q}||prox_{\lambda f}(x)||^p - \langle J_p^E(prox_{\lambda f}(x), z\rangle \leq -\langle J_p^E(x), z\rangle \\ + \langle J_p^E(x), prox_{\lambda f}(x)\rangle - \frac{1}{p}||prox_{\lambda f}(x)||^p, \end{aligned}$$

which implies that

$$\begin{aligned} \frac{1}{q}||prox_{\lambda f}(x)||^p + \frac{1}{p}||prox_{\lambda f}(x)||^p - \langle J_p^E(x), prox_{\lambda f}(x)\rangle \leq -\langle J_p^E(x), z\rangle \\ + \langle J_p^E(prox_{\lambda f}(x), z\rangle. \end{aligned} \tag{28}$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, then $p = (p-1)q$ and by Lemma (2), we have that

$$\begin{aligned} \frac{1}{p}||prox_{\lambda f}(x)||^p + \frac{1}{q}||prox_{\lambda f}(x)||^{(p-1)q} &\geq ||prox_{\lambda f}(x)||^{p-1}||prox_{\lambda f}(x)|| \\ &= ||prox_{\lambda f}(x)||^p \\ &= \langle J_p^E(prox_{\lambda f}(x), prox_{\lambda f}(x)\rangle. \end{aligned} \tag{29}$$

Therefore from (28) and (29), we have

$$\langle J_p^E(prox_{\lambda f}(x)), prox_{\lambda f}(x)\rangle - \langle J_p^E(x), prox_{\lambda f}(x)\rangle \leq \langle J_p^E(prox_{\lambda f}(x), z\rangle - \langle J_p^E(x), z\rangle,$$

which implies that

$$\langle J_p^E(prox_{\lambda f}(x)) - J_p^E(x), prox_{\lambda f}(x)\rangle \leq \langle J_p^E(prox_{\lambda f}(x)) - J_p^E(x), z\rangle,$$

thus

$$\langle J_p^E(prox_{\lambda f}(x)) - J_p^E(x), prox_{\lambda f}(x) - z\rangle \leq 0.$$

Therefore, we have

$$\langle J_p^E(x) - J_p^E(prox_{\lambda f}(x)), prox_{\lambda f}(x) - z\rangle \geq 0.$$

□

We now prove the convergence of our main theorem in this paper.

Theorem 2. *Let E_1 and E_2 be two p-uniformly convex and uniformly smooth real Banach spaces. Let C be a nonempty, closed and convex subset of E_1. Let $f : E_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : E_2 \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions and let $A : E_1 \to E_2$ be a bounded linear operator. Let T be an R-BSNE mapping from C into C such that $\hat{F}(T) = F(T)$ and $\Gamma = S(f,g) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \delta_n = 1$. Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \Pi_C\Big(prox_{\lambda f}\Big(J_q^{E_1^*}(J_p^{E_1}(x_n) - \mu_n A^* J_p^{E_2}(I - prox_{\lambda g})Ax_n)\Big)\Big), \\ x_{n+1} = \Pi_C\Big[J_q^{E_1^*}[\alpha_n J_p^{E_1^*}(u) + \beta_n J_p^{E_1}(y_n) + \delta_n J_p^{E_1}(Ty_n)]\Big], \quad n \geq 1. \end{cases} \tag{30}$$

Let the stepsize μ_n be choosen in such a way that for a small $\epsilon > 0$

$$\mu_n \in \Big(\epsilon, \Big(\frac{q||(I - prox_{\lambda g})Ax_n||^p}{C_q||A^* J_p^{E_2}(I - prox_{\lambda g})Ax_n||^q} - \epsilon\Big)^{\frac{1}{q-1}}\Big), \quad n \in \Omega, \tag{31}$$

where the index set $\Omega := \{n \in \mathbb{N} : (I - prox_{\lambda g})Ax_n \neq 0\}$ otherwise $\mu_n = t$ (t being any nonnegative value). Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = 0$,

(iii) $(1 - \alpha_n)a < \delta_n$, $\alpha_n \leq b < 1$, $a \in (0, \frac{1}{2})$.

Then $\{x_n\}$ converges strongly to $\Pi_\Gamma u$, where Π_Γ is the Bregman projection onto Γ.

Proof. Let $w \in \Gamma$. Then from (22) and (30), we have

$$\begin{aligned} D_p(y_n, w) &\leq D_p(prox_{\lambda f} J_q^{E_1^*}[J_p^{E_1}(x_n) - \mu_n A^* J_p^{E_2}(I - prox_{\lambda g})Ax_n], w) \\ &\leq D_p(J_q^{E_1^*}[J_p^{E_1}(x_n) - \mu_n A^* J_p^{E_2}(I - prox_{\lambda g})Ax_n], w) \\ &= \frac{1}{q}||J_p^{E_1}(x_n) - \mu_n A^* J_p^{E_2}(I - prox_{\lambda g})Ax_n||^q - \langle J_p^{E_1}(x_n) \\ &\quad -\mu_n A^* J_p^{E_2}(I - prox_{\lambda g})Ax_n, w\rangle + \frac{1}{p}||w||^p \\ &\leq \frac{1}{q}||J_p^{E_1}(x_n)||^q - \mu_n\langle J_p^{E_2}(I - prox_{\lambda g})Ax_n, Ax_n\rangle \\ &\quad + \frac{C_q}{q}\mu_n^q||A^* J_p^{E_2}(I - prox_{\lambda g})Ax_n||^q \\ &\quad -\langle J_p^{E_1}(x_n), w\rangle + \langle J_p^{E_2}(I - prox_{\lambda g})Ax_n, Aw\rangle + \frac{1}{p}||w||^p \\ &= \frac{1}{q}||x_n|||^q - \langle J_p^{E_1}(x_n), w\rangle + \frac{1}{p}||w||^p - \mu_n\langle J_p^{E_2}(I - prox_{\lambda g})Ax_n, Ax_n - Aw\rangle \end{aligned}$$

$$
\begin{aligned}
&+\frac{C_q}{q}\mu_n^q||A^*J_p^{E_2}(I-prox_{\lambda g})Ax_n||^q\\
=\ & D_p(x_n,w)-\mu_n\langle J_p^{E_2}(I-prox_{\lambda g})Ax_n, Ax_n-Aw\rangle\\
&+\frac{C_q}{q}\mu_n^q||A^*J_p^{E_2}(I-prox_{\lambda g})Ax_n||^q. \qquad (32)
\end{aligned}
$$

