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On semigroups of endomorphisms of universal algebras

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Abstract. In the present article the left ideals of the semigroup of endomorphisms $End(G)$ of a universal algebra G are studied. The lattice $Spec^s(G)$ of saturated left ideals and the lattice $Spec^f(G)$ of full ideals of the semigroup of endomorphisms $End(G)$ of a universal algebra G are introduced and characterized (Theorem 2, Corollaries 7 and 8). In a free universal algebra any left ideal is a full left ideal. Theorem 1 describes the cyclic universal algebras. Theorem 3 affirms that any semigroup with unity is isomorphic to a semigroup of endomorphisms $End(G)$ of some cyclic free universal algebra G.

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1 Introduction

Let $\mathbb{N} = \{1, 2, ...\}$ be the set of natural numbers and $n \in \omega = \{0, 1, 2, ...\}$. The n-ary Cartesian power of a set X is denoted by X^n. If the set X is empty, then the set X^n is empty too. If the set X is non-empty, then the set X^0 is a singleton. The discrete sum $\Omega = \oplus\{\Omega_n : n \in \omega = \{0, 1, 2, ...\}\}$ of the pairwise disjoint discrete spaces $\{\Omega_n : n \in \omega\}$ is called a signature. A topological Ω-algebra or a topological universal algebra of the signature Ω is a family $\{G, e_{nG} : n \in \omega\}$, where G is a non-empty topological space and $e_{nG} : \Omega_n \times G^n \to G$ is a continuous mapping for each $n \in \omega$. The concept of universal algebra was created by Alfred North Whitehead in 1898 as a generalization of Boole's logical algebras. The term universal algebra was proposed by James Joseph Sylvester [28]. Between 1935 and 1950 important works were published by Garrett Birkhoff, in which he introduced the notions of variety, quasi-variety, free algebra, congruences and proved some homomorphism theorems [1–3]. After 1950, due to applications in mathematical logic, model theory, geometric algebras, theoretical and computer physics, the theory of universal algebras began to develop fruitfully [2, 3, 10, 11, 22, 27]. As in [23–27] we continue the study of semigroups of endomorphisms of universal topological algebras.

Let A, B and C be three universal algebras of signature Ω. The function $f : A \longrightarrow B$ is called a morphism or homomorphism if $f(u(x)) = u(f^n(x))$ for any $n \in \omega$, any $u \in \Omega_n$ and any element $x = (x_1, x_2, ..., x_n) \in G^n$, where $f^n(x) = (f(x_1), f(x_2), ..., f(x_n))$. The composition of the functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is the function $h = f \cdot g$: $A \longrightarrow C$, where $h(x) = g(f(x))$ for any $x \in A$. The composition of two morphisms is always a morphism. A morphism that is a bijective function is called an isomorphism. If an isomorphism can be established between two universal algebras, they are called isomorphic. Two isomorphic universal algebras are identified. Morphisms, respectively isomorphisms, between a universal algebra and itself are called endomorphisms, respectively automorphisms.

Subalgebras and Cartesian products of topological Ω-algebras are defined in traditional way [1–6, 9, 13, 27, 28].

Let G be a topological space and $n \in \mathbb{N}$. A continuous mapping $\lambda : G^n \to G$ is called an n-ary operation on G.

If G is a topological Ω-algebra and $u \in \Omega_n$, then $u : G^n \to G$, where $u(x) = e_{nG}(u, x)$ for every $x \in G^n$, is an n-ary operation on G.

For any topological universal algebra G denote by $Sub(G)$ the set of all subalgebras of G and by $Sub_c(G)$ we denote the set of all closed subalgebras of the algebra G. Relative to the operation of inclusion, $Sub(G)$ and $Sub_c(G)$ are ordered sets with the maximal element G and the minimal element $\emptyset$. Hence $\cap Sub(G)$ and $\cap Sub_c(G)$ are subalgebras of G. Algebra G and $\emptyset$ are considered as improper subalgebras of G. Therefore $\emptyset$ is not a universal algebra which is a subalgebra of any universal algebra.

Any subset $A \subseteq G$ generates the subalgebra $s_G(A) = \cap\{H \in Sub(G) : A \subseteq H\}$.

Definition 1. A nonempty set L together with two binary operations $\vee$ and $\wedge$ on L is called a lattice if it satisfies the following identities:

L1: (a) $x \vee y = y \vee x$; (b) $x \wedge y = y \wedge x$ (commutative laws).

L2: (a) $x \vee (y \vee z) = (x \vee y) \vee z$; (b) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (associative laws).

L3: (a) $x \vee x = x$; (b) $x \wedge x = x$ (idempotent laws).

L4: (a) $x = x \vee (x \wedge y)$; (b) $x = x \wedge (x \vee y)$ (absorption laws).

If L is a lattice, then define the order $\leq$ on L by $a \leq b$ if and only if $a = a \wedge b$.

For any non-empty subset A of the ordered set L the supremum $c = \vee A = supA$ is an element of L with the properties: $x \leq c$ for any $x \in A$; if $b \in L$ and $x \leq b$ for any $x \in A$, then $c \leq b$. Similarly the infimum $\wedge A = infA$ of A in L is defined.

An ordered set L is a lattice if and only if for every $a, b \in L$ both $a \vee b = sup\{a, b\}$ and $a \wedge b = inf\{a, b\}$ exist in L.

A mapping $f : A \longrightarrow B$ of an ordered set A into an ordered set B is an order homomorphism if $x, y \in A$ and $x \leq y$ implies $f(x) \leq f(y)$.

