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On solutions of the kinetic McKean system

Sergey Dukhnovsky

Abstract. In this article, we apply the Painlevé expansion for the kinetic McKean system. This system does not pass the Painlevé test. It leads to the singularity manifold constraint. The singularity manifold conditions are satisfied by the n-dimensional Bateman equation. This allows to get some new solutions.

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1 Introduction

Extensive literature is devoted to the study of discrete kinetic systems [1, 2, 4, 6, 9, 13, 16]. The basic equation for the dynamics of rarefied gas is the Boltzmann equation. This equation is an integro-differential partial differential equation, which makes it difficult to study this equation. To simplify the Boltzmann equation, several nonlinear models have been proposed, among which are discrete velocity models, where unknown functions are the density of particles of a given type at a given point in space-time. These models have interesting conceptual and mathematical features.

Exact solutions of discrete kinetic equations were found in [8–10]. Stationary solutions of the kinetic Carleman and Broadwell models were found by O. V. Il'yin [11,12]. It was proved in [14,16–18] that stationary solutions of the Carleman, Godunov-Sultangazin and Broadwell systems are asymptotically stable. In recent works, a numerical analysis of the Carleman and Godunov-Sultangazin systems was carried out for periodic initial data by O. A. Vasil'eva [19,20]. In paper [4], O. Lindblom and N. Euler construct solutions of two discrete velocity Boltzmann equations in (1+1)- and (1+2)-dimensions using truncated Painlevé expansions. In this paper, we obtain in a similar way exact solutions for the McKean system.

2 The Painlevé test and solutions

We consider the kinetic McKean system [6]:

$$\partial_t u + \partial_x u = \frac{1}{\varepsilon}(w^2 - uw), x \in \mathbb{R}, t > 0, \tag{1}$$

$$\partial_t w - \partial_x w = -\frac{1}{\varepsilon}(w^2 - uw), \tag{2}$$

where $\varepsilon \in \mathbb{R}$ is an analogue of the free path length.

The McKean system describes particles of two groups, namely, the first group of particles moves at a unit speed along the axis Ox, and the second group moves at a unit speed in the opposite direction. Particles of the first and second groups colliding cause a reaction that transfers into two particles of the first group. In turn, two particles of the first group transfers into particles of the first and second groups.

We perform the Painlevé test [3]. One seeks solution of (1), (2) in the form of the Painlevé expansion

$$u(x,t) = \frac{1}{\varphi^p(x,t)} \sum_{j=0}^{\infty} u_j(x,t)\varphi^j(x,t),$$

$$w(x,t) = \frac{1}{\varphi^\beta(x,t)} \sum_{j=0}^{\infty} w_j(x,t)\varphi^j(x,t).$$

For $j=0$, we have

$$u(x,t) = u_0(x,t)\varphi^{-p}(x,t), \tag{3}$$
$$w(x,t) = w_0(x,t)\varphi^{-\beta}(x,t). \tag{4}$$

Substituting the Painlevé expansion (3)–(4) into our system (1)–(2), we obtain

$$u_{0,t}\varphi^{-p} - p\varphi^{-p-1}\varphi_t u_0 + u_{0,x}\varphi^{-p} - p\varphi^{-p-1}\varphi_x u_0 = \frac{1}{\varepsilon}(w_0^2\varphi^{-2\beta} - u_0 w_0\varphi^{-p-\beta}),$$

$$w_{0,t}\varphi^{-\beta} - \beta\varphi^{-\beta-1}\varphi_t u_0 - w_{0,x}\varphi^{-\beta} + \beta\varphi^{-\beta-1}\varphi_x w_0 = -\frac{1}{\varepsilon}(w_0^2\varphi^{-2\beta} - u_0 w_0\varphi^{-p-\beta}).$$

Since $p=1, \beta=1$. In this case

$$-\varphi_t u_0 - \varphi_x u_0 = \frac{1}{\varepsilon}(w_0^2 - u_0 w_0),$$

$$-\varphi_t w_0 + \varphi_x w_0 = -\frac{1}{\varepsilon}(w_0^2 - u_0 w_0).$$

We find leading terms

$$u_0(x,t) = -\varepsilon\frac{(\varphi_t - \varphi_x)^2}{2\varphi_t}, w_0(x,t) = \varepsilon\frac{(\varphi_t - \varphi_x)(\varphi_t + \varphi_x)}{2\varphi_t}. \tag{5}$$