But by Lemma 5 (ii), we have

$$
\begin{aligned}
\langle J_p^{E_2}(I-prox_{\lambda g})Ax_n, Ax_n-Aw\rangle &= \langle J_p^{E_2}(I-prox_{\lambda g})Ax_n,\\
&\quad Ax_n-prox_{\lambda g}Ax_n+prox_{\lambda g}Ax_n-Aw\rangle\\
&= ||Ax_n-prox_{\lambda g}Ax_n||^p\\
&\quad +\langle J_p^{E_2}(I-prox_{\lambda g})Ax_n, prox_{\lambda g}Ax_n-Aw\rangle\\
&\geq ||Ax_n-prox_{\lambda g}Ax_n||^p. \qquad (33)
\end{aligned}
$$

Therefore from (32) and (33), we have

$$
\begin{aligned}
D_p(y_n,w) &\leq D_p(x_n,w)\\
&\quad -\mu_n\Big[||Ax_n-prox_{\lambda g}Ax_n||^p-\frac{C_q\mu_n^{q-1}}{q}||A^*J_p^{E_2}(I-prox_{\lambda g})Ax_n||^q\Big], \quad (34)
\end{aligned}
$$

and by the condition on μ_n, it follows that

$$D_p(y_n,w)\leq D_p(x_n,w). \qquad (35)$$

Also from (30) and (35), we have

$$
\begin{aligned}
D_p(x_{n+1},w) &\leq \alpha_nD_p(u,w)+\beta_nD_p(y_n,w)+\delta_nD_p(Ty_n,w)\\
&\leq \alpha_nD_p(u,w)+\beta_nD_p(y_n,w)+\delta_nD_p(y_n,w)\\
&= \alpha_nD_p(u,w)+(1-\alpha_n)D_p(y_n,w)\\
&\leq \alpha_nD_p(u,w)+(1-\alpha_n)D_p(x_n,w)\\
&\leq \max\{D_p(u,w),D_p(x_n,w)\}\\
&\ \vdots\\
&\leq \max\{D_p(u,w),D_p(x_1,w)\}. \qquad (36)
\end{aligned}
$$

Thus $D_p(x_n,w)$ is bounded and consequently, $\{x_n\}$ and $\{y_n\}$ are bounded.

Setting $w_n = J_q^{E_1^*}\Big[\alpha_nJ_p^{E_1}(u)+\beta_nJ_p^{E_1}(y_n)+\delta_nJ_p^{E_1}(Ty_n)\Big]$, for each $n\geq 1$, then from (18), we have

$$
\begin{aligned}
D_p(x_{n+1},w) &\leq D_f(J_q^{E_1^*}[\alpha_nJ_p^{E_1}(u)+\beta_nJ_p^{E_1}y_n+\delta_nJ_p^{E_1}Ty_n],w)\\
&= V_p(\alpha_nJ_p^{E_1}(u)+\beta_nJ_p^{E_1}y_n+\delta_nJ_p^{E_1}Ty_n],w)\\
&\leq V_p(\alpha_nJ_p^{E_1}(u)+\beta_nJ_p^{E_1}y_n+\delta_nJ_p^{E_1}Ty_n-\alpha_n(J_p^{E_1}(u)-J_p^{E_1}(w)),w)\\
&\quad -\langle-\alpha_n(J_p^{E_1}(u)-J_p^{E_1}(w)),J_q^{E_1^*}[\alpha_nJ_p^{E_1}(u)
\end{aligned}
$$

$$\begin{aligned}
& +\beta_n J_p^{E_1} y_n + \delta_n J_p^{E_1} T y_n] - w\rangle \\
= \;& V_p(\alpha_n J_p^{E_1}(w) + \beta_n J_p^{E_1} y_n + \delta_n J_p^{E_1} T y_n, w) \\
& +\alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(w), w_n - w\rangle \\
= \;& D_p(J_p^{E_1^*}[\alpha_n J_p^{E_1}(w) + \beta_n J_p^{E_1} y_n + \delta_n J_p^{E_1} T y_n], w) \\
& +\alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(w), w_n - w\rangle \\
= \;& \alpha_n D_p(w, w) + \beta_n D_p(y_n, w) + \delta_n D_p(T y_n, w) \\
& +\alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(w), w_n - w\rangle \\
\leq \;& \beta_n D_p(y_n, w) + \delta_n D_p(y_n, w) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(w), w_n - w\rangle \\
= \;& (1-\alpha_n) D_p(y_n, w) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(w), w_n - w\rangle \\
\leq \;& (1-\alpha_n) D_p(x_n, w) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(w), w_n - w\rangle. \qquad (37)
\end{aligned}$$

We now divide the remaining part of the proof into two cases.

Case I: Suppose that there exists $n_1 \in \mathbb{N}$ such that $\{D_p(x_n, w)\}$ is nonincreasing, then $\{D_p(x_n, w)\}$ converges and thus $D_p(x_n, w) - D_p(x_{n+1}, w) \to 0$ as $n \to \infty$. Setting $t_n = J_q^{E_1^*}\Big(\frac{\beta_n}{1-\alpha_n} J_p^{E_1}(y_n) + \frac{\delta_n}{1-\alpha_n} J_p^{E_1}(T y_n)\Big)$, then

$$\begin{aligned}
D_p(t_n, w) \;&=\; D_p\Big(J_q^{E_1^*}\Big[\frac{\beta_n}{1-\alpha_n} J_p^{E_1}(y_n) + \frac{\delta_n}{1-\alpha_n} J_p^{E_1}(T y_n)\Big], w\Big) \\
&\leq\; \frac{\beta_n}{1-\alpha_n} D_p(y_n, w) + \frac{\delta_n}{1-\alpha_n} D_p(T y_n, w) \\
&=\; \frac{\beta_n + \delta_n}{1-\alpha_n} D_p(y_n, w) \\
&=\; D_p(y_n, w). \qquad (38)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
0 \;&\leq\; D_p(x_n, w) - D_p(t_n, w) \\
&=\; D_p(x_n, w) - D_p(x_{n+1}, w) + D_p(x_{n+1}, w) - D_p(t_n, w) \\
&\leq\; D_p(x_n, w) - D_p(x_{n+1}, w) + \alpha_n D_p(u, w) + (1-\alpha_n) D_p(t_n, w) - D_p(t_n, w) \\
&=\; D_p(x_n, w) - D_p(x_{n+1}, w) + \alpha_n [D_p(u, w) - D_p(t_n, w)] \to 0, \text{ as } n \to \infty. \qquad (39)
\end{aligned}$$

