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Localization of singular points of meromorphic functions based on interpolation by rational functions

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Abstract. In this paper we examine two algorithms for localization of singular points of meromorphic functions. Both algorithms apply approximation by interpolation with rational functions. The first one is based on global interpolation and gives the possibility to determine the singular points of the function on a domain that includes a simple closed contour on which the values of the function are known. The second algorithm, based on piecewise interpolation, establishes the poles and the discontinuity points on the contour where the function values are given.

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1 Introduction and problem formulation

Let consider a function $f(z)$ of a complex variable z, which is meromorphic on a finite domain $\Omega \subset \mathbb{C}$ so that the point $0 \in \Omega$. The finite values $f_j := f(z_j)$ of the function $f(z)$ are known on the set $\{z_j\}$, where the points z_j belong to a simple closed contour $\Gamma \subset \Omega$, that contains the point $z = 0$. We admit that points z_j form a dense set on Γ. The function $f(z)$ has a finite number of singular points of polar type on the domain Ω, but their number $s < \infty$ and locations z_j^p are not known. Also, on the contour Γ the function $f(z)$ can have both poles and jump discontinuity points (which can be considered as removable singularities). If the function $f(z)$ has poles on the contour Γ, in order to avoid the computation difficulties, we consider that the values of the function $f(z)$ at the points $z_j \in \Gamma^\rho$ are given (here Γ^ρ represents a small perturbation of the contour Γ). We aim to determine the locations of the singular points on Ω, in particular those on the contour Γ.

An algorithm for the localization of singularities is proposed in [1]. This algorithm is based on algebraic relations, which were obtained from the residue theorem and the properties of Laurent coefficients and it can be applied in the case when the contour Γ coincides with the circle $\Gamma_0 := \{t \in \mathbb{C} : |t| = 1\}$ and the poles of the function $f(z)$ belong to the domain inside of Γ_0.

In paper [2] the function $f(z)$ is expressed as $B(z)/A(z)$, and the coefficients of the functions $B(z)$ and $A(z)$ are determined as a solution of the linear least-squares problem $\sum_{i=1}^{M} |A(s_i) f(s_i) - B(s_i)|^2 \to \min$, where $s_i \in \Gamma_0$. The zeros of the function $A(z)$ are the poles of the function $f(z)$.

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In paper [3] an algebraic method for the localization of simple poles of the meromorphic function is proposed. This method is based on the values of the function on a simple arc from the domain where the function is defined. The algorithm handles both the case when the arc is closed and the case when the arc is open. The accuracy of the pole approximations depends on whether the arc includes inside the poles of the function and on the distance of the poles from the points of the arc.

In our earlier works we have already examined the Padé approximation with Laurent polynomials [4] and the Padé approximation with Faber polynomials [5]. In both cases we have considered the meromorphic functions on a finite domain of the complex plane with given values at the points of a simple closed contour from this domain. On the basis of the obtained results the numerical algorithms have been proposed for the localization of poles of the function, including jump discontinuity points on the contour.

In this paper we propose two algorithms for localizing the singular points of meromorphic function $f(z)$, both are based on the approximation by rational functions. For $M, N \in \mathbb{N}$ let $\mathcal{R}_{N,M}$ be the set of rational functions of order (N, M), $r_{N,M}(z) = P_N(z)/Q_M(z)$, where $P_N(z)$ and $Q_M(z)$ are polynomials of degree M and, respectively, N (we denote by $\mathcal{P}_n$ the set of polynomials of degree at most n). According to the interpolation problem with rational functions, we determine a function $r_{N,M} \in \mathcal{R}_{N,M}$, that takes the given values $f_1, ..., f_n$ at distinct points $z_1, ..., z_n$, where $n = M + N + 1$.

If $M = 0$, then one obtains the problem of polynomial interpolation and it is known that its solution exists and is unique. But for arbitrarily fixed M and N, there may not exist a rational function $r_{N,M} \in \mathcal{R}_{N,M}$, that satisfies the conditions of the interpolation problem [6]. In the case when the function $f(z)$ is meromorphic inside or outside the unit circle $\Gamma_0 := \{z \in \mathbb{C} : \ |z| = 1\}$ and the number of its poles M is known, we find out from [7, 8] that for large enough N there exists a unique rational function $r_{N,M} \in \mathcal{R}_{N,M}$ satisfying the interpolation conditions $r_{N,M}(z_j) = f_j, \ j = 1, ..., N + M + 1$ at the points $z_j = e^{2\pi j i/(N+M+1)} \in \Gamma_0$. For values $N \to \infty$, theoretically the poles of the rational function $r_{N,M}(z)$ converge to poles of the function $f(z)$, accordingly. For $N \to \infty$ the sequence $r_{N,M}(z)$ converges to $f(z)$ on the domain Ω', obtained from the domain Ω where the meromorphic function is defined, by eliminating the poles of $f(z)$. The convergence is uniform on any closed subset of Ω'.

