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Almost Periodic and Almost Automorphic Solutions of Monotone Differential Equations with a Strict Monotone First Integral*

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Abstract. The paper is dedicated to the study of problem of Poisson stability (in particular periodicity, quasi-periodicity, Bohr almost periodicity, almost automorphy, Levitan almost periodicity, pseudo-periodicity, almost recurrence in the sense of Bebutov, recurrence in the sense of Birkhoff, pseudo-recurrence, Poisson stability) and asymptotical Poisson stability of motions of monotone non-autonomous differential equations which admit a strict monotone first integral. This problem is solved in the framework of general non-autonomous dynamical systems.

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1 Introduction

In this paper we study the problem of Bohr/Levitan almost periodicity, almost automorphy, almost recurrence in the sense of Bebutov, recurrence in the sense of Birkhoff and Poisson stability of solutions of monotone non-autonomous differential equations

$$x'(t) = f(t, x(t)) \tag{1}$$

having a strongly monotone first integral. We show that under some conditions every bounded on semi-axis solution of equation (1) has a limiting regime which has the same character of recurrence in time t as the right-hand side f of equation (1).

These types of problems for Bohr almost periodic differential equations were studied by many authors [26, 34, 38] (see also the bibliography therein) and for periodic/almost periodic difference equations in the works [17, 24].

We solve this problem in the framework of abstract non-autonomous dynamical systems with discrete time.

Our paper is organized as follow.

In the second section we collect some notions and facts from the non-autonomous dynamical systems: cocycles, skew-product dynamical systems, some semigroups related with non-autonomous dynamical systems, Poisson stable motions and their

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comparability by character of recurrence, monotone non-autonomous dynamical systems.

The third section is dedicated to the study the asymptotic behavior of monotone non-autonomous dynamical systems having a strict monotone first integral.

In the fourth section we study different classes of Poisson stable (periodic, almost periodic, almost automorphic, almost recurrent, recurrent, pseudo-periodic, pseudo-recurrent and Poisson stable) and asymptotically Poisson stable motions of non-autonomous dynamical systems admitting a strict monotone first integral.

The fifth section is dedicated to some applications of our general results obtained in the third and fourth sections to the study of asymptotic behavior of solutions of monotone non-autonomous differential equations having a strict monotone first integral.

2 Some general properties of non-autonomous dynamical systems

In this section we collect some notions and facts from the non-autonomous dynamical systems [7] (see also [10, Ch.IX]) which we will use below.

2.1 Cocycles.

Let Y be a complete metric space, $\mathbb{R}$ ($\mathbb{Z}$) be a group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be a semigroup of nonnegative real (integer) numbers, $\mathbb{S}$ be one of two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a subsemigroup of additive group $\mathbb{S}$, where $\mathbb{S}_+ := \{s \in \mathbb{S} :\ s \geq 0\}$. Let $(Y, \mathbb{S}, \sigma)$ be an autonomous two-sided dynamical system on Y and E be a real or complex Banach space with the norm $|\cdot|$.

Definition 1. (Cocycle on the state space E with the base $(Y, \mathbb{Z}, \sigma)$). The triplet $\langle E, \phi, (Y, \mathbb{S}, \sigma)\rangle$(or briefly ϕ) is said to be a cocycle (see, for example,[10] and [25]) on the state space E with the base $(Y, \mathbb{S}, \sigma)$ if the mapping $\phi : \mathbb{S}_+ \times Y \times E \to E$ satisfies the following conditions:

1. $\phi(0, u, y) = u$ for all $u \in E$ and $y \in Y$;
2. $\phi(t + \tau, u, y) = \phi(t, \phi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{S}_+, u \in E$ and $y \in Y$;
3. the mapping ϕ is continuous.

Definition 2. (Skew-product dynamical system). Let $\langle E, \phi, (Y, \mathbb{S}, \sigma)\rangle$ be a cocycle on $E, X := E \times Y$ and π be a mapping from $\mathbb{S}_+ \times X$ to X defined by equality $\pi = (\phi, \sigma)$, i.e., $\pi(t, (u, y)) = (\phi(t, \omega, u), \sigma(t, y))$ for all $t \in \mathbb{S}_+$ and $(u, y) \in E \times Y$. The triplet $(X, \mathbb{S}_+, \pi)$ is an autonomous dynamical system and it is called [25] a skew-product dynamical system.

Definition 3. (Non-autonomous dynamical system.) Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two subsemigroups of the group $\mathbb{S}, (X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two autonomous dynamical systems and $h : X \to Y$ be a homomorphism from $(X, \mathbb{T}_1, \pi)$ to $(Y, \mathbb{T}_2, \sigma)$ (i.e.,

$h(\pi(t,x)) = \sigma(t,h(x))$ for all $t \in \mathbb{T}_1$, $x \in X$ and h is continuous), then the triplet $\langle (X,\mathbb{T}_1,\pi),\ (Y,\ \mathbb{T}_2,\ \sigma),h\rangle$ is called (see [3] and [10]) a non-autonomous dynamical system.

Example 1. (The non-autonomous dynamical system generated by cocycle ϕ.) Let $\langle E,\phi,(Y,\mathbb{S},\sigma)\rangle$ be a cocycle, $(X,\mathbb{S}_+,\pi)$ be a skew-product dynamical system ($X = E\times Y, \pi = (\phi,\sigma)$) and $h = pr_2 : X \to Y$, then the triplet $\langle (X,\mathbb{S}_+,\pi),\ (Y,\mathbb{S},\sigma),h\rangle$ is a non-autonomous dynamical system.

We give below some general facts about non-autonomous dynamical systems without proofs. The readers can find more details and the proofs in [7] (see also [11, Ch.III]).

Definition 4. The point $y \in Y$ is called positively (respectively, negatively) stable in the sense of Poisson if there exists a sequence $t_k \to +\infty$ (respectively, $t_k \to -\infty$) such that $\sigma(t_k,y) \to y$. If the point y is Poisson stable in the both directions, in this case it is called Poisson stable.

Denote by $\mathfrak{N}_y = \{\{t_k\}|\ \sigma(t_k,y) \to y\}$, $\mathfrak{N}_y^{+\infty} := \{\{t_k\} \in \mathfrak{N}_y|\ t_k \to +\infty\}$ and $\mathfrak{N}_y^{-\infty} := \{\{t_k\} \in \mathfrak{N}_y|\ t_k \to -\infty\}$.

Definition 5. Let (X,h,Y) be a fiber space [18], i.e., X and Y be two metric spaces and $h : X \to Y$ be a homomorphism from X into Y. The subset $M \subseteq X$ is said to be conditionally precompact [7, 10, 11] if the pre-image $h^{-1}(Y')\bigcap M$ of every pre-compact subset $Y' \subseteq Y$ is a pre-compact subset of X. In particularly $M_y = h^{-1}(y)\bigcap M$ is a precompact subset of X_y for every $y \in Y$. The set M is called conditionally compact if it is closed and conditionally precompact.

Let $\langle (X,\mathbb{T},\pi),(Y,\mathbb{S},\sigma),h\rangle$ be a non-autonomous dynamical system and $y \in Y$ be a positively Poisson stable point. Denote by

$$\mathcal{E}_y^{\pm} := \{\xi|\quad \exists \{t_k\} \in \mathfrak{N}_y^{\pm\infty} \quad \text{such that} \quad \pi^{t_k}|_{X_y} \to \xi\},$$

where $X_y := \{x \in X|\quad h(x) = y\}$ and $\to$ means the pointwise convergence.

Let X^X denote the Cartesian product of X copies of the space X equipped with Tykhonov topology. The set X^X can be provided with a semigroup structure with respect to composition of the maps from X^X (for more details see, for example,[3, ChI] and [15]).

Lemma 1. *[11, ChIII] Suppose that the following conditions are fulfilled:*

1. *$y \in Y$ is a two-sided Poisson stable point;*
2. *$\langle (X,\mathbb{S},\pi),\ (Y,\mathbb{S},\sigma),h\rangle$ is a two-sided non-autonomous dynamical system;*
3. *X is a conditionally compact space;*
4. $$\inf_{t\le 0} \rho(x_1t,x_2t) > 0$$

   for any $x_1,x_2 \in X_y \quad (x_1 \ne x_2)$.

Then the following statement hold:

1. *$\mathcal{E}_y^-$ and $\mathcal{E}_y^+$ are subgroups of the semigroup $\mathcal{E}_y$;*

2. *for any pair of points $x_1, x_2 \in X_y$ with $x_1 \neq x_2$ there are some sequences $\{t_k^-\} \in \mathfrak{N}_y^{-\infty}$ and $\{t_k^+\} \in \mathfrak{N}_y^{+\infty}$ such that*

$$\lim_{k\to\infty} \pi(t_k^{\pm}, x_i) = x_i \ (i = 1, 2).$$

Definition 6. Let $\langle E, \phi, (Y, \mathbb{S}, \sigma)\rangle \quad (respectively, (X, \mathbb{S}_+, \pi))$ be a cocycle (respectively, one-sided dynamical system). The continuous mapping $\nu : \mathbb{S} \to E$ (respectively, $\gamma : \mathbb{S} \to X$) is called an entire trajectory of cocycle ϕ (respectively, of dynamical system $(X, \mathbb{S}_+, \pi)$) passing through the point $(u, y) \in E \times Y$ (respectively, $x \in X$) for $t = 0$ if $\phi(t, \nu(s), \sigma(s, y)) = \nu(t + s)$ and $\nu(0) = u$ (respectively, $\pi(t, \gamma(s)) = \gamma(t + s)$ and $\gamma(0) = x$) for all $t \in \mathbb{S}_+$and $s \in \mathbb{S}$.

Denote by

– $C(\mathbb{S}, X)$ the space of all continuous functions $f : \mathbb{S} \mapsto X$ equipped with the compact-open topology;

– Φ_x the family of all entire trajectories of $(X, \mathbb{S}_+, \pi)$ passing through the point $x \in X$ at the initial moment $t = 0$ and $\Phi := \bigcup\{\Phi_x : \ x \in X\}$.

Remark 1. Note that:

1. the compact-open topology (which coincides, in this case, with the point-wise convergence) on the space $C(\mathbb{S}, X)$ is metrizable, for example, by distance
$$d(\varphi, \psi) := \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(\varphi, \psi)}{1 + d_i(\varphi, \psi)},$$
where $d_i(\varphi, \psi) := \max_{|t|\leq i} \rho(\varphi(t), \psi(t))$;

2. if $\gamma \in \Phi_x$ then $\gamma^\tau \in \Phi_{\pi(\tau,x)}$, where $\gamma^\tau(t) := \gamma(t + \tau)$ for any $t \in \mathbb{S}$ and, consequently, Φ is a translation invariant subset of $C(\mathbb{S}, X)$;

3. if $\gamma_k \in \Phi_{x_k}$ and $\gamma_k \to \gamma$ as $k \to \infty$ in $C(\mathbb{S}, X)$ then $\gamma \in \Phi_x$, where $x := \lim_{k\to\infty} x_k$ and, consequently, Φ is a closed subset of $C(\mathbb{S}, X)$.

2.2 Structure of the ω–limit set

Let $x_0 \in X$. Denote by ω_{x_0} the omega-limit set of the point x_0, i.e., $\omega_{x_0} := \{x \in X :$ there exists a sequence $\{t_k\}$ such that $t_k \to +\infty$ as $k \to \infty$ and $\lim_{k\to\infty} \pi(t_k, x_0) = x\}$.

Theorem 1. *[11, Ch.III],[12] Let $\langle(X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ be a non-autonomous dynamical system, $x_0\in X$, $\Sigma^+_{x_0}:=\{\pi(t,x_0):\ t\ge 0\}$ be a conditionally precompact set and $\omega_{y_0}\neq\emptyset$, where $y_0:=h(x_0)$. Then the following statements hold:*

1. *$\omega_{x_0}\bigcap X_q\neq\emptyset$ for any $q\in\omega_{y_0}$ and, consequently, $\omega_{x_0}\neq\emptyset$;*
2. *$h(\omega_{x_0})=\omega_{y_0}$;*
3. *the set ω_{x_0} is conditionally compact;*
4. *$\pi(t,\omega^q_{x_0})=\omega^{\sigma(t,q)}_{x_0}$ for any $t\in\mathbb{S}_+$ and $q\in\omega_{y_0}$, where $\omega^q_{x_0}:=\omega_{x_0}\bigcap X_q$;*
5. *ω_{x_0} is invariant, i.e., $\pi(t,\omega_{x_0})=\omega_{x_0}$ for any $t\ge 0$.*

Let $\langle(X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ be a non-autonomous dynamical system.

Definition 7. A subset $A\subseteq X$ is said to be uniformly stable in the positive direction if for arbitrary $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $\rho(x,a)<\delta$ ($a\in A,\ x\in X$ and $h(a)=h(x)$) implies $\rho(\pi(t,x),\pi(t,a))<\varepsilon$ for any $t\ge 0$.

A point $x_0\in X$ is called positively uniformly stable if the set $\Sigma^+_{x_0}:=\{\pi(t,x_0):\ t\ge 0\}$ is so.