Moreover

$$\begin{aligned}
D_p(t_n, w) \;&\leq\; \frac{\beta_n}{1-\alpha_n} D_p(y_n, w) + \frac{\delta_n}{1-\alpha_n} D_p(T y_n, w) \\
&=\; D_p(y_n, w) - \Big(1 - \frac{\beta_n}{1-\alpha_n}\Big) D_p(y_n, w) + \frac{\delta_n}{1-\alpha_n} D_p(T y_n, w) \\
&\leq\; D_p(x_n, w) + \frac{\delta_n}{1-\alpha_n}\Big[D_p(T y_n, w) - D_p(y_n, w)\Big]. \qquad (40)
\end{aligned}$$

Since $(1-\alpha_n)a < \delta_n$ and $\alpha_n \leq b < 1$, we have

$$a(D_p(y_n, w) - D_p(T y_n, w)) \;<\; \frac{\delta_n}{1-\alpha_n}\Big[D_p(y_n, w) - D_p(T y_n, w)\Big]$$

$$\leq \quad D_p(x_n, w) - D_p(t_n, w) \to 0, \quad \text{as} \quad n \to \infty.$$

Thus

$$D_p(y_n, w) - D_p(Ty_n, w) \to 0, \quad \text{as} \quad n \to \infty. \tag{41}$$

Since T is R-BSNE, we have that $\lim_{n\to\infty} D_p(y_n, Ty_n) = 0$, which implies that

$$\lim_{n\to\infty} ||y_n - Ty_n|| = 0. \tag{42}$$

Furthermore, from (34) and (37), we have

$$\begin{aligned}
\mu_n\Big[||Ax_n - prox_{\lambda g}Ax_n||^p - \frac{C_q\mu_n^{q-1}}{q}||A^*J_p^{E_2}(I - prox - \lambda g)Ax_n||^q\Big] \\
&\leq D_p(x_n, w) - D_p(y_n, w) \\
&\leq D_p(x_n, w) - D_p(x_{n+1}, w) + D_p(x_{n+1}, w) - D_p(y_n, w) \\
&= D_p(x_n, w) - D_p(x_{n+1}, w) \\
&+ \alpha_n[D_p(y_n, w) + \langle J_p^{E_1}(u) - J_p^{E_1}(w)\rangle].
\end{aligned}$$

Therefore, since $D_p(x_n, w) - D_p(x_{n+1}, w) \to 0$ and
$\alpha_n[D_p(y_n, w) + \langle J_p^{E_1}(u) - J_p^{E_1}(w), x_{n+1} - w\rangle] \to 0$ as $n \to \infty$, the above inequality implies that

$$\mu_n\Big[||Ax_n - prox_{\lambda g}Ax_n||^p - \frac{C_q\mu_n^{q-1}}{q}||A^*J_p^{E_2}(I - prox_{\lambda g})Ax_n||^q\Big] \to 0, \quad \text{as} \quad n \to \infty. \tag{43}$$

Using the condition on μ_n, that is,

$$\mu_n^{q-1} < \frac{q||(I - prox_{\lambda g}Ax_n||^p}{C_q||A^*J_p^{E_2}(I - prox_{\lambda g}Ax_n||^q} - \epsilon,$$

which implies that

$$\begin{aligned}
C_q\mu_n^{q-1}||A^*J_p^{E_2}(I - prox_{\lambda g})Ax_n||^q < q||(I - prox_{\lambda g})Ax_n||^p \\
- \epsilon C_q||A^*J_p^{E_2}(I - prox_{\lambda g})Ax_n||^q,
\end{aligned}$$

and then by (43), we have

$$\begin{aligned}
&\frac{\epsilon C_q}{q}||A^*J_p^{E_2}(I - prox_{\lambda g}Ax_n||^q \\
&< ||(I - prox_{\lambda g})Ax_n||^p - \frac{C_q\mu_n^{q-1}}{q}||A^*J_p^{E_2}(I - prox_{\lambda g})Ax_n||^q \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} ||A^*J_p^{E_2}(I - prox_{\lambda g})Ax_n||^q = 0. \tag{44}$$

Furthermore, we obtain from (31), (34), (37) and (44) that

$$\begin{aligned}0 \leq &\epsilon||(I - prox_{\lambda g})Ax_n||^p \leq \mu_n||(I - prox_{\lambda g})Ax_n||^p \\ &< D_p(x_n, w) - D_p(y_n, w) + \frac{C_q\mu_n^q}{q}||A^* J_p^{E_2}(I - prox_{\lambda g})Ax_n||^q \\ &\leq D_p(x_n, w) - D_p(x_{n+1}, w) + \alpha_n[D_p(y_n, w) + \langle J_p^{E_1}(u) - J_p^{E_1}(w), x_{n+1} - w\rangle] \\ &+ \frac{C_q\mu_n^q}{q}||A^* J_p^{E_2}(I - prox_{\lambda g})Ax_n||^q \to 0 \text{ as } n \to \infty,\end{aligned}$$

thus

$$\lim_{n\to\infty} ||(I - prox_{\lambda g})Ax_n|| = 0. \tag{45}$$

Now, let $z_n = J_q^{E_1^*}(J_p^{E_1}(x_n) - \mu_n A^* J_p^{E_2}(I - prox_{\lambda g})Ax_n)$.
Observe that $D_p(z_n, w) \leq D_p(x_n, w)$, then from (24), we have

$$\begin{aligned}D_p(z_n, y_n) &= D_p(z_n, prox_{\lambda f} z_n) \\ &\leq D_p(z_n, w) - D_p(y_n, w) \\ &\leq D_p(x_n, w) - D_p(y_n, w) \\ &\leq D_p(x_n, w) - D_p(x_{n+1}, w) \\ &+ \alpha_n[D_p(y_n, w) + \langle J_p^{E_1} - J_p^{E_2}, x_{n+1} - w\rangle] \to 0, \text{ as } n \to \infty.\end{aligned} \tag{46}$$