The resonance is at 1. Therefore, the truncated Painlevé series have the form

$$u(x,t) = \frac{u_0(x,t)}{\varphi} + u_1(x,t), \tag{6}$$
$$w(x,t) = \frac{w_0(x,t)}{\varphi} + w_1(x,t). \tag{7}$$

Substituting (6)–(7) into (1)–(2) yields

$$u_{0,t}\varphi^{-1}-\varphi^{-2}\varphi_t u_0+u_{1,t}+u_{0,x}\varphi^{-1}-\varphi^{-2}\varphi_x u_0+u_{1,x}=\frac{1}{\varepsilon}I(u_0,w_0,u_1,w_1),$$

$$w_{0,t}\varphi^{-1}-\varphi^{-2}\varphi_t w_0+w_{1,t}-w_{0,x}\varphi^{-1}+\varphi^{-2}\varphi_x w_0-w_{1,x}=-\frac{1}{\varepsilon}I(u_0,w_0,u_1,w_1),$$

where

$$I(u_0,w_0,u_1,w_1)=\frac{w_0^2}{\varphi^2}-\frac{u_0w_0}{\varphi^2}+2\frac{w_0}{\varphi}w_1-\frac{u_0}{\varphi}w_1-\frac{u_1}{\varphi}w_0+w_1^2-u_1w_1.$$

Collecting terms at equal degrees of φ, we obtain the following equations

$$\varphi^{-1}(u_{0,t}+u_{0,x}-\frac{2}{\varepsilon}w_0w_1+\frac{1}{\varepsilon}u_0w_1+\frac{1}{\varepsilon}u_1w_0)+ \\ +\varphi^0(u_{1,t}+u_{1,x}-\frac{1}{\varepsilon}(w_1^2-u_1w_1))+ \\ +\varphi^{-2}(-\varphi_t u_0-\varphi_x u_0-\frac{1}{\varepsilon}(w_0^2-u_0w_0))=0$$

and

$$\varphi^{-1}(w_{0,t}-w_{0,x}+\frac{2}{\varepsilon}w_0w_1-\frac{1}{\varepsilon}u_0w_1-\frac{1}{\varepsilon}u_1w_0)+ \\ +\varphi^0(w_{1,t}-w_{1,x}+\frac{1}{\varepsilon}(w_1^2-u_1w_1))+ \\ +\varphi^{-2}(-\varphi_t w_0+\varphi_x w_0+\frac{1}{\varepsilon}(w_0^2-u_0w_0))=0.$$

Equating terms at equal degrees of φ, it leads to

$$-\varphi_t u_0-\varphi_x u_0-\frac{1}{\varepsilon}(w_0^2-u_0w_0)=0, -\varphi_t w_0+\varphi_x w_0+\frac{1}{\varepsilon}(w_0^2-u_0w_0)=0,$$
$$u_{0,t}+u_{0,x}-\frac{1}{\varepsilon}(2w_0w_1-u_0w_1-u_1w_0)=0,$$
$$w_{0,t}-w_{0,x}+\frac{1}{\varepsilon}(2w_0w_1-u_0w_1-u_1w_0)=0,$$
$$u_{1,t}+u_{1,x}-\frac{1}{\varepsilon}(w_1^2-u_1w_1)=0, w_{1,t}-w_{1,x}+\frac{1}{\varepsilon}(w_1^2-u_1w_1)=0.$$

The first equations give us the already known leading terms of the expansion, which are determined by (5). The last equations of the system are satisfied because resonance arises and u_1, w_1 are arbitrary functions.
Moreover, we have

$$u_{0,t}+u_{0,x}=\frac{1}{\varepsilon}(2w_0w_1-u_0w_1-u_1w_0),$$
$$w_{0,t}-w_{0,x}=-\frac{1}{\varepsilon}(2w_0w_1-u_0w_1-u_1w_0).$$