Let $\big(C(\mathbb{S},X),\mathbb{Z},\lambda\big)$ be the shift dynamical system (Bebutov's dynamical system [3,10,25,30]) on the space $C(\mathbb{S},X)$. By Remark 1 Φ is a closed and invariant (with respect to shifts) subset of $C(\mathbb{S},X)$ and, consequently, on Φ a shift dynamical system $(\Phi,\mathbb{S},\lambda)$ induced by $\big(C(\mathbb{S},X),\mathbb{Z},\lambda\big)$ is defined.

2.3 Poisson stable motions and their comparability by character of recurrence

Let $(X,\mathbb{S},\pi)$ be a dynamical system.

Definition 8. A number $\tau\in\mathbb{S}$ is called an $\varepsilon>0$ shift of x (respectively, almost period of x) if $\rho(x\tau,x)<\varepsilon$ (respectively, $\rho(x(\tau+t),xt)<\varepsilon$ for all $t\in\mathbb{S}$).

Definition 9. A point $x\in X$ is called almost recurrent (respectively, Bohr almost periodic) if for any $\varepsilon>0$ there exists a positive number l such that at any segment of length l there is an ε shift (respectively, almost period) of point $x\in X$.

Definition 10. If the point $x\in X$ is almost recurrent and the set $H(x):=\overline{\{xt\mid t\in\mathbb{S}\}}$ is compact, then x is called recurrent.

Definition 11. A point $x\in X$ of the dynamical system $(X,\mathbb{S},\pi)$ is called Levitan almost periodic [22](see also [3,8] and [21]) if there exists a dynamical system $(Y,\mathbb{S},\sigma)$ and a Bohr almost periodic point $y\in Y$ such that $\mathfrak{N}_y\subseteq\mathfrak{N}_x$.

Definition 12. A point $x\in X$ is called stable in the sense of Lagrange (st.L), if its trajectory $\{\pi(t,x)\ :\ t\in\mathbb{S}\}$ is relatively compact.

Definition 13. A point $x \in X$ is called almost automorphic in the dynamical system $(X, \mathbb{S}, \pi)$, if it is st.L and Levitan almost periodic.

Definition 14. A point $x \in X$ is said [9, ChI],[13] to be asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically Poisson stable) if there exists a stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent, Levitan almost periodic, almost recurrent, Poisson stable) point $p \in X$ such that

$$\lim_{t \to +\infty} \rho(\pi(t, x), \pi(t, p)) = 0.$$

We will present here some notions and results stated and proved by B. A. Shcherbakov [30]-[33] (see also [11, Ch.I]).

Let $(X, \mathbb{S}, \pi)$ and $(Y, \mathbb{S}, \sigma)$ be two dynamical systems.

Definition 15. A point $x \in X$ is said to be comparable with $y \in Y$ by the character of recurrence if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ–shift of y is an ε–shift for x, i.e., $d(\sigma(\tau, y), y) < \delta$ implies $\rho(\pi(\tau, x), x) < \varepsilon$, where d (respectively, ρ) is the distance on Y (respectively, on X).

Denote by $\mathfrak{N}_x^{\infty} := \{\{t_k\} \in \mathfrak{N}_x : \text{such that } t_k \to \infty \text{ as } k \to \infty\}$.

Theorem 2. *The following conditions are equivalent:*

1. *the point $x \in X$ is comparable with y by the character of recurrence;*
2. $\mathfrak{N}_y \subseteq \mathfrak{N}_x$*;*
3. $\mathfrak{N}_y^{\infty} \subseteq \mathfrak{N}_x^{\infty}$.

Denote by $\mathfrak{M}_{x,\tilde{x}} := \{\{t_k\} \subset \mathbb{Z} \text{ such that } \{\pi(t_k, x)\} \to \tilde{x} \text{ as } k \to \infty\}$.

Theorem 3. *Let x be comparable with $y \in Y$. If the point $y \in Y$ is stationary (respectively, τ–periodic, Levitan almost periodic, almost recurrent, Poisson stable), then the point $x \in X$ is so.*

Definition 16. A point $x \in X$ is called *uniformly comparable with $y \in Y$ by character of recurrence* if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that every δ–shift of $\sigma(t, y)$ is an ε–shift for $\pi(t, x)$ for all $t \in \mathbb{Z}$, i.e., $d(\sigma(t + \tau, y), \sigma(t, y)) < \delta$ implies $\rho(\pi(t + \tau, x), x) < \varepsilon$ for all $t \in \mathbb{S}$ (or equivalently, $d(\sigma(t_1, y), \sigma(t_2, y)) < \delta$ implies $\rho(\pi(t_1, x), \pi(t_2, x)) < \varepsilon$ for all $t_1, t_2 \in \mathbb{S}$).

Denote by $\mathfrak{M}_x := \{\{t_k\} \subset \mathbb{Z} \ : \ \text{such that } \{\pi(t_k, x)\} \text{ converges } \}$.

Definition 17. A point $x \in X$ is said [6,9,11] to be strongly comparable by character of recurrence with the point $y \in Y$ if $\mathfrak{M}_y \subseteq \mathfrak{M}_x$.

Theorem 4. *Let y be stable in the sense of Lagrange. The inclusion $\mathfrak{M}_y \subseteq \mathfrak{M}_x$ takes place, if and only if the point x is stable in the sense of Lagrange and the point x is uniformly comparable by character of recurrence with y.*

Theorem 5. *Let X and Y be two complete metric spaces, the point x be uniformly comparable with $y \in Y$ by the character of recurrence. If the point $y \in Y$ is recurrent (respectively, almost periodic, almost automorphic, uniformly Poisson stable), then so is the point $x \in X$.*

Below we present some generalization of B. A. Shcherbakov's results concerning the comparability of points by the character of their recurrence (more details see in [4,5] and also [11, ChI]).

Let $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be two sub-semigroups of group $\mathbb{Z}$ ($\mathbb{T}_i = \mathbb{S}$ or $\mathbb{S}_+$ and $i = 1,2$). Consider two dynamical systems $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$.

Let $\mathfrak{M}_x^{+\infty} := \{\{t_k\} \in \mathfrak{M}_x \ : \ \text{such that } t_k \to +\infty \text{ as } k \to \infty\}$ and $\mathfrak{N}_x^{+\infty} := \{\{t_k\} \in \mathfrak{N}_x \ : \ \text{such that } t_k \to +\infty \text{ as } k \to \infty\}$.

Denote by $\mathfrak{M}_{y,q}^{+\infty} := \{\{t_k\} \in \mathfrak{M}_y^{+\infty} : \ \text{such that } \sigma(t_k, y) \to q \text{ as } k \to \infty\}$.

Theorem 6. *Let $y \in \omega_y$, then the following conditions are equivalent:*

a. $\mathfrak{N}_y^{\infty} \subseteq \mathfrak{N}_x^{\infty}$;

b. $\mathfrak{N}_y^{+\infty} \subseteq \mathfrak{N}_x^{+\infty}$.

Theorem 7. *Let $y \in \omega_y$, then the following conditions are equivalent:*

a. $\mathfrak{M}_y^{\infty} \subseteq \mathfrak{M}_x^{\infty}$ *and* $\mathfrak{N}_y^{\infty} \subseteq \mathfrak{N}_x^{\infty}$;

b. $\mathfrak{M}_y^{+\infty} \subseteq \mathfrak{M}_x^{+\infty}$ *and* $\mathfrak{N}_y^{+\infty} \subseteq \mathfrak{N}_x^{+\infty}$.

2.4 Monotone Non-autonomous Dynamical Systems

Recall that a Banach space E is ordered if it contains a closed convex cone P, that is, a non-empty closed subset P satisfying $P + P \subseteq P$, $\lambda P \subseteq P$ for all $\lambda \geq 0$, and $P \bigcap (-P) = \{0\}$.

Assume that (E, P) is an ordered Banach space with $Int(P) \neq \emptyset$. For $u_1, u_2 \in E$, we write $u_1 \leq u_2$ if $u_2 - u_1 \in P$; $u_1 < u_2$ if $u_2 - u_1 \in P \setminus \{0\}$; $u_1 \ll u_2$ if $u_2 - u_1 \in Int(P)$. Given $u_1, u_2 \in P$ the set $[u_1, u_2] := \{u \in E| \ u_1 \leq u \leq u_2\}$ is called a closed order interval in P, and we write $(u_1, u_2) := \{u \in E| \ u_1 < u < u_2\}$.

A subset U of E is said to be order convex if for any $a; b \in U$ with $a < b$, the segment $\{a + s(b - a)| \ s \in [0,1]\}$ is contained in U. And U is called lower-bounded (resp. upper-bounded) if there exists an element $a \in E$ such that $a \leq U$ (resp. $a \geq U$). Such an a is said to be a lower bound (resp. upper bound) for U. A lower bound α is said to be the greatest lower bound (g.l.b.) or *infimum* if any other lower bound a satisfies $a \leq \alpha$. Similarly, we can define the least upper bound (l.u.b.) or *supremum*.

Let $V = [0; b]_X$ with $b \gg 0$ or $V = P$, and furthermore, V be an order convex subset of E.

Let (X, h, Y) be a local-trivial and normed vector bundle [18] with the norm $|\cdot|$ and $V \subseteq X$ be a nonempty closed subset possessing the following properties:

1. $h(V) = Y$;
2. $V_y := V \bigcap X_y$ is a closed convex cone, that is, a non-empty closed subset V_y satisfying $V_y + V_y \subseteq V_y$, $\lambda V_y \subseteq V_y$ for all $\lambda \geq 0$, and $V_y \bigcap (-V_y) = \{\theta_y\}$ for any $y \in Y$.

We will use the order relation on (X, h, Y). We write $x_1 \leq x_2$ (respectively, $x_1 < x_2$ or $x_1 \ll x_2$) if $h(x_1) = h(x_2) = y$ and $x_2 - x_1 \in V_y$ (respectively, $x_2 - x_1 \in V_y \setminus \{\theta_y\}$ or $x_2 - x_1 \in Int(V_y)$).

Definition 18. A non-autonomous dynamical system $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$ is said to be monotone (respectively, strictly monotone) if $x_1 \leq x_2$ (respectively, $x_1 < x_2$) implies $\pi(t, x_1) \leq \pi(t, x_2)$ (respectively, $\pi(t, x_1) < \pi(t, x_2)$) for any $t > 0$.

Recall that a forward orbit $\{\pi(t, x_0) |\ t \geq 0\}$ of non-autonomous dynamical systems $\langle (X, \mathbb{Z}_+, \pi), (Y, \mathbb{Z}, \sigma), h \rangle$ is said to be uniformly stable if for any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\rho(\pi(t_0, x), \pi(t_0, x_0)) < \delta$ $(h(x) = h(x_0))$ implies $d(\pi(t, x), \pi(t, x_0)) < \varepsilon$ for every $t \geq t_0$.

Below we will use the following assumptions:

(**C1**) For every compact subset K in $V \subseteq X$ and $y \in Y$ the set $K_y := h^{-1}(y) \bigcap K$ has both the greatest lower bound (g.l.b.) $\alpha_y(K)$ and the least upper bound (l.u.b.) $\beta_y(K)$.

(**C2**) For every $x \in X$, the semi-trajectory $\Sigma_x^+ := \{\pi(t, x) :\ t \geq 0\}$ is conditionally precompact and its ω-limit set ω_x is positively uniformly stable.

(**C3**) The non-autonomous dynamical system $\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h \rangle$ is monotone.

Lemma 2. *[12] Suppose that the following conditions are fulfilled:*

1. *the points $x, x_0 \in X$ with $h(x) = h(x_0)$ are proximal, i.e., there is a sequence $t_k \to +\infty$ as $k \to \infty$ such that $\lim\limits_{k \to \infty} \rho(\pi(t_k, x), \pi(t_k, x_0)) = 0$;*
2. *the points $x_0 \in X$ is uniformly positively stable.*

Then the points x, x_0 are asymptotic, that is, $\lim\limits_{t \to +\infty} \rho(\pi(t, x), \pi(t, x_0)) = 0$.

Lemma 3. *[12] Assume that (**C1**)–(**C3**) hold, $x_0 \in X$ such that ω_{x_0} is positively uniformly stable. Let $K := \omega_{x_0}$ be fixed and $y_0 := h(x_0)$. Then if $q \in \omega_q \subseteq \omega_{y_0}$, $\alpha_q := \alpha_q(K)$, $K^1 := \omega_{\alpha_q}$, then the set $K_q^1 := \omega_{\alpha_q} \bigcap X_q$ (respectively, $\omega_{\beta_q} \bigcap X_q$) consists of a single point γ_q (respectively, δ_q), i.e., $K_q^1 = \{\gamma_q\}$ (respectively, $\{\delta_q\}$).*

Theorem 8. *[12] (**Comparability**) Assume that (**C1**)–(**C3**) hold, $x_0 \in X$ such that ω_{x_0} is positively uniformly stable and $y_0 := h(x_0)$. Then the following statements hold:*

1. *if $y_0 \in \omega_{y_0}$, then the point γ_{y_0} (respectively, β_{y_0}) is comparable by character of recurrence with y_0, i.e., $\mathfrak{N}^{+\infty}_{y_0} \subseteq \mathfrak{N}^{+\infty}_{\gamma_{y_0}}$ and*

2. $$\lim_{n\to\infty} \rho(\pi(t,\alpha_{y_0}),\pi(t,\gamma_{y_0})) = 0\ .$$

Corollary 1. *Under the conditions (**C1**) – (**C3**) if the point y_0 is τ-periodic (respectively, Levitan almost periodic, almost recurrent, almost automorphic, recurrent, Poisson stable), then:*

1. *the point γ_{y_0} is so;*
2. *the point α_{y_0} is asymptotically τ-periodic (respectively, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically almost automorphic, asymptotically recurrent, asymptotically Poisson stable).*

Definition 19. A point $x_0 \in X$ is said to be:

– pseudo-recurrent [29,33,35] if for any $\varepsilon > 0$, $p \in \Sigma_{x_0} := \{\pi(t,x_0) :\ t \in \mathbb{S}\}$ and $t_0 \in \mathbb{S}$ there exists $L = L(\varepsilon,t_0) > 0$ such that

$$B(p,\varepsilon) \bigcap \pi([t_0,t_0+L],p) \neq \emptyset,$$

where $B(p,\varepsilon) := \{x \in X :\ \rho(p,x) < \varepsilon\}$ and $\pi([t_0,t_0+L],p) := \{\pi(tp) :\ t \in [t_0,t_0+L]\}$;

– uniformly Poisson stable [1] (or pseudo-periodic [2, ChII,p.32]) if for arbitrary $\varepsilon > 0$ and $l > 0$ there exists a number $\tau > l$ such that $\rho(\pi(t+\tau,x),\pi(t,x)) < \varepsilon$ for any $t \in \mathbb{S}$.