Thus

$$\lim_{n\to\infty} ||z_n - y_n|| = 0. \tag{47}$$

It then follows from the definition of z_n that

$$\begin{aligned}0 &\leq ||J_p^{E_1}(z_n) - J_p^{E_1}(x_n)|| \\ &\leq \mu_n||A^*||||J_p^{E_2}(I - prox_{\lambda g})Ax_n|| \\ &\leq \mu_n||A^*||||(I - prox_{\lambda g})Ax_n||^{p-1} \to 0 \text{ as } n \to \infty.\end{aligned} \tag{48}$$

Since $J_p^{E_2^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have that

$$\lim_{n\to\infty} ||z_n - x_n|| = 0. \tag{49}$$

Therefore, from (47) and (49) we have

$$||y_n - x_n|| \leq ||y_n - z_n|| + ||z_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{50}$$

Since $\{x_n\}$ is bounded in E_1 and E_1 is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\tilde{x}$ in E_1. By (42) and (50), it follows that $\tilde{x} \in F(T)$ since $F(T) = \hat{F}(T)$.
We now show that $\tilde{x} \in S(f, g)$. Since $z_{n_i} - x_{n_i} \to 0$ as $i \to \infty$, it follows from (47) that $\tilde{x} = prox_{\lambda f}\tilde{x}$, hence $\tilde{x}$ is a fixed point of the proximal mapping of f or equivalently $0 \in \partial f(\tilde{x})$. Thus $\tilde{x}$ is a minimizer of f.

Likewise, it follows from (45) that $A\tilde{x} = prox_{\lambda g}A\tilde{x}$, i.e $A\tilde{x}$ is a fixed point of the proximal mapping of g or equivalently $0 \in \partial g(A\tilde{x})$. Thus $A\tilde{x}$ is a minimizer of g. Hence $\tilde{x} \in S(f,g)$.
Therefore $\tilde{x} \in \Gamma = F(T) \cap S(f,g)$.

Next, we show that $\{x_n\}$ converges strongly to $\Pi_\Gamma u$.
From (50), we have

$$\begin{aligned} D_p(w_n, x_n) &= D_p(j_q^{E_1^*}\Big[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(y_n) + \delta_n J_p^{E_1}(Ty_n)\Big], x_n) \\ &\leq \alpha_n D_p(u, x_n) + \beta_n D_p(y_n, x_n) + \delta_n D_p(Ty_n, x_n) \\ &\leq \alpha_n D_p(u, x_n) + (1-\alpha_n) D_p(y_n, x_n) \to 0. \end{aligned}$$

Thus

$$\lim_{n\to\infty} ||w_n - x_n|| = 0. \tag{51}$$

Now let $x^* = \Pi_\Gamma u$, from (37) we have

$$D_p(x_{n+1}, x^*) \leq (1-\alpha_n) D_p(x_n, x^*) + \alpha_n \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), w_n - x^* \rangle. \tag{52}$$

Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^* \rangle = \lim_{j\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_j} - x^* \rangle.$$

Since $x_{n_j} \rightharpoonup \tilde{x}$, it follows from (14) that

$$\begin{aligned} \limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^* \rangle &= \lim_{j\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_j} - x^* \rangle \\ &= \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), \tilde{x} - x^* \rangle \leq 0. \end{aligned} \tag{53}$$

Since $||w_n - x_n|| \to 0$, $n \to \infty$, then

$$\limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), w_n - x^* \rangle \leq 0.$$

Hence by Lemma 4 and (52), we conclude that $D_p(x_n, x^*) \to 0$, $n \to \infty$. Therefore $x_n \to x^* = \Pi_\Gamma u$.
Case II: Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$D_p(x_{n_j}, w) < D_p(x_{n_j+1}, w),$$

for all $j \in \mathbb{N}$. Then by Lemma 3, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ with $m_k \to \infty$ as $n \to \infty$ such that

$$D_p(x_{m_k}, w) \leq D_p(x_{m_k+1}, w), \quad D_p(x_k, w) \leq D_p(x_{m_k+1}, w),$$

for all $k \in \mathbb{N}$. Following the same line of arguments as in Case I, we have that

$$\lim_{k\to\infty} ||Ty_{m_k} - y_{m_k}|| = 0, \quad \lim_{k\to\infty} ||(I - prox_{\lambda g})Ax_{m_k}|| = 0, \quad \lim_{k\to\infty} ||z_{m_k} - y_{m_k}|| = 0,$$

and

$$\limsup_{k\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(w), w_{m_k} - x^*\rangle \leq 0, \tag{54}$$

where $x^* = \Pi_\Gamma u$. From (37), we have

$$D_p(x_{m_k+1}, x^*) \leq (1-\alpha_{m_k})D_p(x_{m_k}, x^*) + \alpha_{m_k}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), w_{m_k} - x^*\rangle. \tag{55}$$

Since $D_p(x_{m_k}, x^*) \leq D_p(x_{m_k+1}, x^*)$, it follows from (55) that

$$\begin{aligned}\alpha_{m_k}D_p(x_{m_k}, x^*) &\leq D_p(x_{m_k}, x^*) - D_p(x_{m_k+1}, x^*)\\ &+ \alpha_{m_k}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), w_{m_k} - x^*\rangle\\ &\leq \alpha_{m_k}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), w_{m_k} - x^*\rangle.\end{aligned} \tag{56}$$

Since $\alpha_{m_k} > 0$, we obtain

$$D_p(x_{m_k}, x^*) \leq \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), w_{m_k} - x^*\rangle.$$

Then from (54), it follows that $D_p(x_{m_k}, x^*) \to 0$ as $k \to \infty$. This together with (55), we obtain $D_p(x_{m_k+1}, x^*) \to 0$ as $k \to \infty$. Since $D_p(x_k, x^*) \leq D_p(x_{m_k+1}, x^*)$ for all $k \in \mathbb{N}$, we have $x_k \to x^*$ as $k \to \infty$, which implies that $x_n \to x^*$ as $n \to \infty$.

Therefore from the above two cases, we conclude that $\{x_n\}$ converges strongly to $x^* = \Pi_\Gamma u$.