Then

$$u_{0,t} + u_{0,x} = \frac{2}{\varepsilon}(2w_0w_1 - u_0w_1 - u_1w_0), \tag{8}$$

$$u_{0,t} + u_{0,x} = -(w_{0,t} - w_{0,x}). \tag{9}$$

These equations are not satisfied. Substituting the dominant terms (5) into (9), we have

$$\varphi_{tt}\varphi_x^2 - 2\varphi_x\varphi_t\varphi_{xt} + \varphi_t^2\varphi_{xt} = 0. \tag{10}$$

Equation (10) is known as the two-dimensional Bateman equation (see [4, 5, 7]). The Painlevé test will be performed only if φ satisfies the equation (10). It's the constraint on the given function. Since u_1, w_1 are arbitrary functions, set $u_1 = w_1 = 0$. Then we obtain equation for finding the function φ:

$$\varphi_x(\varphi_{tt} + \varphi_{xt}) + \varphi_t(\varphi_{tt} - \varphi_{xt} - 2\varphi_{xx}) = 0. \tag{11}$$

The general implicit solution of (10) is

$$f(\varphi) = x + g(\varphi)t, \tag{12}$$

where f and g are arbitrary smooth functions.
We prove the following

Lemma. *For the 2-velocity model* (1)–(2) *the truncated Painlevé expansion*

$$u(x,t) = \frac{u_0(x,t)}{\varphi}, w(x,t) = \frac{w_0(x,t)}{\varphi}, \tag{13}$$

where u_0, w_0 are given by (5)*, yields conditions on φ by* (10) *and by* (11) *with following solutions*

$$\varphi(x,t) = \frac{x + k_0t - c_2}{c_1},$$

where $k_0 \in \mathbb{R} \setminus \{0, \pm 1\}, c_1 \in \mathbb{R} \setminus \{0\}, c_2 \in \mathbb{R}$;

$$\varphi(x,t) = F(x \pm t),$$

where F is an arbitrary invertible function;

$$\varphi(x,t) = \frac{1}{A}\Big(\frac{1}{4}\Big(-\frac{2(c_1 - t)}{x - c_2 - c_1 + t} + \ln\Big(\frac{x - c_2 - c_1 + t}{x - c_2 + c_1 - t}\Big)\Big) + B\Big),$$

where $\{A, c_1\} \in \mathbb{R} \setminus \{0\}$ and $\{c_2, B\} \in \mathbb{R}$.

Proof. Differentiating (12) and substituting into (11), respectively leads to

$$\frac{\varepsilon\Big((1 + g + 2g^2)\frac{df}{d\varphi}\frac{dg}{d\varphi} - t(1 + g + 2g^2)(\frac{dg}{d\varphi})^2 - g(g^2 - 1)(\frac{d^2f}{d\varphi^2} - t\frac{d^2g}{d\varphi^2})\Big)}{g(\frac{\partial f}{\partial\varphi} - t\frac{\partial g}{\partial\varphi})^3} = 0.$$

From here

$$(1+g+2g^2)\frac{df}{d\varphi}\frac{dg}{d\varphi} - g(g^2-1)\frac{d^2 f}{d\varphi^2} = 0,$$

$$-(1+g+2g^2)(\frac{dg}{d\varphi})^2 + g(g^2-1)\frac{d^2 g}{d\varphi^2} = 0.$$

Case 1. Consider $g = \pm 1$. Then

$$\varphi_t = \pm\varphi_x.$$

From (12) we have $\varphi(x,t) = F(x \pm t)$, where F is an arbitrary invertible function. The solution of the Carleman system for $g = 1$ has the form

$$u(x,t) = 0, w(x,t) = 0.$$

For $g = -1$, yields

$$u(x,t) = \frac{2\varepsilon F'(x-t)}{F(x-t)}, w(x,t) = 0.$$

Case 2. Let $g = k_0, k_0 \notin \{0, \pm 1\}$. Then we have equation

$$\frac{\partial^2 f}{\partial \varphi^2} = 0 \Rightarrow f(\varphi) = c_1\varphi + c_2.$$

$$c_1\varphi + c_2 = x + k_0 t \Rightarrow \varphi(x,t) = \frac{x + k_0 t - c_2}{c_1}.$$