Remark 2. 1. Every recurrent (respectively, uniformly Poisson stable) point is pseudo recurrent. The inverse statement, generally speaking, is not true.

2. If $x_0 \in X$ is a pseudo-recurrent point, then $p \in \omega_p$ for any $p \in H(x_0)$.

3. If x_0 is a Lagrange stable point and $p \in \omega_p$ for any $p \in H(x_0)$, then the point x_0 is pseudo-recurrent.

Definition 20. A point $x \in X$ is said to be strongly Poisson stable if $p \in \omega_p$ for any $p \in H(x)$.

Remark 3. Every pseudo-recurrent point is strongly Poisson stable. The inverse statement, generally speaking, is not true.

Theorem 9. *[12] (**Strong comparability**) Assume that (**C1**)–(**C3**) hold, $x_0 \in X$ and $y_0 := h(x_0) \in Y$ is strongly Poisson stable. Then the following statements hold:*

1. *the point γ_{y_0} (respectively, δ_{y_0}) is strongly comparable by character of recurrence with y_0, i.e., $\mathfrak{M}^{+\infty}_{y_0} \subseteq \mathfrak{M}^{+\infty}_{\gamma_{y_0}}$ and*

2.
$$\lim_{t\to+\infty}\rho(\pi(t,\alpha_{y_0}),\pi(t,\gamma_{y_0}))=0.$$

Corollary 2. *Under the conditions (**C1**) – (**C3**) if the point y_0 is τ-periodic (respectively, quasi-periodic, Bohr almost periodic, recurrent, pseudo-recurrent and Lagrange stable, pseudo-periodic and Lagrange stable), then:*

1. *the point u_{y_0} is so;*
2. *the point α_{y_0} is asymptotically τ-periodic (respectively, asymptotically quasi periodic, asymptotically Bohr almost periodic, asymptotically recurrent, asymptotically pseudo-recurrent, asymptotically pseudo-periodic).*

Remark 4. 1. If the point y_0 is recurrent (in the sense of Birkhoff), then Corollary 2 coincides with the results of the work of J. Jiang and X.-Q. Zhao [20].

2. In the works of B. A. Shcherbakov [27,29],[30, ChV, Example 5.2.1] examples of pseudo-recurrent and Lagrange stable motions which are not recurrent (in the sense of Birkhoff) were constructed.

Definition 21. A point $x\in X$ is said to be asymptotically strongly Poisson stable if there exists a strongly Poisson stable point $p\in X$ such that $\lim\limits_{t\to+\infty}\rho(\pi(t,x),\pi(t,p))=0$.

3 Convergence in Non-autonomous Dynamical Systems with a Strict Monotone First Integral.

In this section we study the asymptotic behavior of monotone non-autonomous dynamical systems having a strict monotone first integral.

Lemma 4. *[4] Let $\langle W,\varphi,(Y,\mathbb{S},\sigma)\rangle$ be a cocycle and $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ be a non-autonomous dynamical system associated by cocycle φ. Suppose that $x_0:=(u_0,y_0)\in X:=W\times Y$ and the set $Q^+_{(u_0,y_0)}:=\overline{\{\varphi(t,u_0,y_0)\mid t\in\mathbb{S}_+\}}$ is compact.*

Then the set $H^+(x_0):=\overline{\{\pi(t,x_0)\mid t\in\mathbb{S}_+\}}$ is conditionally compact.

Definition 22. A continuous function $V:X\to\mathbb{R}$ is said to be a first integral for dynamical system $(X,\mathbb{T},\pi)$ (respectively, for the cocycle $\langle E,\varphi,(Y,\mathbb{S},\sigma)\rangle$) if $V(\pi(t,x))=V(x)$ (respectively, $V(\varphi(t,u,y),\sigma(t,y))=V(u,y)$) for any $x\in X$ (respectively, $x=(u,y)\in X=E\times Y$) and $t\in\mathbb{T}$.

Definition 23. Let $\langle (X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ be a monotone non-autonomous dynamical system. A first integral V of dynamical system $(X,\mathbb{S}_+,\pi)$ is called strictly monotone if $x_1<x_2$ $(h(x_1)=h(x_2))$ implies $V(x_1)<V(x_2)$.

Theorem 10. *Assume that (**C1**)–(**C3**) hold, V is a strictly monotone first integral of dynamical system $(X,\mathbb{S}_+,\pi)$ and $x_0\in X$ is a point of X such that ω_{x_0} is positively uniformly stable. Let $K:=\omega_{x_0}$ be fixed and $y_0:=h(x_0)$.*

Then the following statements hold:

1. *if* $q \in \omega_{y_0}$ *is Poisson stable (in the both directions),* $\alpha_q := \alpha_q(K)$, $K^1 := \omega_{\alpha_q}$ *then the set* $K^1_q := \omega_{\alpha_q} \bigcap X_q$ *(respectively,* $\omega_{\beta_q} \bigcap X_q$*) consists of a single point* γ_q *(respectively,* δ_q*), i.e.,* $K^1_q = \{\gamma_q\}$ *(respectively,* $\{\delta_q\}$*);*
2. $\gamma_q \le \alpha_q \le \beta_q \le \delta_q$*;*
3. $$\gamma_q = \alpha_q \tag{2}$$
   and
   $$\beta_q = \delta_q. \tag{3}$$

Proof. The first and second statements of Lemma follow from Lemma 3.

We will establish the third statement. We only prove equality (2) since a similar argument applies to (3). For any $q \in \omega_{x_0}$ and $x \in \omega^q_{x_0} := \omega_{x_0} \bigcap h^{-1}(q)$ we have
$$\alpha_q \le x$$
and hence
$$\pi^t \alpha_q \le \pi^t x$$
for any $t \ge 0$. By Theorem 1 $\pi^t \omega^q_{x_0} = \omega^{\sigma(t,q)}_{x_0}$ for all $t \in \mathbb{S}_+$ and, consequently, we obtain
$$\pi^t \alpha_q \le \nu_{\sigma(t,q)} \le \pi^t x \tag{4}$$
for any $t \ge 0$ and $x \in \omega^q_{x_0}$.

Now we will prove that $\gamma_q = \alpha_q$. Let $x \in \omega^q_{x_0}$ be an arbitrary point, then there is a sequence $\{t_k\} \in \mathfrak{N}^{+\infty}_q$ such that
$$\pi(t_k, x_0) \to x \ \text{ as } k \to \infty.$$
Since $K = \omega_{x_0}$ is conditionally compact we can suppose that the sequence $\pi(t_n, \cdot)\big|_{K_q}$ is pointwise convergent and denote by ξ its limit. Note that $\xi \in \mathcal{E}^{+\infty}_q$ and taking into account that by Lemma 1 $\mathcal{E}^{+\infty}_q$ is a group, then $\xi(K_q) = K_q$. Thus, for any point $x \in K_q$ and $\xi \in \mathcal{E}^{+\infty}_q$ there exists a (unique) point $\tilde{x} \in K_q$ such that $\xi(\tilde{x}) = x$.

By (4) we have
$$\pi^{t_k} \alpha_q \le \pi^{t_k} \tilde{x} \tag{5}$$
for any $k \in \mathbb{N}$.

Passing to the limit in (5) as $k \to \infty$ we obtain
$$\tilde{\alpha}_q \le x, \tag{6}$$
where $\tilde{\alpha}_q = \xi(\alpha_q)$ and $x = \xi(\tilde{x})$. Since x is an arbitrary point from $K_q = \omega^q_{x_0}$, then from (6) we obtain
$$\tilde{\alpha}_q \le \alpha_q.$$
Note that $V(\tilde{\alpha}_q) = \lim\limits_{k\to\infty} V(\pi^{t_k}\alpha_q) = V(\alpha_q)$, $h(\tilde{\alpha}_q) = h(\alpha_q) = q$ and V is a strictly monotone first integral, then we conclude that $\tilde{\alpha}_q = \alpha_q$. On the other hand $\tilde{\alpha}_q \in \omega_{\alpha_q} \bigcap h^{-1}(q) = \{\gamma_q\}$ and, consequently, $\alpha_q = \gamma_q$. Theorem is completely proved. ☐

Below we suppose that the non-autonomous dynamical system $\langle(X,\mathbb{S}_+,\pi), (Y,\mathbb{S},\sigma), h\rangle$ is finite-dimensional, i.e., (X,h,Y) is a real finite-dimensional vectorial fiber with the fiber $\mathbb{R}^n$. This means that for every point $y\in Y$ there exists a neighborhood U $(y\in U)$ such that $h^{-1}(U)$ is isomorphic to $\mathbb{R}^n\times U$.

Let $P: X\to X$ be a projection, i.e., $P(X_y)\subseteq X_y$ for any $y\in Y$ and $P^2=P$. Let $P_i: X\to X$ $(i=1,\dots,n)$ be one-dimensional (i.e., $P_i(X_y)$ is one-dimensional subspace of X_y for any $y\in Y$) projection such that $P_iP_j=\Theta$ and $P_1+\dots+P_n=I$, where $\Theta, I: Y\to X$ are such that $\Theta_y(x)=0$ and $I_y(x)=x$ for any $y\in Y$ and $x\in X_y$.

Let $x^1,x^2\in X$. We will write $x^1<_i x^2$ $(i=1,\dots,n)$ if $h(x^1)=h(x^2)$, $x^1\le x^2$ and $x^1_i<x^2_i$, $x_i:=P_i(x)$ and $x=(x_1,\dots,x_n)\in X$.

Condition (**C4**): A non-autonomous dynamical system $\langle(X,\mathbb{S}_+,\pi),(Y,\mathbb{S},\sigma),h\rangle$ is said to be componentwise monotone if it is monotone and if $x^1<_i x^2$ $(h(x^1)=h(x^2))$ implies $\pi(t,x^1)<_i\pi(t,x^2)$ for any $t>0$ $(i=1,2,\dots,n)$.

Denote by 2^X the family of any compact subsets of X equipped with the Hausdorff metric.

Definition 24. Let $K\in 2^X$ and $\mathcal{E}\subseteq K^K$ be a compact subsemigroup. A subset $A\subseteq K$ is said to be:

- $\mathcal{E}$-*invariant* if $A\mathcal{E}\subseteq A$, where $A\mathcal{E}:=\bigcup\{A\xi:\ \xi\in\mathcal{E}\}$ and $A\xi:=\{x\xi:=\xi(x)|\ x\in A\}$;
- $\mathcal{E}$-*minimal* if M is nonempty, $\mathcal{E}$-invariant, closed and it does not contain an own closed $\mathcal{E}$-invariant subset.

Lemma 5. *[11, Ch.IV] If $A\subseteq K$ is a compact and $\mathcal{E}$-invariant subset, then A contains a nonempty compact $\mathcal{E}$-minimal subset $M\subseteq A$.*

Theorem 11. *Assume that (**C1**)$-$(**C4**) hold, V is a strictly monotone first integral of dynamical system $(X,\mathbb{S}_+,\pi)$ and $x_0\in X$. Let $K:=\omega_{x_0}$ be fixed and $y_0\in\omega_{y_0}$, where $y_0:=h(x_0)$.*

Then the following statements hold:

1. $\gamma_{y_0}=\alpha_{y_0}\in\omega_{x_0}$ $(\beta_q=\delta_q\in\omega_{x_0})$;
2. $$\lim_{t\to+\infty}\rho(\pi(t,x_0),\pi(t,\alpha_{y_0}))=0.$$

Proof. Since $\Sigma^+_{x_0}$ is conditionally compact and $y_0\in\omega_{y_0}$, then the set $\omega^{y_0}_{x_0}=\omega_{y_0}\bigcap X_{y_0}$ is nonempty and compact. By Lemma 5 there exists a compact $\mathcal{E}^{+\infty}_{y_0}$ minimal subset $M_{y_0}\subseteq\omega^{y_0}_{x_0}$. Denote by $\mathcal{M}=H(M_{y_0}):=\overline{\{\pi(t,M_{y_0}):\ t\in\mathbb{S}_+\}}$. It is not difficult to check that $\mathcal{M}$ is the smallest conditionally compact invariant set containing M_{y_0}.

