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Free left k-nilpotent n-tuple semigroups

Anatolii V. Zhuchok*, Yuliia V. Zhuchok, and Oksana O. Odintsova

Abstract. We introduce left (right) k-nilpotent n-tuple semigroups which are analogs of left (right) nilpotent semigroups of rank p considered by Schein, and construct the free left (right) k-nilpotent n-tuple semigroup of rank 1. We prove that the free left (right) k-nilpotent n-tuple semigroup of rank $m > 1$ is a subdirect product of the free left (right) k-nilpotent semigroup with m generators and the free left (right) k-nilpotent n-tuple semigroup of rank 1. We also characterize the least left (right) k-nilpotent congruence on the free n-tuple semigroup.

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1 Introduction

In [8], Koreshkov introduced n-tuple algebras of associative type, using n-tuple semigroups. The n-tuple semigroups now play an important role in different areas of algebra: they contain varieties of commutative dimonoids [10,16,17,25] and commutative trioids [11,18,26], they occur in the theory of interassociative semigroups [1–3,5], and in recent advances in doppelsemigroup theory [19,22,24,27,28,33], they apply in the theory of n-tuple algebras of associative type [6–8]. In addition to their widespread appearance, n-tuple semigroups are connected to duplexes [13], g-dimonoids [12,34], and restrictive bisemigroups [15], while 1-tuple semigroups are semigroups.

One of the central tools of universal algebra is the free object in a variety. The variety of n-tuple semigroups behave well with respect to the typical subvarieties. Recently, free systems in the varieties of n-tuple semigroups [20], commutative n-tuple semigroups [29], k-nilpotent n-tuple semigroups [31], and rectangular n-tuple semigroups [23] were constructed. The free product of arbitrary n-tuple semigroups was given in [29].

In this paper, we introduce the variety of left (right) k-nilpotent n-tuple semigroups. Such algebras are analogs of left (right) nilpotent semigroups of rank p considered by Schein [14], left (right) k-dinilpotent dimonoids [32], left (right) k-dinilpotent doppelsemigroups [19], and left (right) k-trinilpotent trioids [30]. Free objects in the varieties of left (right) k-dinilpotent dimonoids, left (right) k-dinilpotent

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doppelsemigroups, and left (right) k-trinilpotent trioids were constructed in [19, 32] and [30], respectively. The focus of this paper is to construct the free object in the variety of left (right) k-nilpotent n-tuple semigroups.

The paper is organized as follows. In Section 2, we present the relevant definitions and notations, and give some background results from [20]. Section 3 contains new results. We begin with constructing the free left k-nilpotent n-tuple semigroup of rank 1 and the proof that the free left k-nilpotent n-tuple semigroup of rank $m > 1$ is a subdirect product of the free left k-nilpotent semigroup with m generators and the free left k-nilpotent n-tuple semigroup of rank 1. Then we count the cardinality of the free left k-nilpotent n-tuple semigroup for a finite case, establish that the semigroups of the free left k-nilpotent n-tuple semigroup are isomorphic and its automorphism group is isomorphic to the symmetric group. We also characterize the least left k-nilpotent congruence on the free n-tuple semigroup. The description of free right k-nilpotent n-tuple semigroups and characterization of the least right k-nilpotent congruence on the free n-tuple semigroup are obtained in a dual way.

The results obtained in the present paper extend some results in [19].

2 Preliminaries

Following Schein [14], a semigroup T is called a left (right) nilpotent semigroup of rank p if the product of any p elements from this semigroup gives a left (right) zero. Right nilpotent semigroups appear in automata theory, namely, such semigroups are semigroups of self-adaptive automata (see [4, 9]). The class of all left nilpotent semigroups of rank p is characterized by the identity $g_1g_2\dots g_pg_{p+1} = g_1g_2\dots g_p$. The least such p will be called the left nilpotency index of a semigroup T. As usual, we denote the set of all positive integers by $\mathbb{N}$. For $k \in \mathbb{N}$ a left nilpotent semigroup of left nilpotency index $\leq k$ is said to be a left k-nilpotent semigroup. Right k-nilpotent semigroups are defined dually. The class of all left (right) k-nilpotent semigroups forms a subvariety of the variety of semigroups. A semigroup which is free in the variety of left (right) k-nilpotent semigroups will be called a free left (right) k-nilpotent semigroup. Recently, analogs of a left (right) nilpotent semigroup of rank p were introduced in the varieties of dimonoids [32], doppelsemigroups [19], and trioids [30].