Remark 1. The ordered sets $Sub(G)$ and $Sub_c(G)$ are complete lattices.
If $A, B \in Sub_c(G)$, then $inf\{A, B\} = A \cap B$ and the infimum is the same in lattices $Sub(G)$ and $Sub_c(G)$. In general, the supremum $sup\{A, B\}$ is not obligatorily the same in lattices $Sub(G)$ and $Sub_c(G)$. The mapping $cl_G : Sub(G) \longrightarrow Sub_c(G)$, where $\Im cl_G A = cl_G A$ is the closure of the set A in the space G, is an order homomorphism of the lattice $Sub(G)$ onto the lattice $Sub_c(G)$.

Example 1. Let G be the semigroup of all rational numbers with the binary operation $\{+\}$. Let A be the subsemigroup of G generated by the set $\{n + 2^{-n-3} : n \in \mathbb{N}\}$ and B be the subsemigroup of G generated by the set $\{-n - 3^{-n-3} : n \in \mathbb{N}\}$. The sets $G^0 = \{0\}$, $G^+ = \{x \in A : x > 0\}$ and $G^- = \{x \in A : x < 0\}$ are subsemigroups of G.

For any two subsemigroups $P, Q \in Sub(G)$ of the semigroup G we have $P \cup Q \subseteq sup\{P, Q\} \subseteq sup\{cl_G P, cl_G Q\} \subseteq cl_G(sup\{P, Q\}) \subseteq cl_G(sup\{P, Q\}) = cl_G(sup\{cl_G P, cl_G Q\})$ and $P \cap Q = inf\{P, Q\} \subseteq inf\{cl_G P, cl_G Q\} = ccl_G P \cap cl_G Q$.

The sets A and B are closed in G and the set $C = \{x + y : x \in A, y \in B\}$ is not a closed subsemigroup of G. In $Sub(G)$ we have the supremum $sup\{A, B\} = C$ and in $Sub_c(C)$ we have $C \subset sup\{A, B\} = cl_G C \neq C$. Hence $Sub_c(G)$ is not a sublattice of the lattice $Sub(G)$.

The sets G^+ and G^- are not closed in G,
$\emptyset = G^+ \cap G^- = inf\{G^+, G^-\} \subseteq G^0 = inf\{cl_G G^+, cl_G^-\} = cl_G G^+ \cap cl_G^-$.
Hence the mapping $cl_G : Sub(G) \longrightarrow Sub_c(G)$ is an order homomorphism and is not a lattice homomorphism of the lattice $Sub(G)$ onto the lattice $Sub_c(G)$.

2 Ideals and spectral spaces of algebras

The spectrum of a non-empty set X, denoted $Exp(X)$, is the set of the all subsets of X, equipped with the Zariski topology, for which the base of closed sets are the sets

$$V(A) = \{P \in Exp(X) : A \subseteq P\},\ A \in Exp(X).$$

Let G be a topological universal algebra of the signature Ω, $n \geq 2$, $1 \leq i \leq n \in \mathbb{N}$ and $u \in \Omega_n$. The universal algebra G is (u, i)-divisible if for any elements $a_1, a_2, ..., a_i, ..., a_n \in G$ the equation $u(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_n) = a_i$ has some solution in G. A subset $A \subseteq G$ is called a (u, i)-ideal of the algebra G if $A \in Sub(G)$ and $u(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_n) \in A$ for all $x \in G$ and $a_1, ..., a_{i-1}, a_{i+1}, ..., a_n \in A$. The empty set $\emptyset$ is considered a (u, i)-ideal. If the algebra G is (u, i)-divisible, then G has not proper (u, i)-ideals. The $(u, 1)$-ideal is called a left u-ideal and a (u, n)-ideal is called a right u-ideal. If A is a (u, i)-ideal for any $i \leq n$, then A is a u-ideal, or a bilateral u-ideal for $n = 2$. Denote by $Spec(G, u, i)$ the set of all (u, i)-ideals and by $Spec_c(G, u, i)$ the set of all closed (u, i)-ideals of the topological universal algebra G. The sets $Spec(G, u, i)$ and $Spec_c(G, u, i)$ are complete lattices and the mapping $cl_G : Spec(G, u, i) \longrightarrow Spec_c(G, u, i)$ is a homomorphism of the lattice $Spec(G, u, i)$ onto $Spec_c(G, u, i)$. The algebraical proprieties describe the arithmetic properties, and the properties of the topological spaces $Spec(G, u, j)$, $Spec_c(G, u)$ describe the geometric properties of the universal algebra G.

Let S be a multiplication semigroup. A non-empty subset A of S is called a left (respectively right) ideal of S if $S \cdot A \subseteq A$ (respectively $A \cdot S \subseteq A$).

3 Order proprieties of spaces of ideals

Let G be a topological universal algebra of the signature Ω. The set $End(G)$ of all endomorphisms of universal algebra G and the set $End_c(G)$ of all continuous endomorphisms of universal algebra G with binary composition operation are called the semigroups of endomorphisms and continuous endomorphisms, in which the identical automorphism is their unity. The semigroups $End(G)$ and $End_c(G)$

are powerful and convenient tools in the study of algebra G. Thus the theory of semigroups [6–8, 10, 13, 18–20, 22] can be applied in the study of universal algebras.

From [4, 5, 5, 6, 12, 13, 19] it follows that the following definitions are correct.

Definition 2. Let Ω be a fixed signature. A universal algebra G is a free universal algebra in some class of universal algebras if there is given a subset $I = I_G \subset G$ with the properties:

1) the algebra G is generated by the set I, i.e. $G = s_G(I)$, and I is called the space of generators of G;

2) for any mapping $f : I \longrightarrow G$ there exists a (unique) endomorphism $\hat{f} : G \longrightarrow G$ such that $f(x) = \hat{f}(x)$ for each $x \in I$.