We will show that $\mathcal{M}\bigcap X_{y_0}=M_{y_0}$. To prove this equality it is sufficient to establish the inclusion $\mathcal{M}\bigcap X_{y_0}\subseteq M_{y_0}$ because the reverse inclusion is evident. If y_0 is a periodic point, then this statement is evident. Taking into account this fact without loss of generality we can suppose that is not periodic. Let $x \in \mathcal{M}\bigcap X_{y_0}$, then $x \in X_{y_0}$ and there exist sequences $\{t_k\} \in \mathfrak{N}^{+\infty}_{y_0}$ and $\{x_k\} \subseteq M_{y_0} \subseteq \omega^{y_0}_{x_0}$ such that

$$x = \lim_{k\to\infty} \pi(t_k, x_k) \ . \tag{7}$$

Since $\{x_k\} \subseteq M_{y_0}$ and M_{y_0} is a compact subset of $\omega^{y_0}_{x_0}$, then without loss of generality we can suppose that the sequences $\{x_k\}$ and $\{\pi^{t_k}\big|_{\omega^{y_0}_{x_0}}\}$ converge. Denote by $\bar{x} := \lim\limits_{k\to\infty} x_k$ and

$$\xi := \lim_{k\to\infty} \pi^{t_k}\big|_{\omega^{y_0}_{x_0}}.$$

The convergence in the last equality is pointwise. Since the set ω_{x_0} is positively uniformly stable, then $\{\pi^{t_k}\}$ converges to ξ uniformly on $\omega^{y_0}_{x_0}$. This means, in particular, that the map $\xi : \omega^{y_0}_{x_0} \to \omega^{y_0}_{x_0}$ is continuous and

$$\lim_{k\to\infty} \rho(\pi(t_k, x_k), \xi(x_k)) = 0. \tag{8}$$

Hence, we have

$$\rho(\xi(\bar{x}), \pi(t_k, x_k)) \le \rho(\xi(\bar{x}), \xi(x_k)) + \rho(\xi(x_k), \pi(t_k, x_k)) \tag{9}$$

for any $k \in \mathbb{N}$. Passing to the limit in (9) as $k \to \infty$ and taking into consideration (8) and the continuity of ξ on $\omega^{y_0}_{x_0}$ we obtain

$$\xi(\bar{x}) = \lim_{k\to\infty} \pi(t_k, x_k) \ . \tag{10}$$

From (7) and (10) we obtain $x = \xi(\bar{x}) \in \xi(M_{y_0}) \subseteq M_{y_0}$, i.e., $\mathcal{M}\bigcap X_{y_0} = M_{y_0}$.

Let $\tilde{\alpha}_{y_0} := \alpha_{y_0}(\mathcal{M})$. By Lemma 3 the set $\omega_{\tilde{\alpha}_{y_0}}\bigcap X_{y_0}$ consists of a single point $\{\tilde{\gamma}_{y_0}\}$ and by Theorem 10 we have the equality $\tilde{\alpha}_{y_0} = \tilde{\gamma}_{y_0}$.

It follows that for each $i = 1, \dots, n$, there is $x^i \in M_{y_0}$ such that

$$x^i_i = P_i(\tilde{\alpha}_{y_0}) \ . \tag{11}$$

By Theorem 1, we have

$$\pi^t \tilde{\alpha}_{\sigma^{-t}y_0} = \tilde{\alpha}_{y_0} \tag{12}$$

for any $t \in \mathbb{S}_+$. Note that

$$\tilde{\alpha}_{\sigma^{-t}y_0} \le \pi^{-t}x^i \tag{13}$$

and

$$P_i(\pi^t \tilde{\alpha}_{\sigma^{-t}y_0}) = P_i(\tilde{\alpha}_{y_0}) = x^i_i = P_i(\pi^t\pi^{-t}x^i). \tag{14}$$

By Condition (**C4**),

$$P_i(\tilde{\alpha}_{\sigma^{-t}y_0}) = P_i(\pi^{-t}x^i) \tag{15}$$

for any $t \in \mathbb{S}_+$. Since M_{y_0} is a $\mathcal{E}^{+\infty}_{y_0}$ minimal subset of $\omega^{y_0}_{x_0}$ and $\mathcal{E}^{+\infty}_{y_0} = \mathcal{E}^{-\infty}_{y_0}$, then for any $i = 1, \dots, n$ there exists $\xi^i \in \mathcal{E}^{-\infty}_{y_0}$ such that $\xi^i(x^i) = x$. On the other hand for $\xi^i \in \mathcal{E}^{-\infty}_{y_0}$ there exists a sequence $\{t^i_k\} \in \mathfrak{N}^{-\infty}_{y_0}$ $(i = 1, \dots, n)$ such that

$$x = \lim_{k\to\infty} \pi^{t^i_k}x^i \quad \text{for any } i = 1, \dots, n. \tag{16}$$

From (11)-(16) we have

$$P_i(\tilde{\alpha}_{y_0}) = P_i(\xi^i(x^i)) = P_i(x) \ \ \forall\, x \in M_{y_0} \ \text{ and } \ i = 1,\dots,n,$$

i.e., $\alpha_{y_0} = x$ for any $x \in M_{y_0}$. This means that $M_{y_0} = \{\tilde{\alpha}_{y_0}\}$.

Let now $t_k \to +\infty$ such that $\pi^{t_k} x_0 \to \alpha_{y_0} \in M_{y_0} \subseteq \omega_{x_0}^{y_0}$ as $k \to \infty$. Since $y_0 \in \omega_{y_0}$, then it is easy to see that $\{t_k\} \in \mathfrak{N}_{y_0}^{+\infty} \subseteq \mathfrak{N}_{\tilde{\alpha}_{y_0}}^{+\infty}$. Then we obtain

$$\rho(\pi(t_k, x_0), \pi(t_k, \tilde{\alpha}_{y_0})) \le \rho(\pi(t_k, x_0), \tilde{\alpha}_{y_0}) + \rho(\tilde{\alpha}_{y_0}, \pi(t_k, \tilde{\alpha}_{y_0})) \tag{17}$$

for any $k \in \mathbb{N}$. Passing to the limit in (17) as $k \to \infty$ and taking into account the arguments above we obtain

$$\lim_{k \to \infty} \rho(\pi(t_k, x_0), \pi(t_k, \tilde{\alpha}_{y_0})) = 0. \tag{18}$$

Note that $\tilde{\alpha}_{y_0} \in M_{y_0} \subseteq \omega_{x_0}^{y_0}$ and, consequently, $\omega_{\tilde{\alpha}_{y_0}} \subseteq \omega_{x_0}$. Taking into consideration that the set ω_{x_0} is positively uniformly stable and equality (18), by Lemma 2 we obtain

$$\lim_{t \to +\infty} \rho(\pi(t, x_0), \pi(t, \tilde{\alpha}_{y_0})) = 0$$

and, consequently, $\omega_{\tilde{\alpha}_{y_0}} = \omega_{x_0}$. This means, in particular, that $\tilde{\alpha}_{y_0} = \alpha_{y_0}$. Theorem is completely proved. □

Let $U \subseteq \mathbb{R}^n$, $V \in C^1(U, \mathbb{R})$ and denote by $\nabla V := \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n}\right)$.

Lemma 6. *Assume that the following conditions are fulfilled:*

(i) $\langle \mathbb{R}^n_+, \varphi, (Y, \mathbb{S}, \sigma) \rangle$ *is a monotone cocycle;*

(ii) there exists a first integral $H \in C^1(\mathbb{R}^n_+, \mathbb{R})$ *for cocycle* φ *with* $\nabla H(x) \gg 0$ *for any* $x \in \mathbb{R}^n_+$*;*

(iii) $\varphi(t, u_0, y_0)$ $((u_0, y_0) \in \mathbb{R}^n_+ \times Y)$ *is a bounded motion of the cocycle* φ*.*

Then $\varphi(t, u_0, y_0)$ *is positively uniformly stable.*

Proof. This statement may be proved using the same ideas as in the proof of Lemma 3.1 from [34]. Below we will present the details of this proof. Let $e := (1, \dots, 1) \in \mathbb{R}^n_+$. Since $\varphi(t, u_0, y_0)$ is bounded on $\mathbb{S}_+$, we can choose a sufficiently large real number $r > 0$ such that

$$0 \le \varphi(t, u_0, y_0) \le q_0 := re$$

for any $t \ge 0$. Denote by

$$M := \max_{1 \le i \le n} \big\{ \max_{0 \le x \le q_0 + e} H_{x_i}(x) \big\},\ m := \min_{1 \le i \le n} \big\{ \min_{0 \le x \le q_0 + e} H_{x_i}(x) \big\}.$$

Condition (ii) implies that $M, m > 0$. From the equality

$$H(y) - H(z) = \sum_{i=1}^{n} \int_0^1 H_{x_i}(z + s(y - z)) ds (y_i - z_i) \ \ (\forall\, y, z \in \mathbb{R}^n_+)$$

it follows that

$$|H(y) - H(z)| \leq nM\|y - z\|, \quad \forall\, 0 \leq z, y \leq q_0 + e, \tag{19}$$

and

$$|H(y) - H(z)| \geq m\|y - z\|, \quad \forall\, 0 \leq z \leq y \leq q_0 + e, \tag{20}$$

where $\|x\| := \sum_{i=1}^{n} |x_i|$.

Let $\varepsilon_0 := \min\{1, \frac{m}{2nM}\}$. For any given $0 < \varepsilon \leq \varepsilon_0$, there is $0 < \delta(\varepsilon) \leq \varepsilon/2$ such that

$$\varphi(\tau, u_0, y_0) - \delta(\varepsilon)e \leq \varphi(\tau, u_0, y_0) \leq \varphi(\tau, u_0, y_0) + \delta(\varepsilon)e \leq q_0 + e \tag{21}$$

for any $t \geq 0$. Put

$$p(\varepsilon, \tau) := (\max(\varphi_1(\tau, u_0, y_0) - \delta(\varepsilon), 0), \ldots, \varphi_n(\tau, u_0, y_0) - \delta(\varepsilon), 0))$$

and $q(\varepsilon, t) := \varphi(t, u_0, y_0) + \delta(\varepsilon)e$. Note that

$$0 \leq q(\varepsilon, \tau) - p(\varepsilon, \tau) = [\varphi(t, u_0, y_0) - \delta(\varepsilon)e] - p(\varepsilon, \tau) + 2\delta(\varepsilon)e \leq 2\delta(\varepsilon)e$$

for all $\tau \geq 0$ and taking into consideration (19) and (21) we obtain

$$|H(p(\varepsilon, \tau)) - H(q(\varepsilon, \tau))| \leq nM\varepsilon \tag{22}$$

for all $t \geq 0$. For given $\tau \geq 0$, let

$$U(\varepsilon, \tau) := \{z \in \mathbb{R}^n_+ |\ p(\varepsilon, \tau) \leq z \leq q(\varepsilon, \tau)\}.$$

Since the cocycle φ is monotone, then we will have

$$\varphi(t, p(\varepsilon, \tau), \sigma(\tau, y_0)) \leq \varphi(t, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0)) \leq \varphi(t, q(\varepsilon, \tau), \sigma(\tau, y_0))$$

and

$$\varphi(t, p(\varepsilon, \tau), \sigma(\tau, y_0)) \leq \varphi(t, z, \sigma(\tau, y_0)) \leq \varphi(t, q(\varepsilon, \tau), \sigma(\tau, y_0))$$

for all $t \geq 0$ and $z \in U(\varepsilon, \tau)$. Taking into consideration that H is a first integral for the cocycle φ and inequality (22), we obtain

$$|H(\varphi(t, q(\varepsilon, \tau), \sigma(\tau, y_0))) - H(\varphi(t, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0)))| =$$
$$|H(q(\varepsilon, \tau)) - H(\varphi(\tau, u_0, y_0))| \leq |H(q(\varepsilon, \tau)) - H(p(\varepsilon, \tau))| \leq nM\varepsilon$$

for all $t \geq 0$.

By (20) and (21), we have

$$\|\varphi(t, q(\varepsilon, \tau), \sigma(\tau, y_0)) - \varphi(t, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0))\| \leq \frac{nM}{m}\varepsilon$$

for all $t \geq 0$ with $\varphi(t, q(\varepsilon, \tau), \sigma(\tau, y_0)) \in [0, q_0 + e]$. We will show that

$$\varphi(t, q(\varepsilon, \tau), \sigma(\tau, y_0)) \in [0, q_0 + e] \tag{23}$$

for all $t \geq 0$. If it is not true, then there exists a real number $t^* > 0$ such that $\varphi(t, q(\varepsilon,\tau), \sigma(\tau, y_0)) \in [0, q_0 + e]$ for any $t \in [0, t^*]$ and

$$\|\varphi(t^*, q(\varepsilon,\tau), \sigma(\tau, y_0))\| \geq q_0 + 1. \tag{24}$$

On the other hand from inequality (23) we have

$$\|\varphi(t^*, q(\varepsilon,\tau), \sigma(\tau, y_0))\| \leq \|\varphi(t^*, q(\varepsilon,\tau), \sigma(\tau, y_0)) - \varphi(t^*, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0))\| + \\ \|\varphi(t^*, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0))\| \leq \|q_0\| + \tfrac{nM}{m}\varepsilon \leq \|q_0\| + \tfrac{1}{2}$$

which contradicts (24).

