

An analogue of Bremermann’s theorem on finding the Bergman kernel for the Cartesian product of the classical domains $\mathfrak{R}_I(m, k)$ and $\mathfrak{R}_{II}(n)$

Jonibek Sh. Abdullayev

Abstract. In this paper, an analogue of Bremermann’s theorem on finding the Bergman kernel is obtained for the Cartesian product of classical domains. For this purpose, the groups of automorphisms of the considered domains are used, i.e., the Bergman kernels are constructed for the Cartesian product of classical domains, without applying complete orthonormal systems.

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1 Introduction, preliminaries and problem statement

In domains of \mathbb{C}^n and in matrix domains of $\mathbb{C}[m \times m]$, finding the kernels of integral representations of holomorphic functions is a rather difficult problem (see [1–3]). In classical theory, such kind kernels are usually constructed in bounded symmetric domains (see [4, 5]). Some of these domains are classical domains and matrix balls associated with classical domains. A number of problems were set for these domains (see [3]): finding the transitive group of automorphisms in these matrix balls, calculating the Bergman and Cauchy-Szegő kernels for these domains, finding the Carleman formula, restoring the values of holomorphic functions in classical domains and in matrix balls from the values of the function on some boundary sets of uniqueness (see [6–9]).

In homogeneous domains, automorphism groups are used for finding integral formulas ([2, 10]). Domains with rich automorphism groups are often implemented as matrix domains ([3, 4]). They are useful in solving various problems in function theory. Writing out explicitly the transitive group of automorphisms of classical domains and matrix balls associated with classical domains, a direct calculation can be used to find the Bergman and Cauchy-Szegő kernels for these domains. And then (using the properties of the Poisson kernel) we can find the Carleman formula, which restores the values of holomorphic function in this domain from its values on some boundary sets of uniqueness (see [11]). In this case, the scheme for finding the Bergman and Cauchy-Szegő kernels from [4, 12] is used. In [13] the volumes of a matrix ball of the third type and a generalized Lie ball are calculated. The total volumes of these domains are necessary for finding the kernels of integral formulas

for these domains (the Bergman, Cauchy–Szegő, Poisson kernels, etc.) (see, for example, [7, 8, 14]). In addition, it is used for the integral representation of functions holomorphic in these domains, in the mean value theorem and in other important concepts. In [15] definitions of holomorphic and pluriharmonic functions are given for classical domains of the first type by E. Cartan, the Laplace and Hua Luogeng operators are studied and a connection is found between these operators.

Definition 1. The domain $D \subset \mathbb{C}^n$ is called homogeneous if the group $Aut(D)$ of automorphisms of this domain is transitive, that is, for any pair of points $z_1, z_2 \in D$ there exists an automorphism $\varphi \in Aut(D)$ such that $\varphi(z_1) = z_2$.

Definition 2. The homogeneous domain $D \subset \mathbb{C}^n$ is called symmetric if for any point $\zeta \in D$ there exists such an automorphism $\varphi \in Aut(D)$ that:

- 1) $\varphi(\zeta) = \zeta$ but $\varphi(z) \neq z$ if $z \in D$ is different from ζ ;
- 2) $\varphi \circ \varphi = e$, where $e \in Aut(D)$ is the identity mapping.

Definition 3. The domain $D \subset \mathbb{C}^n$ is called an irreducible domain if it is not a direct product of bounded symmetric domains of lower dimension.

Definition 4. The bounded domain $D \subset \mathbb{C}^n$ is called classical if the complete group of its holomorphic automorphisms is a classical Lie group¹ and it is transitive on D .

According to E. Cartan's classification, there are four types of irreducible classical domains (see [4, 16]):

$$\begin{aligned} \mathfrak{R}_I(m, k) &= \left\{ Z \in \mathbb{C}[m \times k] : I^{(m)} - ZZ^* > 0 \right\}, \\ \mathfrak{R}_{II}(n) &= \left\{ Z \in \mathbb{C}[n \times n] : I^{(n)} - Z\bar{Z} > 0, \forall Z' = Z \right\}, \\ \mathfrak{R}_{III}(n) &= \left\{ Z \in \mathbb{C}[n \times n] : I^{(n)} + Z\bar{Z} > 0, \forall Z' = -Z \right\}, \\ \mathfrak{R}_{IV}(n) &= \left\{ z \in \mathbb{C}^n : |zz'|^2 - 2\bar{z}z' + 1 > 0, |zz'| < 1 \right\}, \end{aligned}$$

here $I^{(m)}$ is the identity matrix of order m , Z^* is the complex conjugate matrix of the transposed matrix Z'^2 ($H > 0$ for a Hermitian matrix H means, as usual, that H is positive definite). All these domains are homogeneous symmetric convex complete circular domains centered at O (O is zero matrix).