This completes the proof. □

Corollary 1. *Let E_1 and E_2 be two p-uniformly convex and uniformly smooth real Banach spaces. Let C be a nonempty, closed and convex subset of E_1.*
Let $f : E_1 \to \mathbb{R}\cup\{+\infty\}$ and $g : E_2 \to \mathbb{R}\cup\{+\infty\}$ be proper lower semicontinuous functions and let $A : E_1 \to E_2$ be a bounded linear operator. Let T be an R-BSNE mapping from C into C such that $\hat{F}(T) = F(T)$ and $\Gamma = S(f,g)\cap F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be sequences in $(0,1)$. For a fixed $u, x_1 \in E_1$, let $\{x_n\}$ be generated by

$$\begin{cases} y_n = \Pi_C\Big(prox_{\lambda f}\Big(J_q^{E_1^*}(J_p^{E_1}(x_n) - \mu_n A^* J_p^{E_2}(I - prox_{\lambda g})Ax_n)\Big)\Big),\\ x_{n+1} = \Pi_C J_q^{E_1^*}\Big[\alpha_n J_p^{E_1^*}(u) + (1-\alpha_n)J_p^{E_1}(Ty_n)\Big], \quad n \geq 1,\end{cases} \tag{57}$$

where the stepsize μ_n is choosen in such a way that for a small $\epsilon > 0$

$$\mu_n \in \Big(\epsilon, \Big(\frac{q||(I-prox_{\lambda g})Ax_n||^p}{C_q||A^*J_p^{E_2}(I-prox_{\lambda g})Ax_n||^q} - \epsilon\Big)^{\frac{1}{q-1}}\Big), \quad n \in \Omega, \tag{58}$$

where the index set $\Omega := \{n \in \mathbb{N} : (I - prox_{\lambda g})Ax_n \neq 0\}$ otherwise $\mu_n = t$ (t being any nonnegative value). Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty}\alpha_n = 0$,

(iii) $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n < 1$.

Then $\{x_n\}$ converges strongly to $\Pi_\Gamma u$.

Putting $p = 2 = q$, then Theorem 2 becomes:

Corollary 2. *Let E_1 and E_2 be two real Hilbert spaces with subsets C and Q respectively. Let $f : E_1 \to \mathbb{R}\cup\{+\infty\}$ and $g : E_2 \to \mathbb{R}\cup\{+\infty\}$ be proper lower semicontinuous functions and let $A : E_1 \to E_2$ be a bounded linear operator. Let T be an R-BSNE mapping from C into C such that $\hat{F}(T) = F(T)$ and $\Gamma = S(f, g) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \delta_n = 1$. Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = P_C\Big(prox_{\lambda f}\Big(x_n - \mu_n A^*(I - prox_{\lambda g})Ax_n\Big)\Big), \\ x_{n+1} = P_C\Big[\alpha_n u + \beta_n y_n + \delta_n T y_n\Big], \quad n \geq 1. \end{cases} \tag{59}$$

Let the stepsize μ_n be choosen in such a way that for a small $\epsilon > 0$

$$\mu_n \in \Big(\epsilon, \frac{2||(I - prox_{\lambda g})Ax_n||^2}{||A^*(I - prox_{\lambda g})Ax_n||^2} - \epsilon\Big), \quad n \in \Omega, \tag{60}$$

where the index set $\Omega := \{n \in \mathbb{N} : (I - prox_{\lambda g})Ax_n \neq 0\}$ otherwise $\mu_n = t$ (t being any nonnegative value). Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = 0$,

(iii) $(1 - \alpha_n)a < \delta_n$, $\alpha_n \leq b < 1$, $a \in (0, \frac{1}{2})$.

Then $\{x_n\}$ converges strongly to $P_\Gamma u$, where P_Γ is the metric projection onto Γ.

4 Applications

In this section, we give applications of our main result to approximation of solutions of some other nonlinear problems.

4.1 Split Feasibility Problems

Taking $f = i_C$ and $g = i_Q$ the indicator functions of the nonempty closed and convex sets $C \subseteq E_1$ and $Q \subseteq E_2$ respectively, then the SMP (5) reduces to the SFP (1). Thus, we have the following theorem for approximating common solution of SFP and fixed point problem of R-BSNE mappings.

Theorem 3. *Let E_1 and E_2 be two p-uniformly convex and uniformly smooth real Banach spaces. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_2$ be a bounded linear operator.*
Suppose $\Theta := \{x \in C : Ax \in Q\}$ and let $T : E_1 \to E_2$ be an R-BSNE mapping such

that $\hat{F}(T) = F(T)$ *and* $\Gamma = \Theta \cap F(T) \neq \emptyset$. *Let* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\delta_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \delta_n = 1$. *Let the sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} y_n = P_C\Big[J_q^{E_1^*}(J_p^{E_1}(x_n) - \mu_n A^* J_p^{E_2}(I - P_Q)Ax_n)\Big], \\ x_{n+1} = \Pi_C J_q^{E_1^*}\Big[\alpha_n J_p^{E_1^*}(u) + \beta_n J_p^{E_1}(y_n) + \delta_n J_p^{E_1}(Ty_n)\Big], \quad n \geq 1. \end{cases} \tag{61}$$

Let the stepsize μ_n *be choosen in such a way that for a small* $\epsilon > 0$

$$\mu_n \in \Big(\epsilon, \Big(\frac{q||(I - P_Q)Ax_n||^p}{C_q||A^* J_p^{E_2}(I - P_Q)Ax_n||^q} - \epsilon\Big)^{\frac{1}{q-1}}\Big), \quad n \in \Omega, \tag{62}$$

where the index set $\Omega := \{n \in \mathbb{N} : (I - P_Q)Ax_n \neq 0\}$ *otherwise* $\mu_n = t$ *(*t *being any nonnegative value). Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = 0$,

(iii) $(1 - \alpha_n)a < \delta_n$, $\alpha_n \leq b < 1$, $a \in (0, \frac{1}{2})$.

Then $\{x_n\}$ *converges strongly to* $\Pi_\Gamma u$.