Definition 3. Let Ω be a fixed signature. A topological universal algebra G is a topological free universal algebra in some class of universal algebras if there is given a subspace $I = I_G \subset G$ with the properties:

1) the algebra G is generated by the set I, i.e. $G = s_G(I)$, and I is called the space of generators of G;

2) for any continuous mapping $f : I \longrightarrow G$ there exists a (unique) continuous endomorphism $\hat{f} : G \longrightarrow G$ such that $f(x) = \hat{f}(x)$ for each $x \in I$.

Definition 4. Let Ω be a fixed signature. A topological free universal algebra G is almost discrete if the space of generators I_G is a discrete subspace of the space G.

Since any mapping of a discrete space is continuous, each almost discrete topological free universal algebra G is a free universal algebra G. Moreover, $End(G) = End_c(G)$ for any almost discrete topological free universal algebra G.

Lemma 1. *Let G be a topological universal algebra of the signature Ω and M be a subset of G. Then:*

1. The set $End_c(G)_M = \{\varphi \in End_c(G) : \varphi(G) \subseteq M\}$ is a left ideal of the semigroup $End_c(G)$.

2. The set $End(G)_M = \{\varphi \in End(G) : \varphi(G) \subseteq M\}$ is a left ideal of the semigroup $End(G)$.

Proof. If $\varphi, \psi \in End(G)$ and $\varphi(G) \subseteq M$, then $(\psi \cdot \varphi)(G) = \varphi(\psi(G)) \subseteq \varphi(G) \subseteq M$. The proof is complete. □

Definition 5. Let G be a topological universal algebra of the fixed signature Ω. The left ideal $End(G)_M$ is called a G-saturated ideal of the semigroup $End(G)$ and the ideal $End_c(G)_M$ is called a topologically G-saturated ideal of the semigroup $End_c(G)$.

Denote by $Spec^s(S)$ the family of all G-saturated ideals of $End(G)$ and by $Spec^s_c(S)$ the family of all topologically G-saturated ideals of $End_c(G)$ and $End(G)$.

Remark 2. Let $\{M_\gamma : \gamma \in \Gamma\}$ be a family of subsets of a universal algebra G. Then $\cup\{End(G)_{M_\gamma} : \gamma \in \Gamma\} \subseteq End(G)_{\cup\{M_\gamma : \gamma \in \Gamma\}}$ and $\cup\{End_c(G)_{M_\gamma} : \gamma \in \Gamma\} \subseteq End_c(G)_{\cup\{M_\gamma : \gamma \in \Gamma\}}$.

Definition 6. A universal algebra A is called:

– simple if and only if every homomorphism with domain A is either injective or constant;

– weakly simple if every homomorphism $f : A \longrightarrow A$ is either surjective or constant;

– cyclic if there exists a point $a \in G$ such that the set $\{a\}$ generates the algebra G.

Any topological free cyclic universal algebra is almost discrete.

The Abelian group of integers $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ modulo p, where $\mathbb{Z}$ is the group of integers with the addition operation $+$ and p is a prime number, is a simple and cyclic free group. Distinct examples of simple universal algebras are constructed in [16, 17].

Example 2. Let $G = \Omega = \Omega_1 = \{z = (x, y) : x, y \in \mathbb{R}, x^2+y^2 = 1\}$. If $u = (v, w) \in \Omega$ and $z = (x, y) \in G$, then $u(z) = (vx - wy, vy + wx)$. Then G is a cyclic and weakly simple universal algebra. The algebra G is not simple. Any element $a \in G$ generates G.

Example 3. Let $\mathbb{T} = \{z = (x, y) : x, y \in \mathbb{R}, x^2 + y^2 = 1\}$, with the multiplication binary operation $u \cdot z = (vx - wy, vy + wx)$ for all $u = (v, w), z = (x, y) \in \mathbb{T}$, be the multiplicative circle group of all complex numbers with absolute value 1, that is, the unit circle in the complex plane. Then $\mathbb{T}$ is a weakly simple topological group and is not a simple topological group.

Proposition 1. *Let G be a weakly simple universal algebra. Then for any family $\{M_\gamma : \gamma \in \Gamma\}$ of subsets of the algebra G it holds that either $G = \cup\{M_\gamma : \gamma \in \Gamma\}$ or $\cup\{End(G)_{M_\gamma} : \gamma \in \Gamma\} = End(G)_{\cup\{M_\gamma:\gamma\in\Gamma\}}$ and $\cup\{End_c(G)_{M_\gamma} : \gamma \in \Gamma\} = End_c(G)_{\cup\{M_\gamma:\gamma\in\Gamma\}}$.*

Proof. Assume that $G \neq \cup\{M_\gamma : \gamma \in \Gamma\}$ and $\varphi \in End(G)_{\cup\{M_\gamma:\gamma\in\Gamma\}}$. Since $\varphi(G) \subseteq \cup\{M_\gamma : \gamma \in \Gamma\} \neq G$, the mapping φ is constant. Assume that $b \in \varphi(G)$. There exists $\gamma \in \Gamma$ such that $b \in M_\gamma$. Then $\varphi(G) = \{b\} \subseteq M_\gamma$ and $\varphi \in End(G)_{M_\gamma}$. The proof is complete. □

The subset M of the algebra G is called the $a(\sigma)$-subset if M is a union of subalgebras of the algebra G.