If we write the elements of a matrix $Z \in \mathbb{C}[m \times k]$ as a point in the space \mathbb{C}^{mk} :

$$z = \{z_{11}, \dots, z_{1k}, z_{21}, \dots, z_{2k}, \dots, z_{m1}, \dots, z_{mk}\} \in \mathbb{C}^{mk}, \quad (1)$$

then we can assume that Z is an element of the space \mathbb{C}^{mk} , i.e., we arrive at the isomorphism

$$\mathbb{C}[m \times k] \cong \mathbb{C}^{mk}.$$

¹See https://en.wikipedia.org/wiki/Lie_group

² Z' is transposed matrix of the matrix Z

Therefore, the dimensions of the classical four domains above are equal, respectively, to

$$mk, \frac{m(m+1)}{2}, \frac{m(m-1)}{2}, n.$$

The Bergman space on bounded symmetric domains is a fundamental concept in the analysis. It is equipped with a natural projection, i.e. the Bergman projection, determined by the property of reproducing nucleus. On the other hand, the weighted Bergman spaces are also important in harmonic analysis (see, for example [17]).

Definition 5. [1] Let $\{\varphi_\nu(z), \nu = 0, 1, 2, \dots\}$ be a complete orthonormal system of functions in $L^2(D)$. The Bergman kernel (or kernel function³) $K_D(z, \bar{\zeta})$ is the sum of the series

$$\sum_{\mu=1}^{\infty} \varphi_\mu(z) \overline{\varphi_\mu(\bar{\zeta})} = K_D(z, \bar{\zeta}), \quad (2)$$

which is holomorphic in z and antiholomorphic in ζ .

Let us present the following Bremermann's theorem on finding the Bergman kernel for the Cartesian product of two domains.

Theorem 1. [18] *If $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ are bounded domains, then the Bergman kernel K_D for the domain $D = \Omega_1 \times \Omega_2$ has the following form*

$$K_D(w, z, \bar{w}, \bar{\zeta}) = K_{\Omega_1}(w, \bar{w}) \cdot K_{\Omega_2}(z, \bar{\zeta}),$$

where $K_{\Omega_1}(w, \bar{w})$ and $K_{\Omega_2}(z, \bar{\zeta})$ are the Bergman kernel for domains Ω_1 and Ω_2 , respectively.

For example (see [19]), using the Bergman kernel formula $K_{\mathbb{U}^1}(z, \bar{\zeta}) = \frac{R^2}{\pi(R-z\bar{\zeta})^2}$ for the circle $\mathbb{U}^1 = \{z \in \mathbb{C} : |z| < R\}$ and using Theorem 1 we can find the Bergman kernel

$$K_{\mathbb{U}^n}(z, \bar{\zeta}) = \frac{R_1^2 \cdot R_2^2 \cdot \dots \cdot R_n^2}{\pi^n (R_1 - z_1 \bar{\zeta}_1)^2 (R_2 - z_2 \bar{\zeta}_2)^2 \dots (R_n - z_n \bar{\zeta}_n)^2}, \quad (3)$$

for polydisk $\mathbb{U}^n = \{z \in \mathbb{C}^n : |z_1| < R_1, |z_2| < R_2, \dots, |z_n| < R_n\}$.

The following statement is true, for a circular domain.

Theorem 2. [4] *If \mathfrak{R} is a bounded circular domain, then the Bergman kernel for the domain \mathfrak{R} has the following form*

$$K(z, \bar{z}) = \frac{1}{V(\mathfrak{R})} A(z, \bar{z}), \quad (4)$$

where $V(\mathfrak{R})$ is the volume of the domain \mathfrak{R} , and $A(z, \bar{z})$ is the density of the volume \mathfrak{R} , and it is equal to the real Jacobian of the automorphism domain \mathfrak{R} , which transfers the point a to initial point.