It then follows that for every $z \in U(\varepsilon,\tau)$ we have $\varphi(t, z, \sigma(\tau, y_0)) \in [0, q_0 + e]$ for any $t \geq 0$. Similarly, we can show that for any $z \in U(\varepsilon,\tau)$ we have

$$\|\varphi(t, q(\varepsilon,\tau), \sigma(\tau, y_0)) - \varphi(t, z, \sigma(\tau, y_0))\| \leq \frac{nM}{m}\varepsilon, \tag{25}$$

for any $t \geq 0$. Then (23) and (25) imply that for any $z \in U(\varepsilon,\tau)$ and $t \geq 0$,

$$\|\varphi(t, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0)) - \varphi(t, z, \sigma(\tau, y_0))\| \leq \\ \|\varphi(t, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0)) - \varphi(t, q(\varepsilon,\tau), \sigma(\tau, y_0))\| + \\ \|\varphi(t, q(\varepsilon,\tau), \sigma(\tau, y_0)) - \varphi(t, z, \sigma(\tau, y_0))\| \leq \tfrac{nM}{m}\varepsilon + \tfrac{nM}{m}\varepsilon = \tfrac{2nM}{m}\varepsilon. \tag{26}$$

For any $\varepsilon \in (0, \varepsilon_0]$, $\tau \geq 0$, and $y \in \mathbb{R}^n_+$ with $\|\varphi(\tau, u, y_0) - \varphi(\tau, u_0, y_0)\| \leq \delta(\varepsilon)$, we have $\varphi(\tau, u, y_0) \in U(\varepsilon,\tau)$. Then the inequality (26) with $u = \varphi(\tau, u, y_0)$ implies that

$$\|\varphi(t+\tau, u, y_0) - \varphi(t+\tau, u_0, y_0)\| = \\ \|\varphi(t, \varphi(\tau, u, y_0), \sigma(\tau, y_0)) - \varphi(t, \varphi(\tau, u_0, y_0), \sigma(\tau, y_0))\| \leq \tfrac{2nM}{m}\varepsilon$$

for any $t \geq 0$. Thus, $\varphi(t, u_0, y_0)$ is uniformly stable. □

4 Asymptotically almost periodic and asymptotically almost automorphic motions of non-autonomous dynamical systems with a strictly monotone first integral

This section is dedicated to the study of different classes of Poisson stable (periodic, almost periodic, almost automorphic, almost recurrent, recurrent, pseudo-periodic, pseudo-recurrent and Poisson stable) and asymptotically Poisson stable motions of non-autonomous dynamical systems admitting a strictly monotone first integral.

Theorem 12. *Assume that (**C1**) and (**C4**) hold and the following conditions are fulfilled:*

1. *for every $u \in \mathbb{R}^n_+$ and $y \in Y$ the semi-trajectory $\varphi(\mathbb{S}_+, u, y)$ is relatively compact;*

2. *$\langle \mathbb{R}^n_+, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is a monotone cocycle with the fiber $\mathbb{R}^n_+$ over dynamical system $(Y, \mathbb{S}, \sigma)$;*

3. *there exists a first integral $V \in C^1(\mathbb{R}^n_+, \mathbb{R})$ for the cocycle φ with $\nabla V(x) \gg 0$ for any $x \in \mathbb{R}^n_+$;*

4. *the point y is Poisson stable (in the both directions).*

Then

1. *the set $\omega^y_x = \omega_x \bigcap X_y$ ($x = (u, y)$ and $X = \mathbb{R}^n_+ \times Y$) consists of a single point $x^* = (u^*, y)$;*

2. *the point x^* is comparable by character of recurrence with y, i.e., $\mathfrak{N}^{+\infty}_y \subseteq \mathfrak{N}^{+\infty}_{x^*}$;*

3. $$\lim_{t\to+\infty} \rho(\pi(t, x), \pi(t, x^*)) = 0 \ .$$

Proof. Let
$$\langle (X, \mathbb{S}_+, \pi), (Y, \mathbb{S}, \sigma), h\rangle \tag{27}$$
be a non-autonomous dynamical system generated by cocycle φ ($X = \mathbb{R}^n_+ \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$). It is easy to check that under the conditions of Theorem 12 the non-autonomous dynamical system (27) possesses the properties $(C1)-(C3)$. By Lemma 3 the set $K^1 := \omega_{\alpha_{y_0}}$, then the set $K^1_y := \omega_{\alpha_y} \bigcap X_y$ consists of a single point γ_y), i.e., $K^1_y = \{\gamma_y\}$, where $\alpha_y := \alpha_y(K)$ and $K^1 := \omega_{\alpha_y}$.

According to Theorems 10 and 11 we have:

1. $\alpha_y = \gamma_y \in \omega_x$;

2. $\lim\limits_{t\to\infty} \rho(\pi(t, x), \pi(t, \alpha_y)) = 0$.

By Theorem 8 the point γ_y is comparable by character of recurrence with y. To finish the proof of Theorem it is sufficient to put $x^* = \alpha_y$. Theorem is completely proved. □

Corollary 3. *Under the conditions of Theorem 12 if the point y is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then the point x is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically Poisson stable).*

Proof. This statement follows from Theorems 12 and 3. □

Corollary 4. *Under the conditions of Theorem 12 if the point y is almost automorphic (respectively, recurrent), then the point x is asymptotically almost automorphic (respectively, asymptotically τ-periodic, asymptotically Levitan almost periodic, asymptotically recurrent).*

Proof. Let y be almost automorphic (respectively, recurrent). Since an almost automorphic (respectively, recurrent) point is Levitan almost periodic (respectively, almost recurrent), then by Corollary 3 the point x is asymptotically Levitan almost periodic (respectively, asymptotically almost recurrent). On the other hand we note that the point y is almost automorphic (respectively, recurrent) and, consequently, it is Lagrange stable in the positive direction. Taking into consideration that the point x is conditionally Lagrange stable, then it is also Lagrange stable and, hence, it is asymptotically almost automorphic (respectively, asymptotically recurrent). □

Theorem 13. *Assume that (**C1**) and (**C4**) hold and the following conditions are fulfilled:*

1. *for every $u \in \mathbb{R}^n_+$ and $y \in Y$ the semi-trajectory $\varphi(\mathbb{S}_+, u, y)$ is relatively compact;*
2. *$\langle \mathbb{R}^n_+, \varphi, (Y, \mathbb{S}, \sigma)\rangle$ is a monotone cocycle with the fiber $\mathbb{R}^n_+$ over dynamical system $(Y, \mathbb{S}, \sigma)$;*
3. *there exists a first integral $V \in C^1(\mathbb{R}^n_+, \mathbb{R})$ for the cocycle φ with $\nabla V(x) \gg 0$ for any $x \in \mathbb{R}^n_+$;*
4. *the point y is strongly Poisson stable (in the both directions).*

Then

1. *for any $q \in \omega_y$ the set $\omega_x^q = \omega_x \bigcap X_q$ ($x = (u, y)$ and $X = \mathbb{R}^n_+ \times Y$) consists of a single point $x^q = (u^q, q)$;*
2. *the point $x^* := x^y$ is strongly comparable by character of recurrence with y, i.e., $\mathfrak{M}_y^{+\infty} \subseteq \mathfrak{M}_{x^*}^{+\infty}$ and $\mathfrak{N}_y^{+\infty} \subseteq \mathfrak{N}_{x^*}^{+\infty}$;*
3. $$\lim_{t\to+\infty} \rho(\pi(t, x), \pi(t, x^*)) = 0.$$

Proof. Consider the non-autonomous dynamical system (27) generated by cocycle φ. It is not difficult to check that under the conditions of Theorem 12 the non-autonomous dynamical system (27) possesses the properties $(\mathbf{C1})-(\mathbf{C3})$. Let $K^1 := \omega_{\alpha_{y_0}}$. By Lemma 3 for any $q \in \omega_y$ the set $K^1_q := \omega_{\alpha_y} \bigcap X_q$ consists of a single point γ_q), i.e., $K^1_q = \{\gamma_q\}$, where $\alpha_q := \alpha_q(K)$. According to Theorems 10 and 11 we have:

1. $\alpha_q = \gamma_q =\in \omega_x$ for any $q \in \omega_y$;
2. $\lim\limits_{t\to\infty} \rho(\pi(t, x), \pi(t, \alpha_y)) = 0$.

By Theorem 9 the point γ_y is strongly comparable by character of recurrence with y. To finish the proof oh Theorem it is sufficient to put $x^* = \alpha_y$. Theorem is completely proved. □

Corollary 5. *Under the conditions of Theorem 13 if the point y is stationary (respectively, τ-periodic, quasi-periodic, Bohr almost periodic, almost automorphic, recurrent, pseudo-recurrent and Lagrange stable, pseudo-periodic and Lagrange stable, strongly Poisson stable and Lagrange stable), then the point x is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically quasi-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically pseudo-recurrent, asymptotically pseudo-periodic, asymptotically strongly Poisson stable).*

Proof. This statement follows from Theorems 13 and 5. □

5 Applications

5.1 Ordinary Differential Equations

Let $\mathbb{R}^n$ be an n-dimensional real Euclidean space with the norm $|\cdot|$. Let us consider a differential equation

$$u' = f(t,u), \tag{28}$$

where $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Along with equation (28) we consider its H-class [3, 22, 25, 30, 33], i.e., the family of equations

$$v' = g(t,v), \tag{29}$$

where $g \in H(f) = \overline{\{f_\tau : \tau\in\mathbb{R}\}}$, $f_\tau(t,u) = f(t+\tau,u)$ for all $(t,u)\in\mathbb{R}\times\mathbb{R}^n$ and by bar we denote the closure in $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$.

Condition (**A1**). The function f is said to be regular if for every equation (29) the conditions of existence, uniqueness and extendability on $\mathbb{R}_+$ are fulfilled.

Denote by $\varphi(\cdot,v,g)$ the solution of equation (29), passing through the point $v\in\mathbb{R}^n$ at the initial moment $t=0$. Then we correctly defined a mapping $\varphi : \mathbb{R}_+\times E^n\times H(f)\to\mathbb{R}^n$, verifying the following conditions (see, for example,[3, 25]):

1) $\varphi(0,v,g) = v$ for all $v\in\mathbb{R}^n$ and $g\in H(f)$;

2) $\varphi(t,\varphi(\tau,v,g),g_\tau) = \varphi(t+\tau,v,g)$ for every $v\in\mathbb{R}^n$, $g\in H(f)$ and $t,\tau\in\mathbb{R}_+$;

3) the mapping $\varphi : \mathbb{R}_+\times\mathbb{R}^n\times H(f)\to\mathbb{R}^n$ is continuous.

Denote by $Y := H(f)$ and $(Y,\mathbb{R},\sigma)$ a dynamical system of translations on Y, induced by dynamical system of translations $(C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{R},\sigma)$. The triplet $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{R},\sigma)\rangle$ is a cocycle on $(Y,\mathbb{R},\sigma)$ with the fiber $\mathbb{R}^n$. Thus the equation (28) generates a cocycle $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{R},\sigma)\rangle$ and a non-autonomous dynamical system $\langle(X,\mathbb{R}_+,\pi), (Y,\mathbb{R},\sigma),h\rangle$, where $X := \mathbb{R}^n\times Y$, $\pi := (\varphi,\sigma)$ and $h : pr_2 : X\to Y$.

Remark 5. Let F be a mapping from $H(f)\times\mathbb{R}^n\to\mathbb{R}^n$ defined by equality $F(g,x) = g(0,x)$, then F possesses the following properties:

1. F is continuous;

2. $F(g^\tau, x) = g(\tau, x)$ for any $(g, x, \tau) \in H(f) \times \mathbb{R}^n \times \mathbb{R}$;

3. equation (28) (and its H-class (29)) can be rewritten as follows

$$x' = F(\sigma(t, g), x) \quad (g \in H(f)). \tag{30}$$

Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : \text{ such that } x_i \geq 0 \ (x := (x_1, \dots, x_n)) \text{ for any } i = 1, 2, \dots, n\}$ be the cone of nonnegative vectors of $\mathbb{R}^n$. By $\mathbb{R}^n_+$ on the space $\mathbb{R}^n$ a partial order. Namely: $u \leq v$ if $v - u \in \mathbb{R}^n_+$ is defined.

Condition (**A2**). Equation (28) is monotone. This means that the cocycle $\langle \mathbb{R}^n, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ (or shortly φ) generated by (28) is monotone, i.e., if $u, v \in \mathbb{R}^n$ and $u \leq v$ then $\varphi(t, u, g) \leq \varphi(t, v, g)$ for all $t \geq 0$ and $g \in H(f)$.

Let K be a closed cone in $\mathbb{R}^n$. The dual cone to K is the closed cone K^* in the dual space $(\mathbb{R}^n)^*$ of linear functions on $\mathbb{R}^n$, defined by

$$K^* := \{\lambda \in (\mathbb{R}^n)^* : \ \langle \lambda, x\rangle \geq 0 \text{ for any } x \in K\},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$.

Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$. Recall [36, 37] that the function $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be quasimonotone if for any $(t, u), (t, v) \in \mathbb{T} \times \mathbb{R}^n$ and $\phi \in (\mathbb{R}^n_+)^*$ we have: $u \leq v$ and $\phi(u) = \phi(v)$ implies $\phi(f(t, u)) \leq \phi(f(t, v))$.