Proposition 2. *Let G be a universal algebra. The family $Sub_{a(\sigma)}(G)$ of all $a(\sigma)$-subsets of the algebra G is a complete sublattice of the lattice $Exp(G)$ of all subsets of G.*

Proof. For any element $a \in G$ denote by C_a the cyclic subalgebra generated by the element a. A subset M of G is an $a(\sigma)$-subset if and only if $C_a \subseteq M$ for any element $a \in M$. Hence for any family $\{M_\gamma : \gamma \in \Gamma\}$ of $a(\sigma)$-subsets of G the sets $\cup\{M_\gamma : \gamma \in \Gamma\} = \cup\{C_a : a \in \cup\{M_\gamma : \gamma \in \Gamma\}\}$ and $\cap\{M_\gamma : \gamma \in \Gamma\} = \cup\{C_a : a \in \cap\{M_\gamma : \gamma \in \Gamma\}\}$ are $a(\sigma)$-subsets of G. The proof is complete. □

The subset M of the algebra G is called the $e(\sigma)$-subset if
$M = \cup\{\varphi(G) : \varphi \in H\}$ for some subset $H \subseteq End(G)$. The subset M of the algebra G is called the $e_c(\sigma)$-subset if $M = \cup\{\varphi(G) : \varphi \in H\}$ for some subset $H \subseteq End_c(G)$. We have $Sub_{ec(\sigma)}(G) \subseteq Sub_{e(\sigma)}(G) \subseteq Sub_{a(\sigma)}(G)$.

Theorem 1. *Let G be a universal algebra. The following assertions are equivalent:*

1. G is a cyclic universal algebra.

2. $\cup\{End(G)_{M_\gamma} : \gamma \in \Gamma\} = End(G)_{\cup\{M_\gamma : \gamma \in \Gamma\}}$ for any family of $a(\sigma)$-subsets $\{M_\gamma : \gamma \in \Gamma\} \subseteq Sub_{a(\sigma)}(G)$.

Proof. Let G be a cyclic universal algebra with the generator element $b \in G$ and $\{M_\gamma : \gamma \in \Gamma\}$ be a family of $a(\sigma)$-subsets of G. Assume that $\varphi \in End(G)$ and $\varphi(G) \subseteq \cup\{M_\gamma : \gamma \in \Gamma\}$. There exists $\gamma \in \Gamma$ such that $\varphi(b) \in M_\gamma$. Since M_γ is an $a(\sigma)$-subset of G, there exists a subalgebra H of G such that $\varphi(b) \in H \subseteq M_\gamma$. Then the subalgebra B of G generated by the point $\varphi(b)$ is a subalgebra of the algebra H. Since the algebra G is generated by the element b, the subalgebra $\varphi(G)$ is a cyclic algebra generated by the point $\varphi(b)$. Hence $\varphi(G) = B \subseteq H \subseteq M_\gamma$ and $\varphi \in End(G)_{M_\gamma}$. Therefore $End(G)_{\cup\{M_\gamma : \gamma \in \Gamma\}} \subseteq \cup\{End(G)_{M_\gamma} : \gamma \in \Gamma\}$. By virtue of Remark 2, $\cup\{End(G)_{M_\gamma} : \gamma \in \Gamma\} = End(G)_{\cup\{M_\gamma : \gamma \in \Gamma\}}$. Implication $1 \rightarrow 2$ is proved.

Let G be a non-cyclic universal algebra. The point $a \in G$ generates the cyclic subalgebra C_a of G. Then $\{C_a : a \in G\}$ is a family of cyclic subalgebras of algebra G and $G = \cup\{C_a : a \in G\}$. Let $\psi : G \longrightarrow G$ be the identical endomorphism. Since G is not a cyclic universal algebra, $C_a \neq G$ and $\psi \notin End(G)_{C_a}$ for any $a \in G$. Hence $\psi \notin \cup\{End(G)_{M_\gamma} : \gamma \in \Gamma\}$ and $\psi \in End_G = End(G)_{\cup\{M_\gamma : \gamma \in \Gamma\}}$. Implication $2 \rightarrow 1$ is proved. The proof is complete. □

Lemma 2. *Let G be a free universal algebra with the space of generators I. Then $Sub_{e(\sigma)}(G) = Sub_{a(\sigma)}(G)$.*

Proof. Fix a point $a \in G$. Then there exits a unique endomorphism $\varphi_a : G \longrightarrow G$ such that $\varphi_a(x) = a$ for any $x \in I$. Since the algebra G is generated by the set I, the subalgebra $C_a = \varphi_a(G)$ is generated by the set $\varphi_a(I) = \{a\}$ and C_a is a cyclic subalgebra. If H is a subalgebra of the algebra G, then $H = \cup\{C_a : a \in H\}$. Therefore, if M is an $a(\sigma)$-subset of G, then
$M = \cup\{C_a : a \in M\} = \cup\{\varphi_a(G) : a \in M\}$ and M is an $e(\sigma)$-set of G. The proof is complete. □

Corollary 1. *(Theorem [27], p.79). Let G be a free universal algebra. The following assertions are equivalent:*

1. G is a cyclic universal algebra.

2. $\cup\{End(G)_{M_\gamma} : \gamma \in \Gamma\} = End(G)_{\cup\{M_\gamma : \gamma \in \Gamma\}}$ for any family of $e(\sigma)$-subsets $\{M_\gamma : \gamma \in \Gamma\} \subseteq Sub_{e(\sigma)}(G)$.