For fixed values N, M we can consider the linearized problem of interpolation with rational functions, according to which we determine the polynomials $P_N \in \mathcal{P}_N$ and $Q_M \in \mathcal{P}_M$ that satisfy the relation

$$f(z_j)\, Q_M(z_j) - P_N(z_j) = 0, \ \ j = \overline{1, N + M + 1}. \tag{1}$$

Relation (1) is a system of $N+M+1$ homogeneous linear equations with $N+M+2$ unknowns, that has a non-trivial solution. However, if $Q_M(z_j) = P_N(z_j) = 0$ for some $z_j, \ j \in \{1, ..., N + M + 1\}$, then the solution of the system (1) may not define a solution of the interpolation problem with a rational function. If $Q_M(z_j) \neq 0, \ j =$

$1, ..., N+M+1$, then the function $r_{N,M}(z) = P_N(z)/Q_M(z)$ $(\in \mathcal{R}_{N,M})$ satisfies the interpolation conditions $r_{N,M}(z_j) = f_j,\ j = 1, ..., N+M+1$.

Next, in order to determine the poles of the function $f(z)$ on the domain Ω, an approximation algorithm by interpolation with a rational function on Ω is proposed. Also, the poles and discontinuity points which belong to the contour Γ are determined by piecewise approximation on Γ.

2 Localization of singular points based on approximation by global interpolation with rational functions

The poles of the function $f(z)$ on the domain Ω can be determined if it is approximated with a rational function $R(z) := P_N(z)/Q_M(z)$ on Ω, where $P_N(z)$ and $Q_M(z)$ are the polynomials $P_N(z) = \sum_{k=0}^{N} p_k z^k$, $Q_M(z) = \sum_{r=0}^{M} q_r z^r$, and $N, M \in \mathbb{N},\ N \geq M$. To ensure the existence and uniqueness of the rational function $R(z)$ we consider that $q_0 = 1$. The zeros of the polynomial $Q_M(z)$ are approximations of the poles of the function $f(z)$ on Ω.

The coefficients $(p_k)_{k=0}^{N}$ and $(q_r)_{r=1}^{M}$ of the rational function $R(z)$ are determined by applying to the function $f(z)$ the interpolation procedure on the set of nodes $z_j \in \Gamma,\ j = \overline{1, N+M+1}$:

$$R(z_j) = f_j,\ j = \overline{1, N+M+1}. \tag{2}$$

The interpolation conditions (2) are written in linearized form (1) or explicitly in the following form

$$\sum_{k=0}^{N} p_k (z_j)^k - f(z_j) \sum_{r=1}^{M} q_r (z_j)^r = f(z_j),\ \ j = \overline{1, N+M+1}. \tag{3}$$

Relations (3) represent a system $A\bar{\alpha} = \bar{b}$ that contains $N+M+1$ linear algebraic equations and $N+M+1$ unknowns. If we consider the vector of the unknowns in the form $\bar{\alpha} = (p_0, p_1, ..., p_N, q_1, ..., q_M)^T$, then the lines of the coefficient matrix of the system can be written in the form

$$\left(1,\ z_j,\ (z_j)^2,\ \dots,\ (z_j)^N,\ -f(z_j)\, z_j, -f(z_j)\,(z_j)^2,\ \dots, -f(z_j)\,(z_j)^M\right),$$

and the right member of the system $\bar{b} = (f_1, ..., f_{M+N+1})^T$.

Thereafter, we determine the M zeros of the polynomial $Q_M(z) = \sum_{r=0}^{M} q_r z^r$, where $q_0 = 1$, and $q_1, ..., q_M$ are the last M components of the solution $\bar{\alpha}$ to the system (3).