Lemma 7. *[12] Let $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be a regular and quasimonotone function, then the following statements hold:*

1. *if $u \leq v$, then $\varphi(t, u, f) \leq \varphi(t, v, f)$ for any $t \geq 0$;*

2. *any function $g \in H(f)$ is quasimonotone;*

3. *$u \leq v$ implies $\varphi(t, u, g) \leq \varphi(t, v, g)$ for any $t \geq 0$ and $g \in H(f)$;*

4. *equation (28) is monotone.*

Definition 25. A solution $\varphi(t, u_0, f)$ of equation (28) is said to be:

- uniformly Lyapunov stable in the positive direction, if for arbitrary $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|\varphi(t_0, u, f) - \varphi(t_0, u_0, f)| < \delta$ $(t_0 \in \mathbb{R},\ u \in \mathbb{R}^n)$ implies $|\varphi(t, x, f) - \varphi(t, x_0, f)| < \varepsilon$ for any $t \geq t_0$;

- compact on $\mathbb{R}_+$ if the set $Q := \overline{\varphi(\mathbb{R}_+, u_0, f)}$ is a compact subset of $\mathbb{R}^n$, where by bar the closure in $\mathbb{R}^n$ is denoted and $\varphi(\mathbb{R}_+, u_0, f) := \{\varphi(t, u_0, f) : \ t \in \mathbb{R}_+\}$.

Let $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $\sigma(t, f)$ be the motion (in the shift dynamical system $(C(\mathbb{R}\times\mathbb{R}^n, \mathbb{R}^n), \mathbb{R}, \sigma)$) generated by f, $u_0 \in \mathbb{R}^n$, $\varphi(t, u_0, f)$ be the solution of equation (28), $x_0 := (u_0, f) \in X := \mathbb{R}^n \times H(f)$ and $\pi(t, x_0) := (\varphi(t, u_0, f), \sigma(t, f))$ be the motion of skew-product dynamical system $(X, \mathbb{R}_+, \pi)$.

Definition 26. A solution $\varphi(t,u_0,f)$ of equation (28) is called [9,30,33] compatible (respectively, strongly compatible or uniformly compatible) if the motion $\pi(t,x_0)$ is comparable (respectively, strongly comparable or uniformly comparable) by character of recurrence with $\sigma(t,f)$.

Remark 6. If $x_0 := (u_0,y_0) \in X := W \times Y$ and α_{y_0} (respectively, γ_{y_0}) is a point from X defined in Lemma 3 then we denote by α_{u_0} (respectively, γ_{u_0}) a point from W such that $\alpha_{y_0} = (\alpha_{u_0}, y_0)$ (respectively, $\gamma_{y_0} = (\gamma_{u_0}, y_0)$).

Recall that the function $\varphi \in C(\mathbb{R},\mathbb{R}^n)$ (respectively, $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$) possesses the property (A) if the motion $\sigma(\cdot,\varphi)$ (respectively, $\sigma(\cdot,f)$) generated by the function φ (respectively, f) possesses this property in the dynamical system $(C(\mathbb{R},\mathbb{R}^n),\mathbb{R},\sigma)$ (respectively, $(C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{R},\sigma)$).

As property (A) stability in the sense of Lagrange (st. L), uniform stability (un. st. $\mathcal{L}^+$) in the sense of Lyapunov, periodicity, almost periodicity, asymptotical almost periodicity and other may serve.

For example, a function $f \in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ is called *almost periodic (respectively, recurrent, asymptotically almost periodic, etc.) in $t \in \mathbb{R}$ uniformly with respect to (w.r.t.) w on every compact subset from* $\mathbb{R}^n$ if the motion $\sigma(\cdot,f)$ is almost periodic (respectively, recurrent, asymptotically almost periodic, etc.) in the dynamical system $(C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{R},\sigma)$.

Condition (**A3**). $f_i(t,x) \ge 0$ for all $x \in \Gamma_i$, $t \in \mathbb{R}$ and $i = 1,\dots,n$, where $\Gamma_i := \{x \in \mathbb{R}^n_+ :\ x_i = 0\}$.

Remark 7. It is easy to see that if the function f satisfies Condition **(A3)**, then every function $g \in H(f)$ possesses the same property.

Denote by

$$C^{0,1}(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n) := \{f \in C(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n)|\ \exists \frac{\partial f}{\partial x_i} \in C(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n) \text{ for any } i = 1,\dots,n\}$$

equipped with the distance

$$d(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(f,g)}{1+d_k(f,g)},$$

where

$$d_k(f,g) := \max_{|t|+|x|\le k} \big(|f(t,x)-g(t,x)| + \|\frac{\partial f}{\partial x}(t,x) - \frac{\partial g}{\partial x}(t,x)\|\big)$$

and

$$\frac{\partial f}{\partial x}(t,x) = \big(\frac{\partial f_i}{\partial x_j}(t,x)\big)_{i,j=1}^n.$$

Let $(C^{0,1}(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n),\mathbb{R},\sigma)$ be the shift dynamical system on $C^{0,1}(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n)$, where $\sigma(f,\tau) := f_\tau$ for any $f \in C^{0,1}(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n)$, $\tau \in \mathbb{R}$ and $f_\tau(t,x) := f(t+\tau,x)$ for all $t,\tau \in \mathbb{R}$ and $x \in \mathbb{R}^n_+$.

Condition (**A4**). $f \in C^{0,1}(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ and

$$\frac{\partial f_i}{\partial x_j}(t,x) \geq 0 \tag{31}$$

for any $t \geq 0$, $x \in \mathbb{R}^n_+$ and $i \neq j$. If $f \in C^{0,1}(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$, then we denote by $H(f)$ the closure in $C^{0,1}(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ of the family of all translations $\{f_\tau |\ \tau \in \mathbb{R}\}$ of f.

Remark 8. If $f \in C^{0,1}(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ satisfies condition (31), then

1. for any $g \in H(f)$ we have
$$\frac{\partial g_i}{\partial x_j}(t,x) \geq 0 \tag{32}$$
for all $t \geq 0$, $x \in \mathbb{R}^n_+$ and $i \neq j$;

2. the cocycle φ generated by equation (28) is monotone.

The first statement is evident. The second statement follows from the first one and Remark 1.1 [37, Ch.III, p.33].

Theorem 14. *[14, Ch.V] Assume that $f \in C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ is regular, quasi-monotone and $\langle \mathbb{R}^n_+, \varphi, (H(f), \mathbb{R}, \sigma)\rangle$ is the cocycle in $\mathbb{R}^n_+$ generated by equation (28) (respectively, by equation (30)). Then the condition*

$$\mathcal{F}(g,x) \leq F(g,x)$$

for any $(g,x) \in H(f) \times \mathbb{R}^n_+$ implies that

$$\phi(t,x,g) \leq \varphi(t,x,g)$$

for any $t \geq 0$, $g \in H(f)$ and $x \in \mathbb{R}^n_+$, where $\mathcal{F} \in C(H(f) \times \mathbb{R}^n_+, \mathbb{R}^n)$ is some regular function and $\langle \mathbb{R}^n_+, \phi, (H(f), \mathbb{R}, \sigma)\rangle$ (shortly, ϕ) is the cocycle generated by equation

$$x' = \mathcal{F}(\sigma(t,g),x) \quad (g \in H(f));$$

.

Lemma 8. *Assume that the function $f \in C(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ is regular, quasi-monotone and $f(t,0) \geq 0$ for any $t \in \mathbb{R}$. Then $\mathbb{R}^n_+$ is a positively invariant subset of the cocycle φ, generated by equation (28), i.e., $\varphi(t,x,g) \in \mathbb{R}^n_+$ for any $(t,x,g) \in \mathbb{R}_+ \times \mathbb{R}^n_+ \times H(f)$.*

Proof. Let $g \in H(f)$, then it is easy to check that under the condition of Lemma 8 the function g is also regular, quasi-monotone and $g(t,0) \geq 0$ for any $t \in \mathbb{R}$. Note that $F(g,x) = g(0,x) \geq 0$ for any $(x,g) \in \mathbb{R}^n_+ \times H(f)$. By Theorem 14 we have $\phi(t,x,g) \leq \varphi(t,x,g)$ for any $t \geq 0$, $g \in H(f)$ and $x \in \mathbb{R}^n_+$, where φ is the cocycle generated by equation (28) (respectively, equation (30)) and ϕ is the cocycle defined by equation $x' = 0$, i.e., $\phi(t,x,g) = x$ for any $x \in \mathbb{R}^n_+$, $t \geq 0$ and $g \in H(f)$. Thus we have $\varphi(t,x,g) \geq x$ for any $(t,x,g) \in \mathbb{R}_+ \times \mathbb{R}^n_+ \times H(f)$. This means that $\varphi(t,x,g) \geq 0$, i.e., $\varphi(t,x,g) \in \mathbb{R}^n_+$ for any $(t,x,g) \in \mathbb{R}_+ \times \mathbb{R}^n_+ \times H(f)$. Lemma is proved. □

Lemma 9. *[34] If $f \in C^{0,1}(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ satisfies condition (31), then $\varphi_i(t, x^1, f) < \varphi_i(t, x^2, f)$ for each $t > 0$ whenever $x^1 <_i x^2$.*

Corollary 6. *If $f \in C^{0,1}(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ satisfies condition (31), then $\varphi_i(t, x^1, g) < \varphi_i(t, x^2, g)$ for each $t > 0$ and $g \in H(f) \subset C^{0,1}(\mathbb{R} \times \mathbb{R}^n_+, \mathbb{R}^n)$ whenever $x^1 <_i x^2$.*

Proof. This statement directly follows from Lemmas 8 and 9. □

Lemma 10. *Assume that $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is almost automorphic (respectively, recurrent) in $t \in \mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$ and $\varphi \in C(\mathbb{R}, \mathbb{R}^n)$ is a bounded on $\mathbb{R}$ and compatible solution of equation (28). Then φ is also almost automorphic (respectively, recurrent).*

Proof. Since the function f is almost automorphic (respectively, recurrent), then it is Levitan almost periodic (respectively, almost recurrent) in $t \in \mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$. Taking into account that φ is a compatible solution of equation (28), then by Lemma 4 it is Levitan almost periodic (respectively, almost recurrent). Now to finish the proof of Lemma it is sufficient to show that the function φ is Lagrange stable. Note that φ is bounded on $\mathbb{R}$ and, consequently, $K := \overline{\varphi(\mathbb{R})}$ is a compact subset of $\mathbb{R}^n$. Since f is almost automorphic (respectively, recurrent), then it is Lagrange stable. This means, in particular, that f is bounded on $\mathbb{R} \times K$, i.e., there exists a $C > 0$ such that $|f(t, \varphi(t))| \le C$ for any $t \in \mathbb{R}$. Thus we have

$$|\varphi'(t)| = |f(t, \varphi(t))| \le C$$

for all $t \in \mathbb{R}$ and, consequently, φ is uniformly continuous on $\mathbb{R}$. Thus the function φ is Lagrange stable and Levitan almost periodic (respectively, almost recurrent) and, consequently, it is almost automorphic (respectively, recurrent). Lemma is proved. □

Theorem 15. *Suppose that the following assumptions are fulfilled:*

- *the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is positively Poisson stable in $t \in \mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$;*
- *each solution $\varphi(t, u_0, f)$ of equation (28) is bounded on $\mathbb{R}_+$;*
- *there exists a function $V \in C^1(\mathbb{R}^n_+, \mathbb{R})$ with $\nabla V(x) \gg 0$ for any $x \in \mathbb{R}^n_+$ and $(\nabla V(x), f(t, x)) = 0$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}^n_+$.*

Then under the conditions $(\boldsymbol{A1}) - (\boldsymbol{A4})$ the following statements hold:

1. *$\mathbb{R}^n_+$ is invariant with respect to cocycle φ generated by equation (28);*
2. *for any solution $\varphi(t, u, f)$ of equation (28) there exists a solution $\varphi(t, \gamma_u, f)$ of (28) defined and bounded on $\mathbb{R}$ such that:*
   - *(a) $\varphi(t, \gamma_u, f)$ is a compatible solution of (28);*

(b) $\lim_{t\to\infty}|\varphi(t,u,f)-\varphi(t,\gamma_u,f)|=0$;

3. *if the function* $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ *is stationary (respectively,* τ*-periodic, Levitan almost periodic, almost recurrent, almost automorphic, recurrent, Poisson stable) in* $t\in\mathbb{R}$ *uniformly with respect to* u *on every compact subset from* $\mathbb{R}^n$*, then* $\varphi(t,\gamma_u,f)$ *is also stationary (respectively,* τ*-periodic, Levitan almost periodic, almost recurrent, almost automorphic, recurrent, Poisson stable) and*

4. $\varphi(t,u,f)$ *is asymptotically stationary (respectively, asymptotically* τ*-periodic, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically almost automorphic, asymptotically recurrent, asymptotically Poisson stable).*

Proof. Let $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ and $(C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{R},\sigma)$ be the shift dynamical system on $C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$. Denote by $Y:=H(f)$ and $(Y,\mathbb{R},\sigma)$ the shift dynamical system on $H(f)$ induced by $(C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{R},\sigma)$. Consider the cocycle $\langle\mathbb{R}^n,\varphi,(Y,\mathbb{R},\sigma)\rangle$ generated by equation (28) (see Condition (A1)). Now to finish the proof of Theorem it is sufficient to apply Theorems 3, 8, 12 and Lemmas 4, 10 and Corollary 6. Theorem is proved. □

Definition 27. A function f is said to be strongly Poisson stable in $t\in\mathbb{R}$ uniformly with respect to u on every compact subset of $\mathbb{R}^n$ if the point $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ (respectively, the motion $\sigma(t,f)$) is strongly Poisson stable in shift dynamical system $(C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n),\mathbb{R},\sigma)$.