Lemma 3. *Let G be a free topological universal algebra with the space of generators I. Then $Sub_{ec(\sigma)}(G) = Sub_{a(\sigma)}(G)$.*

Proof. Fix a point $a \in G$. Then there exits a unique continuous endomorphism $\varphi_a : G \longrightarrow G$ such that $\varphi_a(x) = a$ for any $x \in I$. Since the algebra G is generated by the set I, the subalgebra $C_a = \varphi_a(G)$ is generated by the set $\varphi_a(I) = \{a\}$ and C_a is a cyclic subalgebra. If H is a subalgebra of the algebra G, then $H = \cup\{C_a : a \in H\}$. Therefore, if M is an $a(\sigma)$-subset of G, then $M = \cup\{C_a : a \in M\} = \cup\{\varphi_a(G) : a \in M\}$ and M is an $e_c(\sigma)$-set of G. The proof is complete. □

Corollary 2. *Let G be a free topological universal algebra. The following assertions are equivalent:*

1. *G is a cyclic universal algebra.*
2. *$\cup\{End_c(G)_{M_\gamma} : \gamma \in \Gamma\} = End_c(G)_{\cup\{M_\gamma : \gamma\in\Gamma\}}$ for any family of $e_c(\sigma)$-subsets $\{M_\gamma : \gamma \in \Gamma\} \subseteq Sub_{ec}(\sigma)(G)$.*

Corollary 3. *Let G be a free topological universal algebra. The family $Sub_{e(\sigma)}(G)$ of all $e(\sigma)$-subsets and the family $Sub_{ec(\sigma)}(G)$ of all $e_c(\sigma)$-subsets of the algebra G are complete sublattices of the lattice $Exp(G)$ of all subsets of G.*

Theorem 2. *Let G be a topological universal algebra. Then $Spec^s_c(G)$ and $Sub_{ec(\sigma)}(G)$ are isomorphic complete lattices.*

Proof. Let $\Lambda_G(M) = End_c(G)_M = \{\varphi \in End_c(G) : \varphi(G) \subseteq M\}$. The mapping $\Lambda_G : Sub_{ec(\sigma)}(G) \longrightarrow Spec^s_c(G)$ is defined correctly.

Property 1. If M, N are $e_c(\sigma)$-subsets of G, then $N \subseteq M$ if and only if $End_c(G)_N \subseteq End_c(G)_M$.

Obviously, from $N \subseteq M$ it follows that $End_c(G)_N \subseteq End_c(G)_M$. Assume that $End_c(G)_N \subseteq End_c(G)_M$. If $a \in N \setminus M$, then there exists $\varphi \in End_c(G)_N$ such that $a \in \varphi(G) \subseteq N$. Then $\varphi \in End_c(G)_N \setminus End_c(G)_M$, a contradiction. Property 1 is proved.

Property 2. If M, N are $e_c(\sigma)$-subsets of G and $N \neq M$, then $End_c(G)_N \neq End_c(G)_M$.

If $a \in N \setminus M$, then there exists $\varphi \in End_c(G)_N$ such that $a \in \varphi(G) \subseteq N$. Then $\varphi \in End_c(G)_N \setminus End_c(G)_M$ and $End_c(G)_N \neq End_c(G)_M$. Property 2 is proved.

From Properties 1 and 2 it follows that Λ_G is an order isomorphism. The proof is complete. □

Corollary 4. *Let G be a universal algebra. Then $Spec^s(G)$ and $Sub_{e(\sigma)}(G)$ are isomorphic complete lattices.*

The following assertion was proved in [23, 24, 27].

Corollary 5. *Let G be a free universal algebra. Then $Spec^s(G)$, $Sub_{a(\sigma)}(G)$ and $Sub_{e(\sigma)}(G)$ are isomorphic complete lattices.*

Corollary 6. *Let G be a free topological universal algebra. Then $Spec^s(G)$, $Spec^s_c(G)$, $Sub_{a(\sigma)}(G)$, $Sub_{e(\sigma)}(G)$ and $Sub_{ec(\sigma)}(G)$ are isomorphic complete lattices.*

Example 4. Let G_1 and G_2 be two non-trivial multiplicative groups, e_1 be the neutral element of G_1, e_2 be the neutral element of G_2 and $G = G_1 \cup G_2$. On G consider binary operation $\{\cdot\}$ such that:

– $\{\cdot\}$ is group multiplication on G_i for any $i \in \{1,2\}$;

– if $x \in G_1$ and $y \in G_2$, then $x \cdot y = y$ and $y \cdot x = x$;

We put $\Omega = \Omega_2 = \{\cdot\}$. Then G is a universal algebra of the signature Ω. By construction, G is a groupoid and is not a semigroup: the multiplication on G is not associative. Instantly, if $x \in G_1$ and $y, z \in G_2$, then $y \cdot (x \cdot z) = y \cdot z$ and $(y \cdot x) \cdot z = x \cdot z = z$. Let $Sub(G_1)$ be the family of all subsemigroups of G_1 and $Sub(G_2)$ be the family of all subsemigroups of G_2. If H_1 is a subsemigroup of G_1 and H_2 is a subsemigroup of G_2, then H_1, H_2 and $H_1 \cup H_2$ are subalgebras of the algebra G of signature Ω. In particular, $\{e_1\}$, $\{e_2\}$ and $\{e_1, e_2\}$ are subalgebras of G and homomorphic images of G. Hence
$Sub(G) = sub(G_1) \cup Sub(G_2) \cup \{H_1 \cup H_2 : H_1 \in Sub(G_1), H_2 \in Sub(G_2)$.

Consider an endomorphism $\psi : G \longrightarrow G$.

Property 1. If $\psi(e_1) \in G_1$, then $\psi(G_1) \subseteq G_1$.

Assume that $x \in G_1$ and $\psi(x) \in G_2$. Then
$\psi(x) = \psi(x \cdot e_1) = \psi(x) \cdot \psi(e_1) = \psi(e_1)$, a contradiction.