Since the polynomial equation $Q_M(z) = 0$ can have multiple roots, and standard iterative methods such as the Newton method are not well suited for calculating multiple roots (numerical accuracy may be low), we intend to apply a specialized algorithm [9]. This algorithm ensures high accuracy in calculating the roots of polynomial equations, including for multiple roots and their multiplicities. The algorithm is derived from the following considerations. Since the multiple zeros of the polynomial $Q_M(z)$ are also zeros of the polynomial $Q'_M(z)$ (here $Q'_M(z)$ is the derivative of $Q_M(z)$), it follows that all the multiple zeros of the polynomial $Q_M(z)$ are zeros of the polynomial $GCD(z)$, that is the greatest common divisor of the polynomials $Q_M(z)$ and $Q'_M(z)$. Therefore, if the polynomial $Q_M(z)$ is divided by the $GCD(z)$, then a polynomial $q(z)$ is obtained, that has M simple zeros that coincide with the zeros of $Q_M(z)$. Thus, for finding the roots of the polynomial equation $Q_M(z) = 0$ the following algorithm can be applied:

1. Determine the derivative $Q'_M(z)$ of the polynomial $Q_M(z)$.
2. Determine the greatest common divisor $GCD(z)$ of the polynomials $Q_M(z)$ and $Q'_M(z)$ by applying Euclid's algorithm for polynomials.
3. Determine the polynomial $q(z)$ by dividing $Q_M(z)$ by $GCD(z)$.
4. Determine the roots $\tilde{z}_k$, $k = \overline{1, M_1}$ $(M_1 \le M)$ of the polynomial equation $q(z) = 0$, applying a standard iterative method (such as Newton). The obtained values for roots are equal to distinct roots of the equation $Q_M(z) = 0$.
5. Establish the multiplicities of the roots $\tilde{z}_k$ of the equation $Q_M(z) = 0$, comparing the absolute values of the polynomial $Q_M(z)$ and its derivative at the points $\tilde{z}_k$ with a sufficiently small value $\delta_2 > 0$:

$$\boxed{\begin{array}{l} nQ := \|Q_M(z)\|_2;\ vQ := |Q_M(\tilde{z}_k)|; \\ m := 0; \\ \text{while } vQ < nQ * \delta_2 \\ Q_M(z) := Q'_M(z);\ vQ := |Q_M(\tilde{z}_k)|;\ m := m + 1; \\ \text{end} \end{array}}$$

We consider that the contour Γ is defined, using the Riemann mapping $z = \psi(w)$, which performs a conformal map of the outside of the circle Γ_0 on the outside of Γ such that $\psi(\infty) = \infty$, $\psi'(\infty) > 0$. It is known that the function $\psi(w)$ transforms the unit circle Γ_0 onto the contour Γ.

The method for the localization of singular points of the function $f(z)$, based on the approximation by interpolation with a rational function $R(z)$, can also be applied to localize the discontinuity points of the function $f(z)$ on Γ. But numerical experiments show that, unlike when we have only poles, higher values for the parameter M are needed to find the discontinuity points.

The simplest approach in computing the number of poles M of the function $f(z)$ is to check the results generated for different values M, starting with a certain value that is incremented if the previous result is not satisfactory. But if we take into account that the problem of numerical computation of the solution to the system (3) is generally poorly conditioned, amplifying the degree M of the polynomial $Q_M(z)$ usually leads to the appearance of spurious poles. However, most of the spurious poles can be removed by applying the residual analysis procedure [10], according to

which the spurious poles have residues that differ significantly from the residues of the true poles (residues are very small or much larger than those for true poles).
Example 1. Consider the Riemann function $z = \psi(w)$ that performs the conformal map of the set $\{w \in \mathbb{C} : \ |w| > 1\}$ on the domain Ω^- from the outside of the contour Γ and $\psi(w) = w + 1/\left(3w^3\right)$. Thus, $\psi(w)$ transforms the circle Γ_0 on the astroid Γ.

The function of a complex variable $f(z)$ is defined as follows:

$$f(z) = \frac{\cos(2z)}{(z - zc_1)(z - zi_1)(z - ze_1)(z - ze_2)},$$

where $zc_1 = \psi(1)$, $zi_1 = 0.2$, $ze_1 = \psi\left(1.5e^{\pi i/6}\right)$, $ze_2 = \psi\left(-1.2e^{2\pi i/5}\right)$ are four simple poles (one inside Γ, one on the contour Γ and two poles outside Γ) of $f(z)$, $i^2 = -1$.