4.2 Split Null Point Problem

Let $A : E_1 \to E_2$ be a bounded linear operator. Let $N : E_1 \to 2^{E_1}$ and $M : E_2 \to 2^{E_2}$ be maximal monotone operators. The split null point problem (SNPP) is to find

$$x^* \in N^{-1}(0) \quad \text{such that} \quad Ax^* \in M^{-1}(0). \tag{63}$$

Several iterative methods have been introduced to approximate the solution of SNPP and related optimization problems in real Hilbert and Banach spaces, see [19,21,22, 43] and the references therein. The resolvent operator $Res_{\lambda M} : E \to 2^E$ associated with a maximal monotone operator M for $\lambda > 0$ is defined by

$$Res_{\lambda M}(x) = \{z \in E : J_p^E(x) \in J_p^E(z) + \lambda M(z)\}.$$

Equivalently, $Res_{\lambda M}(x) := (J_p^E + \lambda M)^{-1} J_p^E(x)$, for all $x \in E$. Moreover, $Res_{\lambda M}$ is single-valued and also $N^{-1}(0) = F(Res_{\lambda M})$ (see Section 5 in [42]). We shall denote the set of solutions of SNPP (63) by $SNPP(N, M)$. It is well known that the resolvent operator $Res_{\lambda M}$ is BFNE, that is

$\langle J_p^E(Res_{\lambda M}(x)) - J_p^E(Res_{\lambda M}(y)), Res_{\lambda M}(x) - Res_{\lambda M}(y)\rangle$
$\leq \langle J_p^E(x) - J_p^E(y), Res_{\lambda M}(x) - Res_{\lambda M}(y)\rangle$, for all $x, y \in C$ (see [32]).

Taking $f = N$ and $g = M$ the maximal monotone operators in E_1 and E_2 respectively, we have the following theorem for approximating solutions of SNPP in real Banach spaces.

Theorem 4. *Let E_1 and E_2 be two p-uniformly convex and uniformly smooth real Banach spaces. Let C and Q be nonempty, closed and convex subsets of E_1 and E_2 respectively. Let $A : E_1 \to E_2$ be a bounded linear operator, $N : E_1 \to 2^{E_1}$ and $M : E_2 \to 2^{E_2}$ be maximal monotone operators and $T : E_1 \to E_2$ be an R-BSNE mapping such that $\hat{F}(T) = F(T)$. Suppose $\Gamma = SNPP(N, M) \cap F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \delta_n = 1$. Let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \Pi_C\Big(Res_{\lambda N}\Big(J_q^{E_1^*}(J_p^{E_1}(x_n) - \mu_n A^* J_p^{E_2}(I - Res_{\lambda M})Ax_n)\Big)\Big), \\ x_{n+1} = \Pi_C J_q^{E_1^*}\Big[\alpha_n J_p^{E_1^*}(u) + \beta_n J_p^{E_1}(y_n) + \delta_n J_p^{E_1}(Ty_n)\Big], \quad n \geq 1. \end{cases} \tag{64}$$

Let the stepsize μ_n be choosen in such a way that for a small $\epsilon > 0$

$$\mu_n \in \Big(\epsilon, \Big(\frac{q||(I - Res_{\lambda M})Ax_n||^p}{C_q||A^* J_p^{E_2}(I - Res_{\lambda M})Ax_n||^q} - \epsilon\Big)^{\frac{1}{q-1}}\Big), \quad n \in \Omega, \tag{65}$$

where the index set $\Omega := \{n \in \mathbb{N} : (I - Res_{\lambda M})Ax_n \neq 0\}$ otherwise $\mu_n = t$ (t being any nonnegative value). Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = 0$,

(iii) $(1 - \alpha_n)a < \delta_n$, $\alpha_n \leq b < 1$, $a \in (0, \frac{1}{2})$.

Then $\{x_n\}$ converges strongly to $\Pi_\Gamma u$.

4.3 Inverse Problem in Signal Processing

Many problems in signal and image processing can be formulated as inverting the equation system

$$z = Ax + \varepsilon, \tag{66}$$

where $x \in \mathbb{R}^N$ are the data to be recovered, $z \in \mathbb{R}^M$ is the vector of noisy observations (or measurements) and ε is an additive noise with bounded variance, $A : \mathbb{R}^N \to \mathbb{R}^M$ is a bounded linear observation operator which is typically ill behaved because it models an acquisition process that encounters loss of information. Two typical problems covered by model (66) are:

(i) Compressed sensing, where A is an $M \times N$ matrix and one only takes $M < N$ measurements of the input signal M.

(ii) Deconvolution or wavelet-based signal restoration, where the operator A can be written as $A = RW$ such that R is the convolution by blurring kernel and W represents an inverse wavelet transformation.

When attempting to find sparse solutions to linear inverse problems of type (66), a successful model is the convex unconstrained minimization problem

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}||z - Ax||^2 + \nu||x||_1, \tag{67}$$

where ν is a postive number, $||\cdot||$ is the Euclidean norm and $||\cdot||_1$ is the l_1 norm. The aim of the l_1 term, which is the convex sparsity-promoting penalty, is to make the small component of x become zero. By means of convex analysis, one is able to show that a minimizer to (67) is actually a solution to the constrained least-square problem

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}||z - Ax||^2 \quad \text{subject to} \quad ||x||_1 < t, \tag{68}$$

for any nonnegative real number t (see [17]). We note that problem (68) is a particular case of SFP (1) where $C = \{x \in \mathbb{R}^N : ||x||_1 \leq t\}$ and $Q = \{z\}$, that is, find $||x||_1 \leq t$ such that $Ax = z$. The projection onto the closed l_1 ball in $\mathbb{R}^N$ of radius t can be computed through soft thresholding:

$$\mathbb{S}_\lambda(u) = \underset{x\in\mathbb{R}^N}{argmin}\ \{||x - u||^2 + 2\lambda||x||_1\},$$

where λ is a certain positive real number and $\mathbb{S}_\lambda$ is the soft thresholding operator with threshold λ defined by $(\mathbb{S}_\lambda(y))_i = S_\lambda(y_i)$, with $1 \leq i \leq N$ with

$$\mathbb{S}_\lambda(y) = \begin{cases} y + \tau, & y < -\lambda, \\ 0, & -\tau \leq y \leq \lambda, \\ y - \tau, & \lambda < y. \end{cases} \tag{69}$$

See [16, 17] for more information on the soft thresholding. Several optimization algorithms have been developed to solve (68) (see, for example [23,44] and references therein).

Let E be a finite − dimensional Hilbert space (typically, the real vector space $\mathbb{R}^N$) equipped with the inner product $\langle\cdot,\cdot\rangle$. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. When $f(x) = \lambda||x||_1$ where $\lambda > 0$, the proximal operator associated with f is given as

$$prox_f(x) = \underset{z\in E}{argmin}\ \{\frac{1}{2}||z - x||^2 + \lambda||z||_1\}.$$

The optimality condition becomes

$$0 \in \nabla(||z - x||^2) + \partial(\lambda||z||_1) \Leftrightarrow 0 \in z - x + \lambda\partial||z||_1. \tag{70}$$

We now consider two cases for the components of the l_1-norm.