³In classical Russian literature, the Bergman kernel is usually called the domain kernel function (see, for example, [1, 19]). At the present time, such a term would not be very apt, since we have three kernels (Bergman, Cauchy-Szegő and Poisson) that could bear this name.

In other words, the Bergman kernel for any transitive circular region is equal to the ratio of the volume density to the Euclidean volume of the domain (we recall that if the domain $D \subset \mathbb{C}^n$ admits the transformation group $z = e^{i\theta}w$, then we call D a circular domain, if, in addition, with a point z , the point rz ($0 \leq r \leq 1$) also lies in D , then we call D a complete circular domain). In Hua Luogeng's book [4] one can find explicit expressions for the Bergman kernel, the automorphism groups of the domains $\mathfrak{R}_I(m, k)$, $\mathfrak{R}_{II}(n)$, $\mathfrak{R}_{III}(n)$ and $\mathfrak{R}_{IV}(n)$. As the main result in this paper, we introduce an analogue of the Bremermann theorem for finding the Bergman kernel for the Cartesian product of the classical domains $\mathfrak{R}_I(m, k)$, $\mathfrak{R}_{II}(n)$. For this, the groups of automorphisms of the considered domains are used, i.e., the Bergman kernels for the Cartesian product of classical domains are constructed, being guided only by this consideration and not using complete orthonormal systems.

2 The Bergman kernel for the Cartesian product of the classical domains $\mathfrak{R}_I(m, k)$ and $\mathfrak{R}_{II}(n)$

It is known [4] that the mapping

$$\Phi_1 = Q(W - P_1) \left(I^{(k)} - P_1^* W \right)^{-1} R^{-1} \quad (5)$$

is an automorphism of the domain $\mathfrak{R}_I(m, k)$ that sends the point P_1 to the initial point, where $W \in \mathfrak{R}_I(m, k)$, and the matrices $Q \in \mathbb{C}[m \times m]$ and $R \in \mathbb{C}[k \times k]$ satisfy the following conditions

$$\overline{Q} \left(I^{(m)} - \overline{P_1} P_1' \right) Q' = I^{(m)}, \quad \overline{R} \left(I^{(k)} - P_1' \overline{P_1} \right) R' = I^{(k)}. \quad (6)$$

Also the mapping

$$\Phi_2 = G(Z - P_2) \left(I^{(n)} - \overline{P_2} Z \right)^{-1} \overline{G}^{-1} \quad (7)$$

is an automorphism of the domain $\mathfrak{R}_{II}(n)$ which transfers the point P_2 to the initial point, where $Z \in \mathfrak{R}_{II}(n)$ and the matrix $G \in \mathbb{C}[n \times n]$ satisfies the following condition:

$$\overline{G} \left(I^{(n)} - \overline{P_2} P_2' \right) G' = I^{(n)}. \quad (8)$$

Let the domain \mathfrak{R} be defined as the Cartesian product of the classical domains $\mathfrak{R}_I(m, k)$ and $\mathfrak{R}_{II}(n)$:

$$\mathfrak{R} = \mathfrak{R}_I(m, k) \times \mathfrak{R}_{II}(n) = \{(W, Z) : W \in \mathfrak{R}_I(m, k), Z \in \mathfrak{R}_{II}(n)\},$$

here

$$\mathfrak{R}_I(m, k) = \left\{ W \in \mathbb{C}[m \times k] : I^{(m)} - WW^* > 0 \right\},$$

and

$$\mathfrak{R}_{II}(n) = \left\{ Z \in \mathbb{C}[n \times n] : I^{(n)} - Z\overline{Z} > 0, \forall Z' = Z \right\}.$$

The skeleton \mathbb{X} of the domain \mathfrak{R} is the Cartesian product of the skeletons \mathbb{X}_I and \mathbb{X}_{II} of the domains $\mathfrak{R}_I(m, k)$ and $\mathfrak{R}_{II}(n)$, i.e.

$$\begin{aligned}\mathbb{X}_I &= \left\{ W \in \mathbb{C}[m \times k] : WW^* = I^{(m)} \right\}, \\ \mathbb{X}_{II} &= \left\{ Z \in \mathbb{C}[n \times n] : Z\bar{Z} = I^{(n)}, \forall Z' = Z \right\}, \\ \mathbb{X} &= \mathbb{X}_I \times \mathbb{X}_{II}.\end{aligned}$$