We consider that there are given values f_k of the examined function $f(z)$ at the points

$$z_k = \psi\left(\rho e^{i\theta_k}\right) \in \Gamma^\rho, \ \theta_k = 2\pi(k-1)/m, \ m \in \mathbb{N}, \ m \geq N + M + 1, \ k = 1, ..., m,$$

where N and M are the degrees of the polynomials $P_N(z)$ and, correspondingly, $Q_M(z)$, which define the rational function $R(z)$. The approximations we obtained for the poles of the function $f(z)$ are presented in Figure 1 and Table 1. Here we use the values $N = 9$, $M = 5$, $\rho = 1.01$.

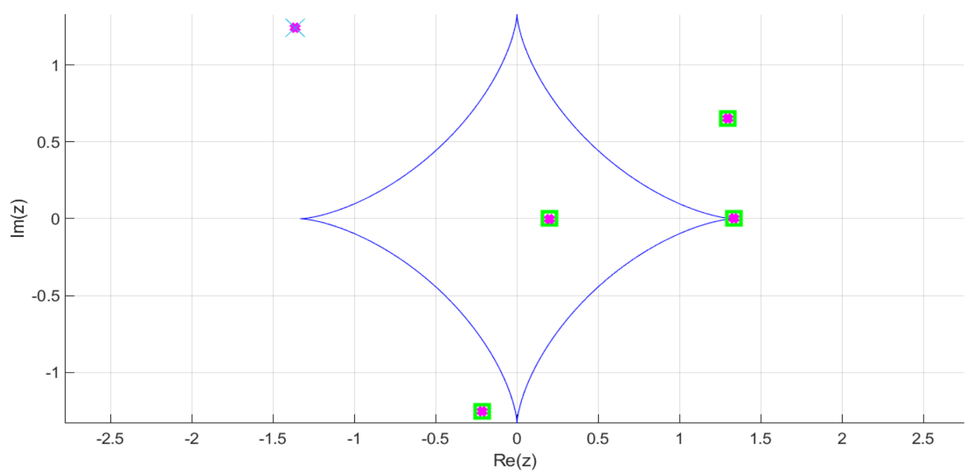

Figure 1: Approximations of the poles in Example 1

Also, the algorithm generated a spurious pole -1.3637 + 1.2430i, which can be easily detected and eliminated, taking into account that its residue differs significantly from the residues of the other determined poles (see Table 1).
Example 2. The Riemann function $z = \psi(w)$ that transforms the circle Γ_0 on the astroid $\Gamma = \left\{ z \in \mathbb{C} \,|\ z = \psi(w) = w + 1/\left(3w^3\right), \, w \in \Gamma_0 \right\}$ coincides with that in Example 1.

Approximations of the poles	*Pole residues*
-1.3637 + 1.2430i	0.0029
1.2992 + 0.6523i	0.9467
1.3333 - 0.0000i	0.6014
-0.2148 - 1.2547i	0.9640
0.2001 - 0.0001i	0.4815

Table 1: The approximations obtained in Example 1

The function of a complex variable $f(z)$ is defined as follows:

$$f(z) = \begin{cases} \frac{z-2i}{(z-zi_1)(z-zc_1)(z-zc_2)^2(z-ze_1)^2} & if \;\; \theta \in [0, \zeta_1] \\ -2\cos(z) \;\; if \;\; \theta \in (\zeta_1, 2\pi] \end{cases},$$

where $\zeta_1 = 8\pi/5,\ zi_1 = 0.2,\ zc_1 = \psi\left(e^{\pi i/8}\right),\ zc_2 = \psi\left(e^{9\pi i/7}\right),\ ze_1 = \psi\left(1.5e^{\pi i/4}\right)$. The function $f(z)$ has a simple pole zi_1 inside Γ, a pole of the second order ze_1 outside Γ and two poles zc_1 (simple) and zc_2 (of second order) on the contour Γ, as well as two jump discontinuity points on Γ $pc_1 = \psi\left(e^{i\zeta_1}\right),\ pc_2 = \psi(1)$ (see Figure 2).

Values f_k of the function $f(z)$ are calculated at the same points $z_k,\ k = 1, ..., m$ belonging to the contour Γ^ρ from Example 1.

The convergence rate of the examined localization algorithm decreases due to the discontinuity points. If the parameter M, defining the degree of the polynomial $Q_M(z)$, is incremented, starting from relatively small values, then the algorithm first generates approximations of the poles inside and on the contour, after that it generates two convergent sequences to the discontinuity points, thereafter, starting with a certain value of M, it generates approximations for the pole from outside of Γ. A relatively good approximation for all singular points is obtained for values $N = 16,\ M = 16,\ \rho = 1.01$ (see Figure 2). For each of the second-order poles of the function $f(z)$, i.e. zc_2 belonging to the contour and ze_1 outside the contour, the algorithm generates two approximations (see Table 2). Also, our algorithm generates spurious poles (see Figure 2), which can be eliminated by residual analysis. From the Table 2 we see that at least the residue of the pole -0.8276 + 0.2296i differs significantly from the residues of the other determined poles.