Case 1: Suppose $z_i = 0$, then the subdifferential of the l_1-norm is the interval $[-1, 1]$, thus (70) becomes

$$0 \in -x_i + \lambda[-1, 1] \Leftrightarrow x_i \in [-\lambda, \lambda]$$

$$\Leftrightarrow |x_i| \leq \lambda. \tag{71}$$

Case 2: Assume $z_i \neq 0$, then $\partial||z_i||_1 = sign(z_i)$, where $sign$ is the signum function, that is,

$$sign(z_i) = \begin{cases} 1, & if\ z_i > 0, \\ 0, & if\ z_i = 0, \\ -1, & if\ z_i < 0. \end{cases}$$

Then we have from (70) that

$$0 = z_i - x_i + \lambda sign(z_i) \Leftrightarrow z_i^* = x_i - \lambda sign(z_i^*), \tag{72}$$

where z_i^* is the optimum point of f. Now if $z_i^* > 0$, then $x_i > \lambda$ and if $z_i < 0$, then $x_i < -\lambda$. Thus $|x_i| > \lambda$ and $sign(z_i^*) = sign(x_i)$. Substituting this in (72) yields

$$z_i^* = x_i - \lambda sign(x_i). \tag{73}$$

Therefore from (71) and (73), we have that

$$\begin{aligned} [prox_f(x)]_i = z_i^* &= \begin{cases} 0, & \text{if } |x_i| \leq \lambda, \\ x_i - \lambda sign(x_i), & \text{if } x_i > \lambda, \end{cases} \\ &= sign(x_i) \max \Big(|x_i| - \lambda, 0 \Big). \end{aligned}$$

This shows that the proximal operator of f is indeed the soft-thresholding operator with threshold λ. Also $\mathbb{S}_\lambda$ is an example of left Bregman strongly nonexpansive mapping (see [26]).
Choosen $C = \{x \in \mathbb{R}^N : ||x||_1 \leq t\}$, $Q = \{z\}$, $f(x) = \lambda||x||_1$ and $g(z) = z$ in Theorem 2, we have the following result for approximating the solution of the constrained least-square problem (68).

Theorem 5. *Let E_1 and E_2 be real Hilbert spaces (typically, the real vector space $\mathbb{R}^N$) and let A be as defined in* (66). *Let $C = \{x \in \mathbb{R}^N : ||x||_1 \leq t\}$, $Q = \{z\}$, $f(x) = \lambda||x||_1$ and $g(z) = z$, where $z \in \mathbb{R}^M$ is an observation vector generated by $z = Ax + \varepsilon$ with noise ε whose variance is bounded and $M < N$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \delta_n = 1$. Choose some initial values $x_1 \in \mathbb{R}^N$ and $u \in \mathbb{R}^N$ and let $\{x_n\}$ be generated iteratively by*

$$\begin{cases} y_n = \mathbb{S}_\lambda(x_n - \mu_n A^*(z - Ax_n)), \\ x_{n+1} = \mathbb{S}_\lambda(\alpha_n u + \beta_n y_n + \delta_n \mathbb{S}_\lambda y_n), \quad n \geq 1, \end{cases} \tag{74}$$

where the stepsize μ_n is choosen in such a way that for a small $\epsilon > 0$

$$\mu_n \in \Big(\epsilon, \frac{2||(z - Ax_n)||^2}{||A^*(z - Ax_n)||^2} - \epsilon\Big), \quad n \in \Omega,$$

where the index set $\Omega := \{n \in \mathbb{N} : z - Ax_n \neq 0\}$ otherwise $\mu_n = t$ (t being any nonnegative value). Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = 0$,

(iii) $(1-\alpha_n)a < \delta_n$, $\alpha_n \leq b < 1$, $a \in (0, \frac{1}{2})$.

Then $\{x_n\}$ *converges strongly to* $\mathbb{S}_\lambda u$ *which is the solution to the least-square problem* (68).

4.4 Numerical Example

We now present a numerical experiment to demonstrate the performance of our algorithm (74). We consider the following simple numerical example to show how the change in initial values affects the number of iterations.

Let $E_1 = \mathbb{R}^N = E_2$, consider $C = \{x \in \mathbb{R}^N : ||x||_1 \leq t\}$ and $Q = \{z\}$. Let $f(x) = ||x||_1$ and $g(z) = z$, then the soft thresholding is given by

$$prox_f(y) = \mathbb{S}_1(x) = \begin{cases} x+1, & x < -1, \\ 0, & |x| \leq 1, \\ x-1, & 1 < x. \end{cases}$$

Choose $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{2n}{3(n+1)}$, and $\delta_n = \frac{n}{3(n+1)}$, then algorithm (74) becomes

$$\begin{cases} y_n = \mathbb{S}_1(x_n - \mu_n A^T(z - Ax_n)), \\ x_{n+1} = \mathbb{S}_1(\frac{1}{n+1}u + \frac{2n}{3(n+1)}y_n + \frac{n}{3(n+1)}\mathbb{S}_\lambda y_n), \quad n \geq 1. \end{cases}$$

Let $Ax = x$, and $z = randn(N,1)$ be random generated vectors whose elements are normally distributed, we make different choices of N as follow: $N = 1000$, $N = 5000$ and $N = 10000$.
Case 1: $u = randn(N,1)$ and $x_1 = randn(N,1)$.
Case II: $u = 3 \times randn(N,1)$ and $x_1 = 0.5 \times randn(N,1)$.
We use $\dfrac{||x_{n+1} - x_n||}{||x_2 - x_1||} < 10^{-6}$ as stopping criterion.