Then from (5) and (7) we obtain the following statement:

Proposition 1. *The mapping $\Phi = (\Phi_1, \Phi_2)$, where Φ_1 is of the form (5) and Φ_2 is of the form (7) with conditions (6) and (8), is an automorphism of the domain $\mathfrak{R} = \mathfrak{R}_I(m, k) \times \mathfrak{R}_{II}(n)$ transferring point $P = (P_1, P_2) \in \mathfrak{R}$ to the beginning point.*

So, the following analogue of Bremerman's theorem [18] on finding the Bergman kernel for the Cartesian product of the classical domains $\mathfrak{R}_I(m, k)$ and $\mathfrak{R}_{II}(n)$ is true.

Theorem 3. *Let $\mathfrak{R}_I(m, k)$ and $\mathfrak{R}_{II}(n)$ be classical domains in spaces of variables $W \in \mathbb{C}[m \times k]$ and $Z \in \mathbb{C}[n \times n]$, respectively, and $\mathfrak{R} = \mathfrak{R}_I(m, k) \times \mathfrak{R}_{II}(n)$, then*

$$K_{\mathfrak{R}}(W, Z, \bar{W}, \bar{Z}) = K_{\mathfrak{R}_I(m, k)}(W, \bar{W}) K_{\mathfrak{R}_{II}(n)}(Z, \bar{Z}). \quad (9)$$

Proof. From Proposition 1 we have an automorphism $\Phi = (\Phi_1, \Phi_2)$ for the domain \mathfrak{R} . First, differentiating (5), we have

$$d\Phi_1 = Q \left[dW \cdot \left(I^{(k)} - P_1^* W \right)^{-1} + (W - P_1) d \left(I^{(k)} - P_1^* W \right)^{-1} \right] R^{-1}.$$

Put $W = P_1$. Then from (6) we have

$$d\Phi_1 = Q \cdot dZ \cdot \left(I^{(k)} - P_1^* P_1 \right)^{-1} R^{-1} = Q \cdot dW \cdot R^*. \quad (10)$$

Now let's put:

$$\begin{aligned}W &= \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1k} \\ w_{21} & w_{22} & \dots & w_{2k} \\ \dots & \dots & \dots & \dots \\ w_{m1} & w_{m2} & \dots & w_{mk} \end{pmatrix}, \quad Q = \begin{pmatrix} q_{11} & q_{12} & \dots & q_{1m} \\ q_{21} & q_{22} & \dots & q_{2m} \\ \dots & \dots & \dots & \dots \\ q_{m1} & q_{m2} & \dots & q_{mm} \end{pmatrix}, \\ R &= \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1k} \\ r_{21} & r_{22} & \dots & r_{2k} \\ \dots & \dots & \dots & \dots \\ r_{k1} & r_{k2} & \dots & r_{kk} \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} f_{11} & f_{12} & \dots & f_{1k} \\ f_{21} & f_{22} & \dots & f_{2k} \\ \dots & \dots & \dots & \dots \\ f_{m1} & f_{m2} & \dots & f_{mk} \end{pmatrix}.\end{aligned}$$

Then the identities (10) are equivalent to

$$df_{sj} = \sum_{l=1}^k \sum_{i=1}^m q_{si} dw_{il} \bar{r}_{lj}, \quad s = 1, 2, \dots, m; j = 1, 2, \dots, k.$$

From these properties it follows that

$$df_{11} \wedge df_{12} \wedge \dots \wedge df_{mk} = \prod_{s,j} \sum_{l=1}^k \sum_{i=1}^m q_{si} dw_{il} \bar{r}_{lj}, \quad s = 1, 2, \dots, m; j = 1, 2, \dots, k,$$

for a holomorphic mapping $\Phi_1 = (f_{11}, \dots, f_{mk})$. Then

$$\begin{aligned} & df_{11} \wedge df_{12} \wedge \dots \wedge df_{mk} = \\ & = \prod_{s,j} (\det Q)^s \cdot dw_{11} \wedge dw_{12} \dots \wedge dw_{mk} \cdot (\det R^*)^j, \quad s = 1, 2, \dots, m; j = 1, 2, \dots, k, \end{aligned}$$