Property 2. If $\psi(e_1) \in G_2$, then $\psi(G_1) \subseteq G_2$.

Assume that $x \in G_1$ and $\psi(x) \in G_1$. Then
$\psi(x) = \psi(x \cdot e_1) = \psi(x) \cdot \psi(e_1) = \psi(e_1)$, a contradiction.

Hence, we have the following four cases:

Case 1. $\psi(G) \subseteq G_1$.

Fix $x \in G_1$ and $y \in G_2$. Since $x \cdot y = y$, we have $\psi(y)= \psi(x \cdot y) = \psi(x) \cdot \psi(y)$ and $\psi(x) = e_1$. Since $y \cdot x = x$, we have $\psi(x)= \psi(y \cdot x) = \psi(y) \cdot \psi(x)$ and $\psi(y) = e_1$. Hence $\psi(G) = \{e_1\}$.

Case 2. $\psi(G) \subseteq G_2$.

In this case $\psi(G) = \{e_2\}$.

Case 3. $\psi(G_1) \subseteq G_1$ and $\psi(G_2) \subseteq G_2$.

The identical endomorphism is one of these endomorphisms. In this case
$\psi_1 = \psi|G_1 : G_1 \longrightarrow G_1$ and $\psi_2 = \psi|G_2 : G_2 \longrightarrow G_2$ are semigroup endomorphisms.

Case 4. $\psi(G_1) \subseteq G_2$ and $\psi(G_2) \subseteq G_1$.

In this case $\psi_1 = \psi|G_1 : G_1 \longrightarrow G_2$ and $\psi_2 = \psi|G_2 : G_2 \longrightarrow G_1$ are semigroup endomorphisms.

Conclusion 1. G_1, G_2 and their non-trivial subsemigroups are subalgebras and are not $e(\sigma)$-subsets of the universal algebra G.

Conclusion 2. $\{e_1\}$, $\{e_2\}$ and $\{e_1, e_2\}$ are subalgebras and $e(\sigma)$-subsets of the universal algebra G and $End(G)_{\{e_1\}} \cup End(G)_{\{e_2\}} \neq End(G)_{\{e_1,e_2\}}$.

Conclusion 3. If G_1 and G_2 are cyclic groups, then G is a bicyclic groupoid.

Example 5. Let $(A, *)$ be a non-trivial cyclic group with the neutral element e and generator element a, $n \geq 2$, $B = \{1, 2, ..., n\}$, $(A \times B) \cap \{\varepsilon, \beta\} = \emptyset$, $\varepsilon \neq \beta$ and
$G = (A \times \{1, 2, ..., n\}) \cup \{\varepsilon, \beta\}$. We put $\mu(n-1) = n$ and $\mu(i) = i + 1$ for $i \leq n - 1$.

On B consider the binary operation $\{\odot\}$, where $i \odot i = i$ and $i \odot j = 1 + |i - j|$ for $i, j \in B$ and $i \neq j$.

Now on G consider the binary operation $\{\cdot\}$ such that:

– $\varepsilon \cdot x = x \cdot \varepsilon = x$ for each $x \in G$;

– if $x, y \in A$ and $i, j \in B$, then $(x, i) \cdot (y, j) = 9x * y, i \odot j)$;

– $\beta \cdot \beta = (a, 1)$ and $\beta \cdot (x, i) = (x, i) \cdot \beta = (x, \mu(i))$ for each $(x, i) \in A \times B$.

We put $\Omega = \Omega_0 \cup \Omega_2$, $\Omega_0 = \{\varepsilon\}$ and $\Omega_2 = \{\cdot\}$. Then G is a universal algebra of the signature Ω. By construction, G is a cyclic groupoid with the unity ε, generator element β and G is not a semigroup.

Let $Sub(A)$ be the family of all subsemigroups of A, $C \subseteq B$ $C \odot C \subseteq C \neq \emptyset$. Then $(H \times C) \cup \{\varepsilon\}$ is a subalgebra of G.

Consider an endomorphism $\psi : G \longrightarrow G$.

Property 1. If $\psi(\beta) = (e, i)$, then $\psi(A \times B) = \{(e, i)\}$.

Property 2. If $\psi(\beta) = (c, i) \in A \times \{i\}$, then $\psi(A \times B) \subseteq A \times \{i\}$ and $\varphi(a, j) = (c^{2+j}, i)$ for any $j \in B$.

4 The semigroup of endomorphisms

G. Gratzer and E. T. Schmidt [14] proved that any complete lattice is isomorphic to the lattice of congruence of some universal algebra. The semigroup $End(G)$ of all endomorphisms and the semigroup $End_c(G)$ of all continuous endomorphisms of a topological universal algebra G are semigroups with unity. In [21] A. I. Mal'cev described the structure of a symmetrical groupoid (semigroup of all transformations of a set).

The following theorem is a generalization and conceptualization of a theorem from ([27], p. 98).

Theorem 3. *For any semigroup with unity S there exist a signature Ω and a universal algebra G_S of signature Ω such that:*

1. The semigroups S and $End(G_S)$ are isomorphic.

2. G_S is a free cyclic universal algebra of signature Ω.

3. $\Omega = \Omega_1$ and there exists a bijection $u : S \longrightarrow \Omega$ such that $u(x) = u_x$ and $u_x \cdot u_y = u_{y \cdot x}$ for any $x, y \in S$. In particular, relative to operation of composition the signature Ω is a semigroup anti-isomorphic with the semigroup S.