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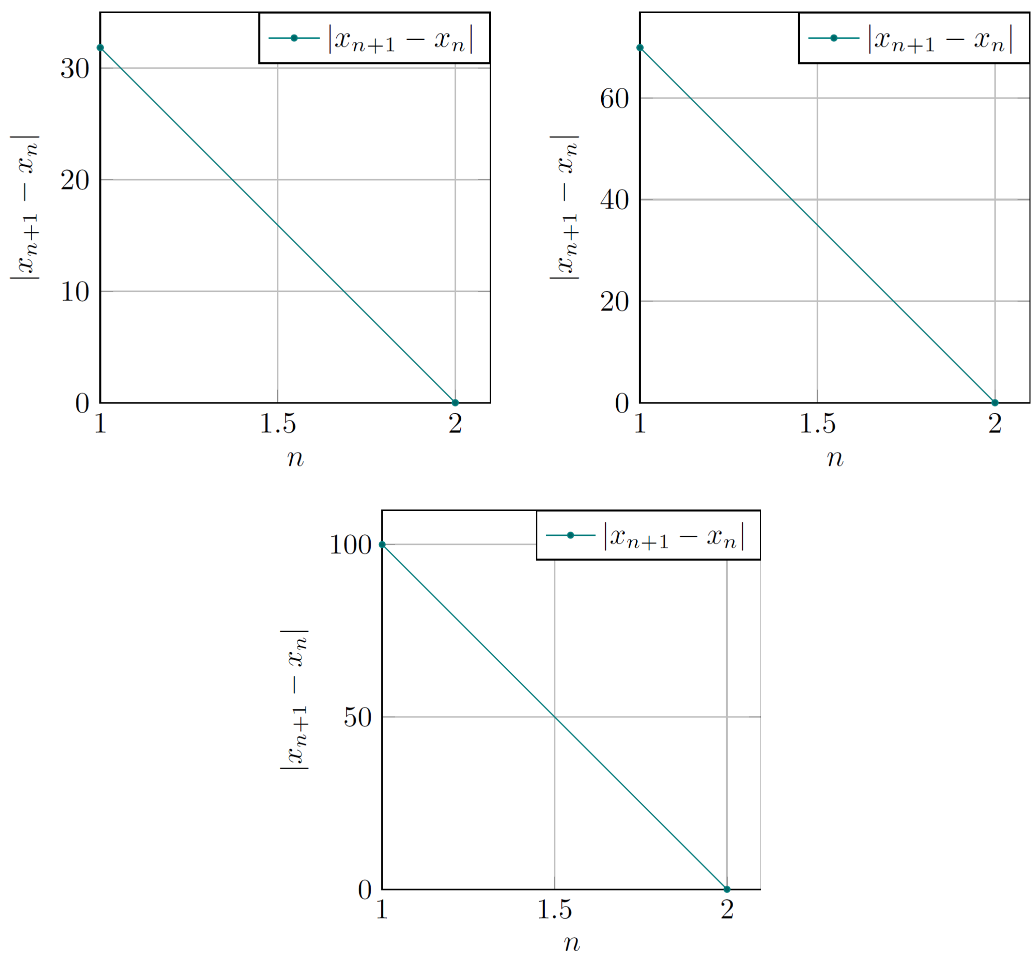

Figure 1. Case I: errors vs number of iterations: $N = 1000$, 0.0027sec (top-left), $N = 5000$, 0.0071sec (top-right), $N = 10000$, 0.0232sec (bottom).

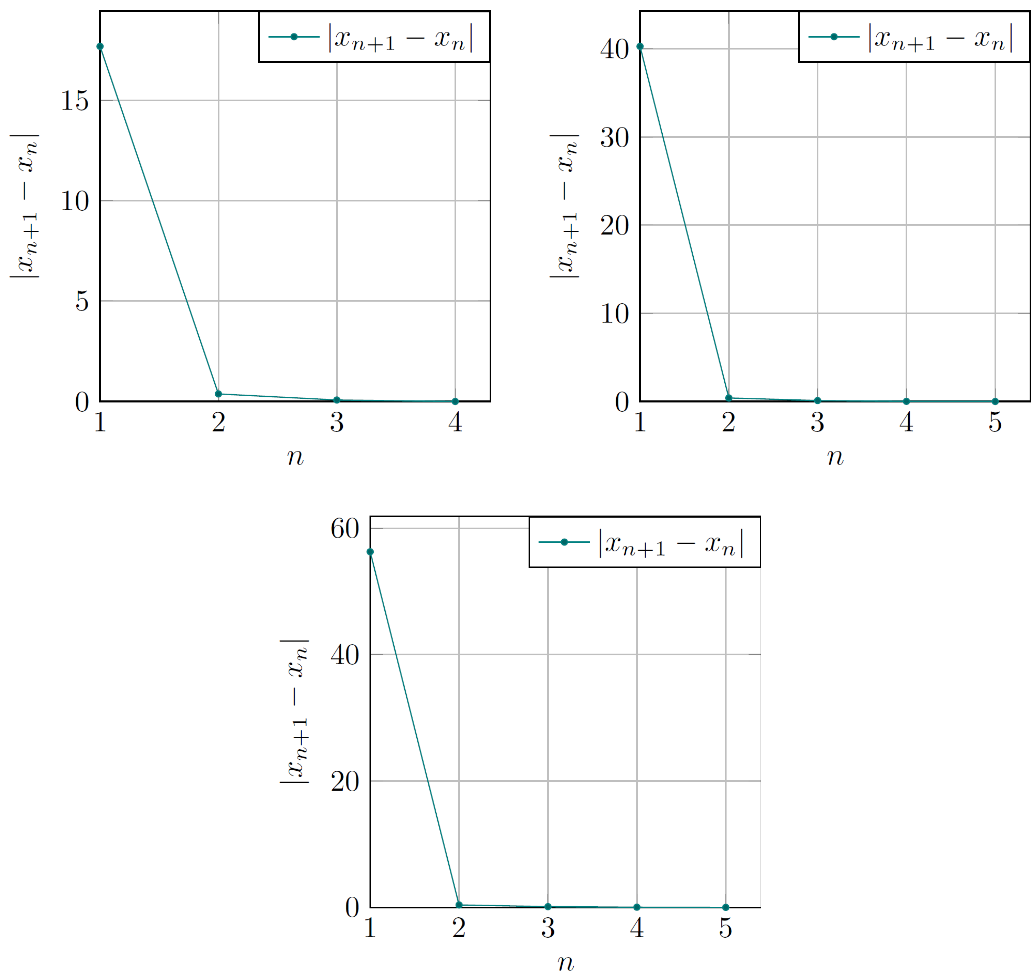

Figure 2. Case I: errors vs number of iterations: $N = 1000$, 0.0038sec (top-left), $N = 5000$, 0.0157sec (top-right), $N = 10000$, 0.0565sec (bottom).

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