i.e.,

$$\begin{aligned} df_{11} \wedge df_{12} \wedge \dots \wedge df_{mk} &= (\det Q)^m \cdot (\det R^*)^k \cdot dw_{11} \wedge dw_{12} \wedge \dots \wedge dw_{mk} = \\ &= J_{\mathbb{C}}(\Phi_1) \cdot dw_{11} \wedge dw_{12} \wedge \dots \wedge dw_{mk}, \end{aligned}$$

where $J_{\mathbb{C}}(\Phi_1)$ is a complex Jacobian and it is equal to

$$J_{\mathbb{C}}(\Phi_1) = (\det Q)^m \cdot (\det R^*)^k.$$

Therefore, from condition (6) we have

$$\dot{\Phi}_1 = |J_{\mathbb{C}}(\Phi_1)|^2 \dot{W} = \left| (\det Q)^m \cdot (\det R^*)^k \right|^2 \dot{W} = \frac{\dot{W}}{\det^{m+n}(I - WW^*)},$$

where \dot{W} and $\dot{\Phi}_1$ are volume elements.

Now differentiating (7), we have

$$d\Phi_2 = G \left[dZ \cdot (I^{(n)} - \bar{P}_2 Z)^{-1} + (Z - P_2) d(I^{(n)} - \bar{P}_2 Z)^{-1} \right] \bar{G}^{-1},$$

Setting $Z = P_2$, from (8) we have

$$d\Phi_2 = G \cdot dZ \cdot (I^{(n)} - \bar{P}_2 P_2)^{-1} \cdot \bar{G}^{-1} = G \cdot dZ \cdot G'. \quad (11)$$

Now put:

$$\begin{aligned} Z &= \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & z_{2n} \\ \dots & \dots & \dots & \dots \\ z_{n1} & z_{n2} & \dots & z_{nn} \end{pmatrix}, \quad G = \begin{pmatrix} g_{11} & g_{12} & \dots & g_{1n} \\ g_{21} & g_{22} & \dots & g_{2n} \\ \dots & \dots & \dots & \dots \\ g_{n1} & g_{n2} & \dots & g_{nn} \end{pmatrix}, \\ \Phi_2 &= \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ h_{21} & h_{22} & \dots & h_{2n} \\ \dots & \dots & \dots & \dots \\ h_{n1} & h_{n2} & \dots & h_{nn} \end{pmatrix}. \end{aligned}$$

Then the identity (11) is equivalent to the following

$$dh_{sj} = \sum_{l=1}^n \sum_{i=1}^n g_{si} dz_{il} g_{lj}, \quad s = 1, 2, \dots, n; j = 1, 2, \dots, n.$$

From these properties it follows that

$$dh_{11} \wedge dh_{12} \wedge \dots \wedge dh_{nn} = \prod_{s,j} \sum_{l=1}^n \sum_{i=1}^n g_{si} dz_{il} g_{lj}, \quad s = 1, 2, \dots, n; j = 1, 2, \dots, n,$$

for a holomorphic mapping $\Phi_2 = (h_{11}, \dots, h_{nn})$. Then by virtue of $dz_{il} = dz_{li}$, the equality

$$\begin{aligned} & dh_{11} \wedge dh_{12} \wedge \dots \wedge dh_{nn} = \\ &= \prod_{s,j} (\det G)^s \cdot dz_{11} \wedge dz_{12} \dots \wedge dz_{nn} \cdot (\det G')^j, \quad s = 1, 2, \dots, n; j = 1, 2, \dots, n, \end{aligned}$$

is equivalent to the equality

$$\begin{aligned} dh_{11} \wedge dh_{12} \wedge \dots \wedge dh_{nn} &= (\det G)^{\frac{n+1}{2}} \cdot (\det G')^{\frac{n+1}{2}} \cdot dz_{11} \wedge dz_{12} \dots \wedge dz_{nn} = \\ &= J_{\mathbb{C}}(\Phi_2) \cdot dz_{11} \wedge dz_{12} \dots \wedge dz_{nn}, \end{aligned}$$

where $J_{\mathbb{C}}(\Phi_2)$ is a complex Jacobian and it is equal to

$$J_{\mathbb{C}}(\Phi_2) = (\det G)^{\frac{(n+1)}{2}} \cdot (\det G')^{\frac{(n+1)}{2}} = (\det G')^{(n+1)}.$$