3 Estimating the number of poles

The examined algorithm for the localization of singular points, based on the approximation with rational functions, generates M approximations of the poles of the function $f(z)$ on domain Ω. Since at the amplification of parameter M the algorithm can generate more and more spurious poles, then an a priori estimation of the number s of true poles (taking into account their multiplicities) of $f(z)$ on domain Ω is required. If we test the values of the parameter M near the previously determined estimate, then we establish more quickly the poles and the discontinuity points of the function $f(z)$.

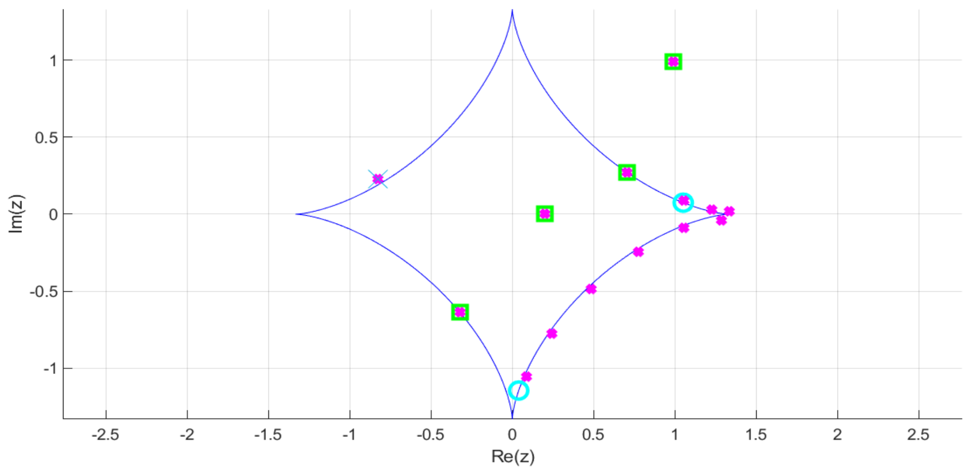

Figure 2: Approximations of the poles and discontinuity points in Example 2

Approximations of the poles	*Pole residues*
0.9908 + 0.9908i	1.069045569935890
0.9908 + 0.9908i	0.329150712708455
1.3360 + 0.0174i	0.016362612794167
1.2882 - 0.0409i	0.027456267135328
1.2295 + 0.0276i	0.032115463246755
0.0876 - 1.0569i	0.000000000273882
1.0566 - 0.0906i	0.005886863566123
1.0562 + 0.0865i	0.004647200386463
-0.8276 + 0.2296i	0.000000000000002
0.7768 - 0.2438i	0.000183044099354
0.2438 - 0.7767i	0.000000083813791
0.7060 + 0.2708i	2.881652529236829
-0.3232 - 0.6372i	1.388868211647166
-0.3232 - 0.6372i	0.536456894220026
0.4854 - 0.4854i	0.000005156848587
0.2000 - 0.0000i	3.206014132656694

Table 2: The approximations obtained in Example 2

Therefore, consider the matrix B of order $t \times t$ $(t = m + N + 1)$

$$B = \begin{pmatrix} 1 & z_1 & \cdots & (z_1)^N & -f(z_1)\, z_1 & \cdots & -f(z_1)(z_1)^m \\ 1 & z_2 & \cdots & (z_2)^N & -f(z_2)\, z_2 & \cdots & -f(z_2)(z_2)^m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_t & \cdots & (z_t)^N & -f(z_t)\, z_t & \cdots & -f(z_t)(z_t)^m \end{pmatrix},$$

that is obtained from the coefficient matrix of the system of equations (3) by replacing there M by m.

Let s_1 and s_2 be values that mean the number of poles of the function $f(z)$ on $\Omega^+ \cup \Gamma$ and, respectively, on $\Omega^- \cup \Gamma$, where Ω^+ (Ω^-) is the domain inside (outside) the contour Γ. The algorithm that computes an estimate for the number of poles s of the function $f(z)$ on domain Ω is based on the statement that the determinant of the considered matrix B is approximately zero for values $m > s_1 + s_2$, and for

$m = s_1 + s_2$ it takes the value $|\det(B)| > \delta$, where $\delta > 0$ is a parameter for defining a non-zero value.