Proof. Let e be the unity of S. We put $G_S = S$ and $\Omega = \Omega_1 = \{u_a : a \in S\}$. For any $a \in S$, $u_a \in \Omega_1$ and any $x \in G_S = S$ we put $u_a(x) = a \cdot x$. If $a, b \in S$, then $(u_a \cdot u_b)(x) = u_b(u_a(x)) = (b \cdot a) \cdot x = u_{b \cdot a}(x)$. Hence, if $u(a) = u_a$ for any $a \in S$, then $u : S \longrightarrow \Omega$ is a bijection and $u(a \cdot b) = u_b \cdot u_a$ for all $a, b \in S$. Therefore $u : S \longrightarrow \Omega$ is an anti-isomorphism.

For any $u_a \in \Omega$ we have $u_a(e) = a$ and $u_e(a) = a$. Hence the universal algebra G_S is generated by the element e and G_S is a cyclic algebra.

For any $a \in S$ consider the function $\varphi_a : G_S \longrightarrow G_S$, where $\varphi_a(x) = x \cdot a$ for any $x \in G_S = S$. We have $\varphi_a(u_b(x)) = (b \cdot x) \cdot a = b \cdot (x \cdot a) = u_b(\varphi_a(x))$ for any $x \in G_S$. Therefore $\varphi_a \in End(G_S)$. Since G_S is a cyclic universal algebra, φ_a is the unique endomorphism of G_S for which $\varphi_a(e) = a$. Moreover if $\varphi : G_S \longrightarrow G_S$ is an endomorphism, then $\varphi = \varphi_{\varphi(e)}$. Hence G_S is a free cyclic universal algebra of signature Ω and $End(G_S) = \{\varphi_a : a \in S\}$.

Consider the correspondence $\psi : S \longrightarrow End(G_S)$, where $\psi(a) = \varphi_a$ for each $a \in S$. We have $(\varphi_a \cdot \varphi_b)(x) = \varphi_b(\varphi_a(x)) = (x \cdot a) \cdot b = x \cdot (a \cdot b) = \varphi_{a \cdot b}(x)$. Therefore $\psi(a \cdot b) = \varphi_{a \cdot b} = \varphi_a \cdot \varphi_b = \psi(a) \cdot \psi(b)$ and ψ is an isomorphism of the semigroups S and $End(G_S)$. The proof is complete. □

We say that two universal algebras are H-equivalent if their semigroups of endomorphisms are isomorphic. In this case, any universal algebra has its imprints in the group of endomorphisms. One of these imprints are saturated ideals. Saturated ideal is not an intrinsic conception of theory of semigroups.

Remark 3. In ([27], p. 98) it was proved that *"a semigroup S is isomorphic with the semigroup $End(G_S)$ of endomorphisms of some free universal algebra G_S of some signature Ω if and only if S contains a right ideal F and a non-empty subset $E \subseteq F$ with the following properties:*

(1) if $x, y \in S$ and $ex = ey$ for each $e \in E$, then $x = y$;

(2) for any mapping $\varphi : E \longrightarrow F$ there exists some element $s \in S$ such that $\varphi(e) = es$ for any $e \in \sharp$

In these conditions $G_S = F$, E is the set of generators of the algebra G and the right translations $F \cdot s$, $s \in S$, are endomorphisms of the algebra G."

In any semigroup S with unity 1 the ideal $F = S$ and the set $E = \{1\}$ are the desired objects. Therefore Theorem 3 is a more concrete formulation of the above Theorem from [27]. Moreover the content and proof of Theorem 3 are simpler, more transparent and present an effective method of construction of the algebra G_S.

5 Lattice of left ideals

For any set X we determine the lattice $Exp_1(X) = Exp(X)$ of all subsets of X, the lattice $Exp_2(X) = Exp_1(Exp_1(X))$ of all subsets of $Exp_1(X)X$ and, by induction, the lattice $Exp_n(X) = Exp_1(Exp_{n-1}(X))$ of all subsets of $Exp_{n-1}(X)X$.

We consider that the structure of a lattice $\mathcal{L}$ is determined if we determine a set X, a natural number n and a lattice $\mathcal{E}$ as the ordered subset of $Exp_n(X)$ such that the lattices $\mathcal{L}$ and $\mathcal{E}$ are isomorphic.

From this point of view, the structure of lattices $Spec^s_c(G)$ and $Spec^s(G)$ were determined.

For any element a of a semigroup S with unity denote by Sa the principal left ideal generated by a. One of the Green's [15] relations is the relation λ_S defined by: $a\lambda_S b$ if and only if $Sa = Sb$. This relation leads us to solve the following equation $x \cdot a = b$ in the semigroup of endomorphisms of any universal algebra.

We put $a \prec_S b$ if and only if $Sa \subseteq Sb$. Obviously, $a\lambda_S b$ if and only if $a \prec_S b$ and $b \prec_S b$.

Fix a signature Ω and a non-trivial universal algebra G of signature Ω. We put $Sub_E(G) = \{\varphi(G) : \varphi \in End(G)\}$. It is obvious that $Sub_E(G) \subseteq Sub(G)$. The set $Sub_E(G)$ is ordered. The subset $L \subseteq Sub_E(G)$ is called an E-set. The E-set $L \subseteq Sub_E(G)$ is a hereditary E-set if from $A \in L$, $B \in Sub_E(G)$ and $B \subseteq A$ follows $B \in L$. The improper families $End_E(G)$ and $\emptyset$ are full hereditary E-sets. The intersection of hereditary E-sets is a hereditary E-set. Therefore the family $Q(G) \cdot \varphi$ of all hereditary E-sets is a complete lattice. We have $Q(G) \subset Exp(Sub_E(G)) \subseteq Exp_2(G)$.