Therefore, by condition (8) we have

$$\dot{\Phi}_2 = |J_{\mathbb{C}}(\Phi_2)|^2 \dot{Z} = \left| (\det G)^{n+1} \right|^2 \dot{Z} = \frac{\dot{Z}}{\det^{n+1}(I - Z\bar{Z})}.$$

It follows that for the mapping $\Phi = (f_{11}, \dots, f_{mk}, h_{11}, \dots, h_{nn})$ the next equality is true

$$\begin{aligned} & df_{11} \wedge df_{12} \wedge \dots \wedge df_{mk} \wedge h_{11} \wedge dh_{12} \wedge \dots \wedge dh_{nn} = \\ &= J_{\mathbb{C}}(\Phi_1) dw_{11} \wedge dw_{12} \wedge \dots \wedge dw_{mk} J_{\mathbb{C}}(\Phi_2) \wedge dz_{11} \wedge dz_{12} \dots \wedge dz_{nn}. \end{aligned}$$

And therefore,

$$J_{\mathbb{R}}(\Phi) = \frac{1}{\det^{m+k}(I - WW^*) \det^{n+1}(I - Z\bar{Z})}.$$

It is known [4] that the Bergman kernel for the domains $\mathfrak{R}_I(m, k)$ and $\mathfrak{R}_{II}(n)$ have the forms

$$K_{\mathfrak{R}_I(m,k)}(z, \bar{z}) = \frac{1}{V(\mathfrak{R}_I(m, k)) \det^{m+k}(I^{(m)} - Z\bar{Z}')},$$

$$K_{\mathfrak{R}_{II}(n)}(z, \bar{z}) = \frac{1}{V(\mathfrak{R}_{II}(n))} \frac{1}{\det^{m+1}(I^{(n)} - Z\bar{Z})},$$

where $V(\mathfrak{R}_I(m, k))$ and $V(\mathfrak{R}_{II}(n))$ are volumes of domains $\mathfrak{R}_I(m, k)$ and $\mathfrak{R}_{II}(n)$, respectively. Hence, by Theorem 2 we have the following relation

$$\begin{aligned} K_{\mathfrak{R}}(W, Z, \bar{W}, \bar{Z}) &= \frac{1}{\det^{m+k}(I - WW^*) \det^{n+1}(I - Z\bar{Z})} \\ &= \frac{1}{V(\mathfrak{R}_I(m, k)) \det^{m+k}(I - WW^*)} \cdot \frac{1}{V(\mathfrak{R}_{II}(n)) \det^{n+1}(I - Z\bar{Z})} = \\ &= K_{\mathfrak{R}_I(m, k)}(W, \bar{W}) \cdot K_{\mathfrak{R}_{II}(n)}(Z, \bar{Z}). \end{aligned}$$

□

Theorem 3 implies immediately the following

Corollary 1. *Let $\mathfrak{R}_I(m, k)$ and $\mathfrak{R}_{II}(n)$ be the classical domains in spaces of variables $W \in \mathbb{C}[m \times k]$ and $Z \in \mathbb{C}[n \times n]$, respectively, and $\mathfrak{R} = \mathfrak{R}_I(m, k) \times \mathfrak{R}_{II}(n)$, then*

$$K_{\mathfrak{R}}(W, Z, \bar{\Theta}, \bar{\Upsilon}) = K_{\mathfrak{R}_I(m, k)}(W, \bar{\Theta}) K_{\mathfrak{R}_{II}(n)}(Z, \bar{\Upsilon}), \quad (12)$$

where $W, \Theta \in \mathfrak{R}_I(m, k)$, $Z, \Upsilon \in \mathfrak{R}_{II}(n)$.

Note that for $m = k = n = 1$, (12) is the formula for computing the Bergman kernel for the polydisk

$$\mathbb{U}^2 = \{z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}.$$

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J. SH. ABDULLAYEV

”M. Ulugbek” National University of Uzbekistan,
Tashkent 100174, Uzbekistan.
E-mail: jonibek-abdullayev@mail.ru

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