We obtain the following solution of the system (1)–(2)

$$u(x,t) = -\varepsilon\frac{(k_0-1)^2}{2k_0(x+k_0t-c_2)}, w(x,t) = \varepsilon\frac{(k_0^2-1)}{2k_0(x+k_0t-c_2)}.$$

Case 3. Consider $g'(\varphi) \neq 0$. Then system can be rewritten as

$$\frac{f''}{f'} = \frac{g''}{g'}, \tag{14}$$

$$-(1+g+2g^2)(\frac{\partial g}{\partial\varphi})^2 + g(g^2-1)\frac{\partial^2 g}{\partial\varphi^2} = 0. \tag{15}$$

Integrating (14), we have

$$f(\varphi) = c_1 g(\varphi) + c_2, c_1, c_2 \in \mathbb{R}, c_1 \neq 0.$$

Using the solution of the Bateman equation (12), we can express a function g :

$$g(x,t) = \frac{x - c_2}{c_1 - t}. \tag{16}$$

Moreover, from equation (15) we obtain that

$$\frac{dg}{d\varphi} = A\frac{(1-g)^2(1+g)}{g}, A \in \mathbb{R} \setminus \{0\}. \tag{17}$$

The solution (17) taking into account (16) is written as

$$\varphi(x,t)=\frac{1}{A}\Big(\frac{1}{4}\Big(-\frac{2(c_1-t)}{x-c_2-c_1+t}+\ln\Big(\frac{x-c_2-c_1+t}{x-c_2+c_1-t}\Big)\Big)+B\Big), \quad (18)$$

where $B\in\mathbb{R}$ is a constant of integration. Thus, the solution of the McKean system (1)-(2) using (5), (13) and (18) has the form

$$u(x,t)=\frac{2(x+t-c_1-c_2)\varepsilon}{(c_1-c_2-t+x)G(x,t)}, w(x,t)=-\frac{2\varepsilon}{G(x,t)},$$

where

$$G(x,t)=2\Big(c_1+2Bc_1-t-2B(x+t-c_2)+(c_1+c_2-t-x)\ln\Big(\frac{x+t-c_1-c_2}{x-t+c_1-c_2}\Big)\Big).$$

□

Proposition. *Solution of the 2-velocity model can be represented as*

$$u(x,t)=u_0H_1(\varphi), w(x,t)=w_0H_2(\varphi),$$

where

$$H_1(\varphi)=H_2(\varphi)+b, b\in\mathbb{R}$$

and H_2 satisfies the equation

$$\frac{dH_2}{d\varphi}=-H_2^2+\frac{b}{2}H_2(\frac{1}{g}-1). \quad (19)$$

Here u_0, w_0 are defined by (5)*, φ satisfies equations* (10) *and* (11)*.*

Proof. We look for solution in the following form

$$u(x,t)=\frac{u_0}{\varphi}f_1(\varphi), w(x,t)=\frac{w_0}{\varphi}f_2(\varphi). \quad (20)$$

After the substitution (20) into (1)–(2), we obtain conditions for finding functions f_1, f_2:

$$\varphi_t(2\varphi f_1'-2f_1+f_2^2+f_1f_2)+\varphi_x(f_2^2-f_1f_2)=0,$$
$$\varphi_t(2\varphi f_2'-2f_2+f_2^2+f_1f_2)+\varphi_x(f_2^2-f_1f_2)=0.$$

By the substitution $f_i(\varphi)=H_i(\varphi)\varphi, i=1,2$, we have

$$\varphi_t(2H_1'+H_2^2+H_1H_2)+\varphi_x(H_2^2-H_1H_2)=0,$$
$$\varphi_t(2H_2'+H_2^2+H_1H_2)+\varphi_x(H_2^2-H_1H_2)=0.$$