Thus, we define a parameter $m > 0$ that means an estimate of the number $s_1 + s_2$, i.e. the number of poles inside and outside the contour Γ plus twice the number of poles and discontinuity points on Γ. If for the initial value m we have the nonzero determinant, then we consider a higher value for m. For values $m, m-1, m-2, ...$ we compute the determinants of the matrix B of order $(m + N + 1) \times (m + N + 1)$, until we have $\det B \neq 0$. A relatively small parameter $\delta > 0$ must be used to evaluate the last condition, for example, $\delta = 10^{-3}$. The first value of the parameter m for which $\det B \neq 0$, represents the evaluation of the number $s_1 + s_2$.

Numerical experiments show that the accuracy of the result depends on the value of the parameters $N \geq m$ and δ as well as on the presence of multiple poles and discontinuity points.

However, this approach does not determine the number of singular points of $f(z)$ on domain Ω, but the estimate of the number $s_1 + s_2$. If we want to evaluate the number of singular points on domains $\Omega^+ \cup \Gamma$ and $\Omega^- \cup \Gamma$, then we can apply the Laurent-Padé approximation algorithm, that is examined in [4]. It generates $m_1 + m_2$ approximations of the poles of the function $f(z)$ on Ω, where m_1 is the estimate of the number of poles s_1 belonging to the domain $\Omega^+ \cup \Gamma$, and m_2 is the estimate of the number of poles s_2 belonging to $\Omega^- \cup \Gamma$. To find an approximation of the value s_1, the algorithm computes the determinants of the matrix

$$C = \begin{pmatrix} c_{N_1} & c_{N_1-1} & \cdots & c_{N_1-(m_1-1)} \\ c_{N_1+1} & c_{N_1} & \cdots & c_{N_1-(m_1-2)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N_1+m_1-1} & c_{N_1+m_1-2} & \cdots & c_{N_1} \end{pmatrix},$$

starting from a value m_1 and decrementing it until the absolute value of the determinant becomes greater than a relatively small value $\delta > 0$. Elements of the matrix C are approximations of the coefficients $c_k = \frac{1}{2\pi i} \int_\Gamma f(z) z^{-k-1} dz$ from the Laurent series expansion for the function $f(z)$.

Thus, for different values of the parameter m_1 the determinants of the matrix C are computed and the number of poles of the function $f(z)$ belonging to the domain $\Omega^+ \cup \Gamma$ is estimated. Subsequently, the resulting value m_1 is subtracted from the estimation of the number $s_1 + s_2$, which is calculated from the determinants of the matrix B. The result is the estimate of the number of singular points on $\Omega^- \cup \Gamma$.

Applying the described algorithm to estimate the number of singular points of the function examined in Example 2 with the initial parameters $\rho = 1.01$, $\delta = 0.001$, $m = 20$, $N = 25$, we obtain $s_1 + s_2 = 13$, $m_1 = 6$, $m_2 = 7$.

4 Localization of singular points based on approximation by piecewise interpolation with rational functions

For functions of a real variable it is known that if the approximation interval is small enough, then the continuous function on this interval can be approximated well enough, using a small degree (1, 2 or 3) interpolation polynomial. This idea underlies the piecewise polynomial approximation, according to which the interval of definition of the function is divided into short length subintervals, and on each of the subintervals consisting of a small number of nodes, the function is approximated with an interpolation polynomial.

In this section we approximate the function $f(z)$, defined on a simple closed contour in the complex plane, with a piecewise rational function. This approach allows us to establish the multiple poles and discontinuity points that belong to the contour Γ, by using small-order rational functions (i.e. with polynomials $P_N(z)$ and $Q_M(z)$ of small degree), thus avoiding the generation of spurious poles.

To perform the steps of the algorithm it is necessary to know the values of the function $f(z)$ at $T := (N+M)\,m+1$ points $z_j \in \Gamma^\rho$. For a given value $m \in \mathbb{N}$, we divide the contour Γ^ρ into m arcs $\Gamma_j^\rho := arc\left(z_{(N+M)(j-1)+1}, z_{(N+M)j+1}\right)$. The curve arc $\Gamma_j^\rho,\ j \in \{1, ..., m\}$ contains the points $z_{(N+M)(j-1)+1}, z_{(N+M)(j-1)+2}, ..., z_{(N+M)j+1}$.