Theorem 16. *Suppose that the following assumptions are fulfilled:*

– *the function* $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ *is strongly Poisson stable in* $t\in\mathbb{R}$ *uniformly with respect to* u *on every compact subset from* $\mathbb{R}^n$*;*

– *each solution* $\varphi(t,v,g)$ *of equation (29) is bounded on* $\mathbb{R}_+$*;*

– *there exists a function* $V\in C^1(\mathbb{R}^n_+,\mathbb{R})$ *with* $\nabla V(x)\gg 0$ *for any* $x\in\mathbb{R}^n_+$ *and* $(\nabla V(x),f(t,x))=0$ *for any* $(t,x)\in\mathbb{R}\times\mathbb{R}^n_+$.

Then under the conditions (***A1***) $-$ (***A4***) *the following statements hold:*

1. $\mathbb{R}^n_+$ *is invariant with respect to cocycle* φ *generated by equation (28);*

2. *for any solution* $\varphi(t,v,g)$ *of equation (29) there exists a solution* $\varphi(t,\gamma_v,g)$ *of (29) defined and bounded on* $\mathbb{R}$ *such that:*

   (a) $\varphi(t,\gamma_u,g)$ *is a strongly compatible solution of (29);*

   (b) $\lim_{t\to\infty}|\varphi(t,v,)g-\varphi(t,\gamma_v,g)|=0$.

3. *if the function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, almost automorphic, recurrent, strongly Poisson stable) in $t \in \mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$, then $\varphi(t, \gamma_u, f)$ is also stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, almost automorphic, recurrent, strongly Poisson stable) and*

4. *$\varphi(t, u, f)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically Levitan almost periodic, asymptotically almost recurrent, asymptotically almost automorphic, asymptotically recurrent, asymptotically strongly Poisson stable).*

Proof. This statement can be proved similar to Theorem 15 using Theorems 5, 9, 13, Lemma 4 and Corollary 6. Theorem is proved. □

Below we give an example of a first integral for a class of differential equations.

Example 2. Consider a system of differential equations

$$\begin{cases} x_1'(t) = \lambda_1 F_1(t, x_1(t), \dots, x_n(t)) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n'(t) = \lambda_n F_n(t, x_1(t), \dots, x_n(t)), \end{cases} \tag{33}$$

where $\lambda_1, \dots, \lambda_n$ are some positive constants.

System (33) can be rewrite in the vectorial form

$$x' = f(t, x), \tag{34}$$

where $f = (\lambda_1 F_1, \lambda_2 F_2, \dots, \lambda_n F_n)$. Along with equation (34) we will consider its H-class

$$x' = g(t, x), \quad (g \in H(f)) \tag{35}$$

where $H(f)$ is the closure of the translations $\{f_\tau |\ \tau \in \mathbb{R}\}$ of f.

Suppose that the functions $f_1, \dots, f_n$ satisfy the following conditions (**A5**):

$$f_1(t, x_1, \dots, x_n) + \ldots + f_n(t, x_1, \dots, x_n) = 0 \tag{36}$$

for any $t \in \mathbb{R}$ and $x := (x_1, \dots, x_n) \in \mathbb{R}^n$.

Lemma 11. *Suppose that the function $f := (f_1, \dots, f_n) \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies the condition (**A5**), then the function $V \in C(\mathbb{R}^n, \mathbb{R})$ defined by equality*

$$V(x_1, \dots, x_n) := \sum_{i=1}^{n} \frac{x_i}{\lambda_i}$$

is a first integral for system of differential equations (33).

Proof. Let $x(t)=(x_1(t),\dots,x_n(t))$ be an arbitrary solution of equation (33), then

$$\Delta V(x(t))=\sum_{i=1}^{n}\frac{\Delta x_i(t)}{\lambda_i}=\sum_{i=1}^{n}\frac{\lambda_i f_i(t,x_1(t),\dots,x_n(t))}{\lambda_i}= \sum_{i=1}^{n} f_i(t,x_1(t),\dots,x_n(t))=0$$

for any $t\in\mathbb{Z}_+$ and, consequently, $V(x(t))=V(x(0))=C$ ($\forall\ t\in\mathbb{Z}_+$). Lemma is proved. □

Corollary 7. *Under the condition (36) every solution $\varphi(t,u,f)$ of (34) passing through the point u at the initial time $t=0$ with $u\in\mathbb{R}^n_+$ is bounded on $\mathbb{R}_+$.*

Proof. This statement directly follows from Lemma 11. In fact, let $u\in\mathbb{R}^n_+$, $u\neq 0$ and

$$u\in M_\alpha:=\{u\in\mathbb{R}^n_+ |\ u_1+u_2+\ldots+u_n=\alpha\},$$

then by Lemma 11 we have $\varphi(t,u,f)\in M_\alpha\subseteq[0,\alpha]^n$ for any $t\in\mathbb{Z}_+$. □

Theorem 17. *Suppose that the function $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ is strongly Poisson stable in $t\in\mathbb{Z}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$.*

*Then under Conditions (**A1**)-(**A5**) the following statements hold:*

1. *$\mathbb{R}^n_+$ is invariant with respect to cocycle φ generated by system of differential equations (34);*
2. *for any solution $\varphi(t,v,g)$ of equation (35) there exists a solution $\varphi(t,\gamma_v,g)$ of (35) defined and bounded on $\mathbb{R}$ such that:*
   - *(a) $\varphi(t,\gamma_u,g)$ is a strongly compatible solution of (35);*
   - *(b) $\lim\limits_{t\to\infty}|\varphi(t,v,)g-\varphi(t,\gamma_v,g)|=0$;*
3. *if the function $f\in C(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable) in $t\in\mathbb{R}$ uniformly with respect to u on every compact subset from $\mathbb{R}^n$, then $\varphi(t,\gamma_u,f)$ is also stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable) and*
4. *$\varphi(t,u,f)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically strongly Poisson stable).*

Proof. This statement follows from Theorem 16, Lemma 11 and Corollaries 6 and 7 . □

Remark 9. In the case, when the right-hand side f of equation (33) is periodic (respectively, almost periodic) in $t\in\mathbb{R}$ Theorem 17 coincides with the result in the work [23] (respectively, in the work [26]).

5.2 Linear Differential Equations

Let $A(t) = (a_{ij}(t))_{i,j=1}^n$ $(t \in \mathbb{R})$ be a matrix possessing the following properties

$$a_{ij}(t) \geq 0 \ \text{ and } \ \sum_{i=1}^n a_{ij}(t) = 0 \tag{37}$$

for any $i, j = 1, \ldots, n$ with $i \neq j$ and $t \in \mathbb{R}$.

Let $[\mathbb{R}^n]$ be the family of all matrices $A = (a_{ij})_{ij=1}^n$ with real coefficients $a_{ij} \in \mathbb{R}$ and $C(\mathbb{R}, [\mathbb{R}^n])$ be the space of all matrix-functions $A(t) = (a_{ij}(t))_{i,j=1}^n$ equipped with the distance

$$d(A, B) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(A, B)}{1 + d_k(A, B)},$$

where $d_k(A, B) := \max\{||A(t) - B(t)|| : \ |t| \leq k\}$. Denote by $(C(\mathbb{R}, [\mathbb{R}^n]), \mathbb{R}, \sigma)$ the shift dynamical system on $C(\mathbb{R}, [\mathbb{R}^n])$, i.e., $\sigma(a, \tau) = A_\tau$ and $A_\tau(t) := A(t + \tau)$ for any $t, \tau \in \mathbb{R}$ and $A \in C(\mathbb{R}, [\mathbb{R}^n])$.

Remark 10. If the matrix $A \in C(\mathbb{R}, [\mathbb{R}^n])$ satisfies condition (37), then every matrix $B \in H(A)$ satisfies conditions (37).

Consider the differential equation

$$x' = A(t)x \tag{38}$$

and its H-class

$$y' = B(t)y \quad (B \in H(A)). \tag{39}$$

Lemma 12. *Suppose that the matrix $A \in C(\mathbb{R}, [\mathbb{R}^n])$ satisfies conditions (37), then the function $V : \mathbb{R}_+^n \to \mathbb{R}$ defined by equality*

$$V(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n \tag{40}$$

is a first integral for equation (38).

Proof. Let $f(t, x) := A(t)x$, then $f_i(t, x) = \sum_{j=1}^n a_{ij}(t)x_j$ $(i = 1, 2, \ldots, n)$. Since the matrix $A(t)$ satisfies condition (37), then we have

$$\sum_{i=1}^n f_i(t, x) = \sum_{i=1}^n \big(\sum_{j=1}^n a_{ij}(t)x_j\big) = \sum_{j=1}^n \big(\sum_{i=1}^n a_{ij}(t)\big)x_j = 0.$$

According to Lemma 11 the function V defined by (40) is a first integral for equation (38). Lemma is proved. □

Condition (**A6**). A matrix $A(t) = (a_{ij}(t))_{i,j=1}^n$ satisfies the following conditions: $a_{ij}(t) \geq 0$ for any $i, j = 1, 2, \ldots, n$ with $i \neq j$ and $t \in \mathbb{R}$.

Lemma 13. *If the matrix $A(t)$ satisfies Condition (**A6**), then any matrix $B \in H(A)$ satisfies the same condition.*

Proof. Assume that the matrix $A(t)$ satisfies Condition (**A6**), then

$$a_{ij}(t) \geq 0 \tag{41}$$

for any $t \in \mathbb{R}$ and $i, j = 1, 2, \ldots, n$ with $i \neq j$. If $B \in H(A)$, then there exists a sequence $\{h_k\} \subseteq \mathbb{R}$ such that $B(t) = \lim\limits_{k\to\infty} A(t + h_k)$ and, consequently, $b_{ij}(t) = \lim\limits_{k\to\infty} a_{ij}(t + h_k)$. From (41) we have

$$a_{ij}(t + h_k) \geq 0 \tag{42}$$

for any $k \in \mathbb{N}$, $t \in \mathbb{R}$ and $i, j = 1, 2, \ldots, n$. Passing to the limit in inequality (42) as $k \to \infty$ we obtain

$$b_{ij}(t) \geq 0.$$

Lemma is proved. □

Lemma 14. *The following statements hold:*

1. *if the matrix $A(t) \geq 0$ for any $t \in \mathbb{R}$, then the cocycle φ, generated by equation (38), is monotone, i.e., $\varphi(t, u, B) \leq \varphi(t, v, B)$ for any $t \in \mathbb{R}_+$ and $B \in H(A)$ whenever $u \leq v$ $(u, v \in \mathbb{R}^n_+)$;*
2. *the cocycle φ is componentwise monotone, i.e., $\varphi(t, u, B)_i < \varphi_i(t, v, B)$ for any $(t, B) \in \mathbb{R}_+ \times H(A)$ whenever $u \leq v$ and $u_i < v_i$ $(i = 1, 2, \ldots, n)$.*

Proof. The first statement follows from Remark 8. The second statement follows from Lemma 9. □

Corollary 8. *If the matrix $A(t) \geq 0$ for any $t \in \mathbb{R}$, then the cone $\mathbb{R}^n_+$ is positively invariant with respect to cocycle φ, generated by equation (38). This means that $\varphi(t, v, B) \in \mathbb{R}^n_+$ for any $t \in \mathbb{R}_+$ whenever $(v, B) \in \mathbb{R}^n_+ \times H(A)$.*

Proof. By Lemma 14 under the conditions of Corollary the cocycle φ is monotone. Let $v \geq 0$ and $B \in H(A)$, then $\varphi(t, v, B) \geq \varphi(t, 0, B) = 0$ for any $t \in \mathbb{R}_+$. □

Theorem 18. *Assume that $A \in C(\mathbb{R}, [\mathbb{R}^n])$ be a matrix possessing property (**A6**) and it is strongly Poisson stable in $t \in \mathbb{R}$.*

Then the following statements hold:

1. *the cone $\mathbb{R}^n_+$ is positively invariant with respect to cocycle $\langle \mathbb{R}^n, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$ (or shortly φ), generated by equation (38) and its H-class (39);*
2. *the cocycle φ is monotone with respect to spacial variable;*
3. *the cocycle φ is componentwise monotone;*
4. *the function $V : \mathbb{R}^n_+ \to \mathbb{R}$, defined by equality (40), is a first integral for non-autonomous dynamical system, generated by equation (38);*

5. *every solution $\varphi(t,v,B)$ of every equation (39) is bounded on $\mathbb{R}_+$ and positively uniformly stable;*

6. *for every solution $\varphi(t,v,B)$ of every equation (39) there exists a unique solution $\varphi(t,\bar{v},B)$ defined and bounded on $\mathbb{R}$;*

7. *the solution $\varphi(t,\bar{v},B)$ is strongly compatible and*
$$\lim_{t\to\infty}|\varphi(t,u,B)-\varphi(t,\bar{u},B)|=0;$$

8. *if the matrix-function $A\in C(\mathbb{R},[\mathbb{R}^n])$ is stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable) in $t\in\mathbb{R}$, then $\varphi(t,\bar{v},B)$ is also stationary (respectively, τ-periodic, Bohr almost periodic, almost automorphic, recurrent, strongly Poisson stable) and*

9. *$\varphi(t,v,B)$ is asymptotically stationary (respectively, asymptotically τ-periodic, asymptotically Bohr almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically strongly Poisson stable).*

Proof. The first statement follows from Corollary 8.