Remark 4. For any subset $\mathcal{L} \subseteq Sub_E(G)$ of E-sets there exists a unique minimal hereditary E-set $\mathcal{E}$ such that $\mathcal{L} \subseteq \mathcal{E}(\mathcal{L})$. We have
$\mathcal{E}(\mathcal{L}) = \cup\{\{H \in Sub_E(G) : H \subseteq L\} : L \in \mathcal{L}\}$. We say that the family $\mathcal{E}(\mathcal{L})$ is generated by the family $\mathcal{L}$.

The family $End(G) \cdot \varphi = \{\psi \cdot \varphi : \psi \in End(G)\}$ is the principal left ideal generated by the endomorphism $\varphi \in End(G)$.

A left ideal A of $End(G)$ is called a full left ideal if for any $f \in A$ and any E-set B with $H \subseteq f(G)$ there exists $g \in A$ such that $g(G) = B$.

Denote by $Spec^f(End(G), l)$ the set of all full left ideals of the semigroup $End(G)$. The intersection of full left ideals is a full left ideal and $Spec^f(End(G), l)$ is a complete lattice of ideals.

Remark 5. Any saturated ideal is a full left ideal.

For any subset $H \subseteq End(G)$ we put $\Phi(H) = \mathcal{E}(\{g(G) : g \in H\})$.

Lemma 4. *Let $H \subseteq End(G)$. The following assertions are equivalent:*

1. H is a full left ideal of $End(G)$.

2. $\Phi(H)$ is a hereditary family of E-sets.

Proof. Follows from the definitions of a full left ideal and a hereditary family of E-sets. □

Lemma 5. *The mapping $\Phi : Spec(End(G), l) \longrightarrow Q(G)$ is an order morphism of the lattice $Spec(End(G), l)$ of left ideals onto a lattice $Q(G)$. Moreover:*

1. If A is a full left ideal, B is a left ideal of the semigroup $End(G)$ and $B \setminus A \neq \emptyset$, then $\Phi(A) \neq \Phi(B)$.

2. $Q(G) = \Phi(Spec^f(End(G), i))$.

Proof. By construction, Φ is a well defined mapping. It is obvious that Φ preserves the order: if $A, B \in Spec(End(G), l)$ and $A \subseteq B$, then $\Phi(A) \subseteq \Phi(B)$. Hence Φ is an order morphism.

Let $A \in Spec^f(End(G), l)$, $B \in Spec(End(G), l)$ and $g \in B \setminus A$, then
$g(G) \in \Phi(B) \setminus \Phi(A)$ and $\Phi(B) \neq \Phi(A)$. Assertion 1 is proved.

Let $\mathcal{L} \in Q(G)$ and $L = \{f \in End(G) : f(G) \in \mathcal{L}\}$. If $f \in L$ and $g \in End(L)$, then $(g \cdot f)(G) = f(g(G)) \subseteq f(G)$ and $g \cdot f \in L$. Hence L is a full left ideal and $\Phi(L) = \mathcal{L}$. Assertion 2 is proved. □

Corollary 7. *Let G be a topological universal algebra. The lattices $Q(G)$ and $Spec^f(End(G), l)$ are isomorphic.*

The assertion 1 of the following proposition was proved in [24, 27].

Proposition 3. *Let G be a free topological universal algebra with the set of generators I. Fix $\varphi, \psi \in End(G)$. Then:*

1. The equation $\xi \cdot \varphi = \psi$ has solution in $End(G)$ if and only if $\psi(G) \subseteq \varphi(G)$.

2. The set of solutions of the equation $\xi \cdot \varphi = \psi$ has the cardinality of the set $\Pi\{\varphi^{-1}(\psi(i)) : i \in I\}$.

Proof. If $h \in End(G)$ is a solution of the equation $\xi \cdot \varphi = \psi$, then $h \cdot \varphi = \psi$ and $\psi(G) = (h \cdot \varphi)(G) = \varphi(h(G)) \subseteq \varphi(G)$.

Assume that $\psi(G) \subseteq \varphi(G)$. Then for any $i \in I$ the set $\varphi^{-1}(\psi(i))$ is non-empty. By virtue of the axiom of choice AC of Zermelo, there exists a choice function $f : I \longrightarrow G$ such that $f(i) \in \varphi^{-1}(\psi(i))$ for any $i \in I$. Since G is a free universal algebra with the generators I, there exists a unique endomorphism $\hat{f} : G \longrightarrow G$ such that $f(i) = \hat{f}(i)$ for each $i \in I$. Consider the endomorphism $h = \hat{f} \cdot \varphi$. We have $h(i) = \varphi(f(i)) = \psi(i)$ for each $i \in I$. Hence $\psi = \hat{f} \cdot \varphi$ and $\hat{f}$ is a solution of the equation $\xi \cdot \varphi = \psi$. Any choice function $f : I \longrightarrow G$ with $f(i) \in \varphi^{-1}(\psi(i))$ is an element of the Cartesian product $\Pi\{\varphi^{-1}(\psi(i)) : i \in I\}$ and any element of this Cartesian product generates a unique solution of the equation $\xi \cdot \varphi = \psi$. Moreover, any solution of the equation $\xi \cdot \varphi = \psi$ is generated by a unique element of the Cartesian product $\Pi\{\varphi^{-1}(\psi(i) : i \in I\}$. The proof is complete. $\square$

Corollary 8. *Let G be a free topological universal algebra. The lattices $Q(G)$, $Spec(End(G), l)$ and $Spec^f(End(G), l)$ are isomorphic.*

Isomorphism of the lattices $Q(G)$ and $Spec(End(G), l)$ was established in [24,27].

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