The piecewise rational function on the contour Γ is defined as follows. On each arc $\Gamma_j^\rho,\ j \in \{1, ..., m\}$ we consider the rational function $R^{\Gamma_j}(z) := P_N^{\Gamma_j}(z) \Big/ Q_M^{\Gamma_j}(z)$, where N and M $(N \geq M)$ take small values, usually, $N, M \in \{2, 3\}$. The coefficients of the functions $R^{\Gamma_j}(z),\ j = \overline{1,m}$ are determined by using the following interpolation conditions:

$$R^{\Gamma_j}(z_i) = f(z_i),\ z_i \in \Gamma_j^\rho,\ i \in I_j, \tag{4}$$

where $I_j := \{(M+N)(j-1)+1, ..., (M+N)\,j+1\}$. For $j = \overline{1,m}$ conditions (4) are written in linearized form

$$f(z_i)\, Q_M^{\Gamma_j}(z_i) - P_N^{\Gamma_j}(z_i) = 0,\ z_i \in \Gamma_j^\rho,\ \ i \in I_j.$$

But if we take into account the representations for the polynomials $P_N^{\Gamma_j}(z) = \sum_{k=0}^{N} p_k^{(\Gamma_j)} z^k$, $Q_M^{\Gamma_j}(z) = 1 + \sum_{r=1}^{M} q_r^{(\Gamma_j)} z^r$, we can write the last conditions in the explicit form

$$\sum_{k=0}^{N} p_k^{(\Gamma_j)} (z_i)^k - f(z_i) \sum_{r=1}^{M} q_r^{(\Gamma_j)} (z_i)^r = f(z_i),\ z_i \in \Gamma_j^\rho,\ \ i \in I_j. \tag{5}$$

Thus, for each $j \in \{1, ..., m\}$ the relation (5) is a system of $N+M+1$ linear algebraic equations with $N+M+1$ unknowns.

For $j = \overline{1,m}$ we determine the solution $\left(p_0^{(\Gamma_j)}, p_1^{(\Gamma_j)}, ..., p_N^{(\Gamma_j)}, q_1^{(\Gamma_j)}, ..., q_M^{(\Gamma_j)}\right)^T$ of the corresponding system (5). The components $q_1^{(\Gamma_j)}, ..., q_M^{(\Gamma_j)}$ are the coefficients of the polynomial $Q_M^{\Gamma_j}(z) = 1 + \sum_{r=1}^{M} q_r^{(\Gamma_j)} z^r,\ z \in \Gamma_j^\rho$. Considering the concatenation of all zeros of the polynomials $Q_M^{\Gamma_j}(z)$ on the arcs $\Gamma_j^\rho,\ j = \overline{1,m}$, we obtain the approximations of the poles and of the discontinuity points of the function $f(z)$ on the contour Γ.

Next we describe a simple method for computing the zeros of polynomial $Q_M^{\Gamma_j}(z)$ on the arc Γ_j^ρ. Let consider the set of polar angles

$$\Theta_j := \left\{ \{\theta_r\}_{r=\overline{1,n_2}} : \ \theta_r := \theta_{(N+M)(j-1)+1} + r\left(\theta_{(N+M)j+1} - \theta_{(N+M)(j-1)+1}\right)/n_2 \right\},$$

which form a grid of n_2 equidistant points on $\left[\theta_{(N+M)(j-1)+1}, \theta_{(N+M)j+1}\right]$, where $\theta_{(N+M)(j-1)+1}$ is the polar angle corresponding to the endpoint $z_{(N+M)(j-1)+1}$ of the arc Γ_j^ρ and $\theta_{(N+M)j+1}$ is the polar angle corresponding to other endpoint $z_{(N+M)j+1}$. On each arc $\Gamma_j^\rho, \ j = \overline{1,m}$ we consider the set

$$T_j := \left\{ \tilde{z}_r \in \Gamma_j^\rho : \ \tilde{z}_r = \psi\left(\rho e^{i\theta_r}\right), \ r = \overline{1,n_2}, \ \theta_r \in \Theta_j \right\},$$

that includes a sufficiently large number n_2 of points from Γ_j^ρ and covers uniformly the arc Γ_j^ρ. For each zero $\tilde{z}_k$ of the polynomial $Q_M^{\Gamma_j}(z)$, determined with algorithm described in previous section, we evaluate its distance to the elements of the set T_j. If in T_j there is an element t_k such that $|t_k - \tilde{z}_k| < \delta_1$, where $\delta_1 > 0$ is a sufficiently small value, then we consider that $\tilde{z}_k$ belongs to the arc Γ_j^ρ and we check the multiplicity of the zero $\tilde{z}_k$, which defines the order of the corresponding pole. If the function $f(z)$ has poles of multiplicity greater than the considered value M, then our algorithm can establish the existence of multiple poles, but it can not determine correctly their multiplicity.