The second and third statements follow from Lemma 14.

The fourth statement directly follows from Lemma 12.

The fifth statement follows from Corollary 7.

The sixth, seventh, eighth and ninth statements follow from Theorem 17. □

5.3 Linear Differential Equations with Constant Matrix

Let $A=(a_{ij})_{i,j=1}^n$ be a matrix with $a_{ij}\in\mathbb{R}$ for any $i,j=1,2,\dots,n$ and $\varphi(t,x):=e^{tA}x$ ($t\in\mathbb{R}$ and $x\in\mathbb{R}^n$). Assume that the matrix $A=(a_{ij})_{i,j=1}^n$ possesses the following property
$$a_{ij}\ge 0\quad\text{and}\quad\sum_{i=1}^n a_{ij}=0 \tag{43}$$
for any $i,j=1,2,\dots,n$ with $i\neq j$.

By Lemma 12 we have
$$\sum_{i=1}^n (e^{tA}x)_i=\sum_{i=1}^n x_i,$$
then
$$\varphi(t,x)\in M:=M_1=\{x\in\mathbb{R}^n_+|\ x_1+x_2+\ldots+x_n=1\}$$
for any $x\in M$ and $t\ge 0$. Thus on the set M a semigroup dynamical system $(M,\mathbb{R}_+,\varphi)$ is defined. Note that the set M is a compact and convex subset of $\mathbb{R}^n_+$.

Let $(X,\mathbb{T},\pi)$ be an arbitrary dynamical system on the complete metric space (X,ρ), where ρ is a distance on X and $\pi:\mathbb{T}\times X\to X$ is a continuous map satisfying the conditions: $\pi(0,x)=x$ and $\pi(t+s,x)=\pi(t,\pi(s,x))$ for any $x\in X$ and $t,s\in\mathbb{T}$.

Recall [11, Ch.I] that a dynamical system is called compact dissipative if there exists a nonempty compact $K \subseteq X$ which attracts every compact subset M from X, i.e.,

$$\lim_{k\to\infty} \beta(\pi(t,M),K) = 0, \tag{44}$$

where $\beta(A,B) := \sup\{\rho(x,B) | \ x \in A\}$ and $\rho(x,B) := \inf\{\rho(x,y) | \ y \in B\}$.

If $(X,\mathbb{T},\pi)$ is compact dissipative and K is a nonempty compact subset figuring in (44), then

$$J = \bigcap_{t\geq 0} \pi(t,K) \tag{45}$$

is a nonempty, compact subset of X and it does not depend of the choice of K. The set J is called Levinson center of the dynamical system $(X,\mathbb{T},\pi)$.

Theorem 19. *[11, Ch.II] If $(X,\mathbb{T},\pi)$ is compact dissipative and J is its Levinson center, then the following statements hold:*

1. *the set J is invariant, i.e., $\pi(t,J) = J$;*
2. *J attracts every compact subset M of X;*
3. *J is orbitally stable, i.e., for arbitrary positive number $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon) > 0$ such that $\rho(\pi(t,x),J) < \varepsilon$ for any $t \geq 0$ whenever $\rho(x,J) < \delta$.*

Let M be a nonempty subset of X and

$$D^+(M) := \bigcap_{\varepsilon>0} \overline{\bigcup_{t\geq 0} \pi(t,(B(M,\varepsilon)))},$$

$B(M,\varepsilon) := \{x \in X | \ \rho(x,M) < \varepsilon\}$. If M consists of a single point m, then we put $D^+(M) = D^+(\{m\}) = D^+_m$.

Lemma 15. *[11, Ch.II] If the set M is compact, then $D^+(M) = \bigcup_{m\in M} D^+_m$.*

Theorem 20. *[11, Ch.II] If the dynamical system $(X,\mathbb{T},\pi)$ is compact dissipative and M is a nonempty compact and positively invariant subset of X, then M is orbitally stable if and only if $D^+(M) = M$.*

Denote by $\Omega := \overline{\{\omega_x | \ x \in X\}}$, where ω_x is the ω-limit set of the point $x \in X$.

Theorem 21. *[11, Ch.II] If $(X,\mathbb{T},\pi)$ is a compact dissipative dynamical system and J is its Levinson center, then $J = D^+(\Omega)$.*

Denote by $Fix(\varphi)$ the set of all stationary points of dynamical system $(M,\mathbb{R}_+,\varphi)$.

Theorem 22. *Let $A = (a_{ij})_{i,j=1}^n \in [\mathbb{R}^n]$ be matrix satisfying condition (43). Then the following statements hold:*

1. *the dynamical system* $(M, \mathbb{R}_+, \varphi)$ *has a nonempty and compact set of stationary points* $Fix(\varphi) \subseteq M$*;*

2. *for every* $x \in M$ *there exists* $\lim_{t\to\infty} \varphi(t, x) = p_x$ *and* $p_x \in Fix(\varphi)$ *for any* $x \in M$*;*

3. *every stationary point* $p \in Fix(\varphi)$ *of* $(M, \mathbb{R}_+, \varphi)$ *is positively stable, i.e, for any positive number* ε *there exists a positive number* $\delta = \delta(\varepsilon)$ *such that* $|\varphi(t, x) - p| < \varepsilon$ *for any* $t \in \mathbb{R}_+$*, whenever* $|x - p| < \delta$ $(x \in M)$*;*

4. *the dynamical system* $(M, \mathbb{R}_+, \varphi)$ *is compact dissipative and its Levinson center* J *coincides with the set* $Fix(\varphi)$*;*

5. $Fix(\varphi) = \bigcap_{t\geq 0} \varphi(t, M)$ *and it is convex;*

Proof. The first, second and third statements of Theorem follow from Theorem 18 (items 5, 7, 8 and 9).

Note that the dynamical system $(M, \mathbb{R}_+, \varphi)$ is compact dissipative. Denote by J its Levinson center, then by (45) we have

$$J = \bigcap_{t\geq 0} \varphi(t, M). \tag{46}$$

Denote by Ω the closure of all ω-limit set points of $(M, \mathbb{R}_+, \varphi)$. Since $\omega_x = p_x \in Fix(\varphi)$ for any $x \in M$ and $Fix(\varphi)$ is a closed subset of M, then $\Omega \subseteq Fix(\varphi)$. On the other hand, $Fix(\varphi) \subseteq \Omega$ and, consequently, $\Omega = Fix(\varphi)$. By Theorem 21 we have $J = D^+(\Omega) = D^+(Fix(\varphi))$. Since the set $Fix(\varphi)$ is compact, then by Lemma 15 we have $D^+(Fix(\varphi)) = \bigcup\{D^+_p |\ p \in Fix(\varphi)\}$ and, consequently,

$$J = \bigcup\{D^+_p |\ p \in Fix(\varphi)\}. \tag{47}$$

Since the fixed point p is positively Lyapunov stable, then by Theorem 20 $D^+_p = \{p\}$ and, consequently, from (47) we obtain

$$J = Fix(\varphi). \tag{48}$$

From (46) and (48) we receive

$$Fix(\varphi) = \bigcap_{t\geq 0} \varphi(t, M).$$

Finally, we note that

$$M \supseteq \varphi(s, M) \supseteq \varphi(t, M)$$

for any $0 \leq s \leq t$ and taking into consideration that every set $\varphi(t, M)$ $(t \in \mathbb{R}_+)$ is compact and convex, then we can conclude that the set $Fix(\varphi)$ is convex. □

Denote by $Int(M)$ the interior of the set M.

Lemma 16. *Let $A = (a_{ij})_{i,j=1}^n \in [\mathbb{R}^n]$. Suppose that the $a_{ij} > 0$ for any $i, j = 1, \dots, n$ with $i \neq j$, then the matrix $B := e^A$ possesses the following properties:*

1. *$B = (b_{ij})_{i,j=1}^n$ is positive, i.e., $b_{ij} > 0$ for any $i, j = 1, 2, \dots, n$;*
2. *B has at most one stationary point p, i.e., $Bp = p$.*

Proof. Let $n \in \mathbb{N}$ be a naturale number and $B_n := e^{A/n}$. Since

$$B_n = E + A/n + \dots, \tag{49}$$

where $E \in [\mathbb{R}^n]$ is the unite matrix, then for sufficiently large n the matrix B_n is positive. Taking into consideration that

$$B = (B_n)^n \tag{50}$$

for any $n \in \mathbb{N}$ we conclude that the matrix B is also positive.

Now we will establish that the matrix $B = e^A$ has at most one fixed point. In fact, if we suppose that it is not true, then there exist two different stationary points p and $\bar{p}$ of B, i.e., $Bp = p$, $B\bar{p} = \bar{p}$ and $p \neq \bar{p}$.

Logically two cases are possible:

a. the vectors p and $\bar{p}$ are linearly dependent, then without loss of generality we can suppose that $\bar{p} = \beta p$ for some $\beta \neq 0$. Since $p, \bar{p} \in M$, then we have

$$1 = \sum_{i=1}^{n} \bar{p}_i = \beta \sum_{i=1}^{n} p_i = \beta,$$

i.e., $\beta = 1$ and, consequently, $\bar{p} = p$. The last equality contradicts the choice of $\bar{p}$ ($\bar{p} \neq p$).

b. the vectors p and $\bar{p}$ are linear independently. Since the matrix B is positive, then it is irreducible and, consequently, it has not two linearly independent nonnegative eigenvectors (see, for example,[16, Ch.XIII] Remark 3, page 342).

From a. and b. we conclude that B has at most one stationary point. Lemma is completely proved. □

Theorem 23. *Let $A = (a_{ij})_{i,j=1}^n \in [\mathbb{R}^n]$. Suppose that*

$$\begin{aligned} &a_{ij} > 0 \quad \text{for any } i, j = 1, \dots, n \quad \text{with } i \neq j \\ &\text{and} \quad \sum_{i=1}^{n} a_{ij} = 0 \quad \text{for any} \quad j = 1, 2, \dots, n, \end{aligned} \tag{51}$$

then the following statements hold:

1. *the dynamical system $(M, \mathbb{R}_+, \varphi)$ has a unique stationary point $p \in M$;*
2. *the vector $p \in M$ is positive, i.e., $p_i > o$ for any $i = 1, \dots, n$;*

3. p is globally asymptotically stable, i.e.,

(a) for any positive number $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $|x - p| < \delta$ ($x \in M$) implies $|\varphi(t, x) - p| < \varepsilon$ for any $t \in \mathbb{R}_+$ and

(b)
$$\lim_{t \to \infty} \varphi(t, x) = p$$
for any $x \in M$.

Proof. To prove Theorem 23, according to Theorem 22, it is sufficient to show that the Levinson center J of the dynamical system $(M, \mathbb{R}_+, \varphi)$ consists of a single point $\{p\}$ and $\{p\} = J \subset Int(M)$.

At first, we will establish that the semi-cascade (M, A) has at most one fixed point. In fact, if we suppose that it is not true, then J contains at least two different points p_i and p_2 ($p_1 \neq p_2$). This means that $\varphi(t, p_i) = p_i$ for any $t \in \mathbb{R}_+$ and $i = 1, 2$. In particular, we have
$$Bp_i = p_i \quad (i = 1, 2), \tag{52}$$
where $B = \varphi(1, \cdot) = e^A$. By Lemma 16 (item (i)) the matrix B is positive. Taking into consideration this fact we see that (52) contradicts Lemma 16 (item (ii)).

Secondly, we will establish that $\{p\} = J \subset Int(M)$ and the vector p is positive. In fact, since the matrix $B = e^A$ is positive, then $B(M) \subseteq Int(M)$ and, consequently, $p = B(p) \in B(M) \subseteq Int(M)$. This means, in particular, that the vector p is positive. Theorem is completely proved. □

Remark 11. Notice that Theorem 23 becames false if we replace condition (51) ($a_{ij} > 0$ for any $i \neq j$) by (43) ($a_{ij} \geq 0$ for any $i \neq j$). Namely, in this case the set $Fix(\varphi)$, generally speaking, is not reduced to a single point.

This statement can be confirmed by following example:

Example 3. Consider the following equation
$$x' = Ax \quad (x \in \mathbb{R}^3) \tag{53}$$
with the matrix
$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Let $(\mathbb{R}^3, \mathbb{R}, \varphi)$ be the dynamical system generated by equation (53) and $(M, \mathbb{R}_+, \varphi)$ be the semi-group dynamical system on $M = \{x \in \mathbb{R}^3_+ | \ x_1 + x_2 + x_3 = 1\}$ induced by $(\mathbb{R}^3, \mathbb{R}, \varphi)$. It is possible to check that $Fix(\varphi) = \{p_\alpha | \ \alpha \in [0, 1]\}$, where
$$p_\alpha = \begin{pmatrix} \alpha \\ (1 - \alpha)/2 \\ (1 - \alpha)/2 \end{pmatrix}.$$

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