Subtracting one from the other equation, we get

$$2\varphi_t(H_1'-H_2')=0, \quad (21)$$

$$\varphi_t(2H_2' + H_2^2 + H_1H_2) + \varphi_x(H_2^2 - H_1H_2) = 0. \tag{22}$$

From (21) it follows that

$$H_1 = H_2 + b, b \in \mathbb{R}. \tag{23}$$

We also take into account that

$$\frac{\varphi_t}{\varphi_x} = g(\varphi).$$

Substituting (23) into (22), one obtains

$$\frac{dH_2}{d\varphi} = -H_2^2 + \frac{b}{2}H_2(\frac{1}{g} - 1). \tag{24}$$

□

Let's consider examples where the above equation (24) gives various solutions for the McKean system.

Example 1. Let $g = 3, b = 1$, $\varphi = x + 3t$, $c_1 = 1, c_2 = 0$. In this case the equation (24) has the form

$$\frac{dH_2}{d\varphi} = -H_2^2 - \frac{1}{3}H_2.$$

Then

$$H_2(\varphi) = \frac{1}{3(C_1 e^{\frac{1}{3}(x+3t)} - 1)}.$$

where C_1 is a constant of integration. Finally, we have

$$u(x,t) = u_0(H_2(\varphi) + 1) = -\frac{2\varepsilon}{3}\Big(\frac{1}{3(Ce^{\frac{1}{3}(x+3t)} - 1)} + 1\Big),$$

$$w(x,t) = w_0 H_2(\varphi) = \frac{4\varepsilon}{9(Ce^{\frac{1}{3}(x+3t)} - 1)}.$$

Example 2. For $g = 1, b = 1$, the equation (19) takes the form

$$\frac{dH_2}{d\varphi} = -H_2^2.$$

Hence

$$H_2(\varphi) = \frac{1}{F(x+t) + C_2}, C_2 \in \mathbb{R}.$$

The solution of the system (1)-(2) has the form

$$u(x,t) = 0, w(x,t) = 0.$$

For $g(\varphi) = -1, b = 1$, we have

$$H_2(\varphi) = -\frac{1}{C_3 e^{-F(x-t)} - 1}, C_3 \in \mathbb{R},$$

$$u(x,t) = 2\varepsilon F'(x-t)\Big(H_2(\varphi) + 1\Big), w(x,t) = 0.$$

Example 3. Consider for $g'(\varphi) \neq 0$. Make a substitution $H_2(\varphi) = \widehat{H}_2(g)$, using (18). We also use the fact that

$$\frac{dH_2}{d\varphi} = \frac{d\widehat{H}_2}{dg}\frac{dg}{d\varphi}.$$

Then the equation (19) can be rewritten in the form

$$A\frac{d\widehat{H}_2}{dg} = -\frac{g}{(1-g)^2(g+1)}\widehat{H}_2^2 + \frac{b}{2(1-g)(g+1)}\widehat{H}_2.$$

Let $b = 0$. In this case

$$\widehat{H}_2(g) = -\frac{4A(g-1)}{2 - 4AC_2 + 4AgC_2 + (g-1)\ln(\frac{g+1}{g-1})}, C_2 \in \mathbb{R}.$$

For $A = 1, b = 4$

$$\widehat{H}_2(g) = -\frac{2(g^2-1)}{-1 + g(2-4C_3) + 2C_3 + 2g^2C_3}, C_3 \in \mathbb{R},$$

the solution of system (1)-(2) takes the form

$$u(x,t) = -\frac{\varepsilon}{2A(c_1 - c_2 - t + x)}(\widehat{H}_2(g) + b),$$

$$w(x,t) = -\frac{\varepsilon}{2A(c_1 + c_2 - t - x)}\widehat{H}_2(g).$$

3 Conclusion

We have investigated the Painlevé analysis for the McKean system and shown that this system does not pass the Painlevé test in order to be integrable. We used the solution of the Bateman equation to solve this problem. The calculations were made using Wolfram Mathematica programme.

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Dukhnovsky S.
Moscow State University Of Civil Engineering (National Research University), 26, Yaroslavskoe shosse, Moscow, 129337, Russian Federation
E-mail: *sdukhnvskijj@gmail.com*

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