Example 3. Determine the poles and discontinuity points that belong to the contour Γ considering the same initial data as in Example 2. The obtained approximations for the poles of the function $f(z)$ with values $M = N = 2, \ \rho = 1.01$, $m = 125, \ \delta_1 = 0.01, \ \delta_2 = 0.001$ are presented in Figure 3 and Table 3.

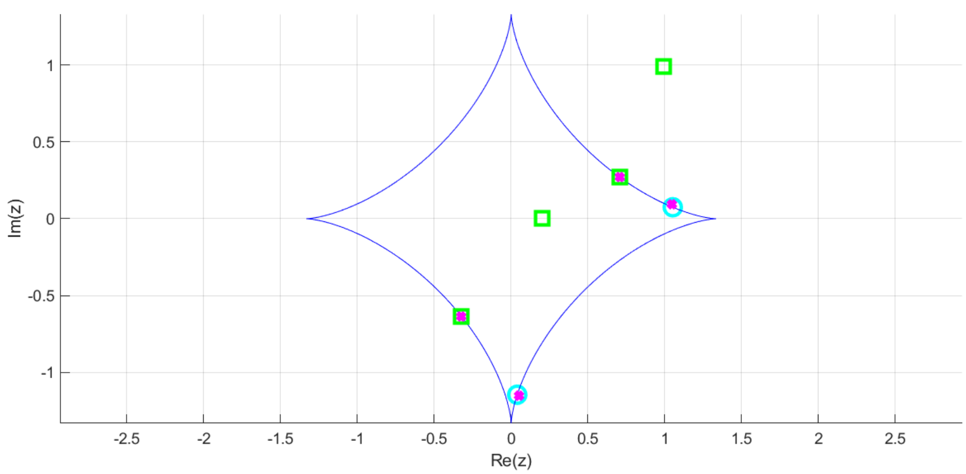

Figure 3: Approximations of the poles and discontinuity points in Example 3

This work is an outcome of research activity performed as a part of the project 20.80009.5007.13 "Deterministic and stochastic methods for solving optimization and control problems".

Approximations of the poles	*Multiplicities established for poles*
1.044744+0.093004i	1
0.706012+0.270767i	1
-0.323168-0.637221i	2
0.050366-1.150733i	1

Table 3: The approximations obtained in Example 3

References

[1] H. Kang, H. Lee *Identification of Simple Poles via Boundary Measurements and an Application of EIT*. Inverse Problems, 2004, **20**, 1853–1863.

[2] Miller K. *Stabilized numerical analytic prolongation with poles*. SIAM J. Appl. Math., 1970, **18**, 346–363.

[3] Nara T., Ando S. *Direct localization of poles of a meromorphic function from measurements on an incomplete boundary*. Inverse Problems, 2010, **26**, 1–26.

[4] Capcelea M., Capcelea T. *Laurent-Padé approximation for locating singularities of meromorphic functions with values given on simple closed contours*. Buletinul Academiei de Stiinte a Republicii Moldova. Matematica, 2020, no. 2(93), 76–87.

[5] Capcelea M., Capcelea T. *Algorithm for the localization of singularities of functions defined on closed contours*. Proceedings of the 4th Conference of Mathematical Society of Moldova, CMSM4-2017, June 28-July 2, 2017, Chisinau, Republic of Moldova, 369–372.

[6] Gonnet P., Pachon R., Trefethen L. *Robust rational interpolation and least-squares*. Electronic Transactions on Numerical Analysis, 2011, **38**, 146–167.

[7] Saff E. *An extension of Montessus de Ballore's theorem on the convergence of interpolating rational functions*. Journal of Approximation Theory, 1972, **6**, 63–67.

[8] Saff E., Walsh J. *On the convergence of rational functions which interpolate in the roots of unity*. Pacific Journal of Mathematics, 1973, **45**, 639–641.

[9] Yan C.-D., Chieng W.-H. *Method for finding multiple roots of polynomials*. Computers and Mathematics with Applications, 2006, **51**, 605–620.

[10] Gonnet P., Guttel S., Trefethen L. N. *Robust Padé approximation via SVD*. SIAM Review, 2013, **55**, no. 1, 101–117.

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