For $n \in \mathbb{N}$ denote the set $\{1, 2, \dots, n\}$ by $\overline{n}$. Recall that an n-tuple semigroup [8] is a nonempty set G equipped with n binary operations $\boxed{1}, \boxed{2}, \dots, \boxed{n}$ satisfying the axioms $(x \boxed{r} y) \boxed{s} z = x \boxed{r} (y \boxed{s} z)$ for all $x, y, z \in G$ and $r, s \in \overline{n}$. For n-tuple semigroups, it is natural to introduce an analog of a left (right) nilpotent semigroup of rank p.

An n-tuple semigroup $(G, \boxed{1}, \boxed{2}, \dots, \boxed{n})$ will be called left nilpotent if for some $m \in \mathbb{N}$, every $x_1, \dots, x_m, x_{m+1} \in G$, and all $i \in \overline{n}$, the following identities hold:

$$(x_1 *_1 \dots *_{m-1} x_m) \boxed{i}\, x_{m+1} = x_1 *_1 \dots *_{m-1} x_m, \tag{2.1}$$

where $*_1, \dots, *_{m-1} \in \{\boxed{1}, \boxed{2}, \dots, \boxed{n}\}$. The least such m will be called the left nilpotency index of $(G, \boxed{1}, \boxed{2}, \dots, \boxed{n})$. For $k \in \mathbb{N}$ a left nilpotent n-tuple semigroup of left nilpotency index $\leq k$ is said to be a left k-nilpotent n-tuple semigroup. Right k-nilpotent n-tuple semigroups are defined dually.

It is clear that operations of any left (right) 1-nilpotent n-tuple semigroup coincide and it is a left (right) zero semigroup, and the class of all left (right) k-nilpotent 1-tuple semigroups coincides with the class of all left (right) k-nilpotent semigroups. The class of all left (right) k-nilpotent n-tuple semigroups forms a subvariety of the variety of n-tuple semigroups. An n-tuple semigroup which is free in the variety of left (right) k-nilpotent n-tuple semigroups will be called a free left (right) k-nilpotent n-tuple semigroup. If ρ is a congruence on an n-tuple semigroup M such that M/ρ is a left (right) k-nilpotent n-tuple semigroup, we say that ρ is a left (right) k-nilpotent congruence.

The free n-tuple semigroup was first given in [20]. Recall this construction.

Let X be an arbitrary nonempty set, and let w be an arbitrary word over X. The length of w is denoted by l_w. Fix $n \in \mathbb{N}$ and let $Y = \{y_1, y_2, \ldots, y_n\}$ be an arbitrary set consisting of n elements. Let further $F[X]$ be the free semigroup on X, let $F^{\theta}[Y]$ be the free monoid on Y, and let $\theta \in F^{\theta}[Y]$ be the empty word. By definition, the length l_θ of θ is equal to 0. Define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on

$$XY_n = \{(w, u) \in F[X] \times F^{\theta}[Y] \,|\, l_w - l_u = 1\}$$

by

$$(w_1, u_1)\boxed{i}(w_2, u_2) = (w_1 w_2, u_1 y_i u_2)$$

for all $(w_1, u_1), (w_2, u_2) \in XY_n$ and $i \in \overline{n}$. The algebra $(XY_n, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$ is denoted by $F_nTS(X)$. By Theorem 2 of [20], $F_nTS(X)$ is the free n-tuple semigroup.

The following lemma is needed for the sequel.

Lemma 2.1. *([20], Lemma 1) In an n-tuple semigroup $(G, \boxed{1}, \boxed{2}, \ldots, \boxed{n})$, for every $m > 1$, $m \in \mathbb{N}$, every $x_i \in G$, $1 \leq i \leq m+1$, and every $*_j \in \{\boxed{1}, \boxed{2}, \ldots, \boxed{n}\}$, $1 \leq j \leq m$, any parenthesizing in*

$$x_1 *_1 x_2 *_2 \ldots *_m x_{m+1}$$

gives the same element of G.

If $f : G_1 \to G_2$ is a homomorphism of n-tuple semigroups, the kernel of f will be denoted by Δ_f. Denote the symmetric group on X by $\Im[X]$ and the automorphism group of an n-tuple semigroup M by $Aut\, M$.

3 Main results

In this section, we construct the free left k-nilpotent n-tuple semigroup of rank 1 and prove that the free left k-nilpotent n-tuple semigroup of rank $m > 1$ is a subdirect product of the free left k-nilpotent semigroup with m generators and the free left k-nilpotent n-tuple semigroup of rank 1. We also count the cardinality of the free left k-nilpotent n-tuple semigroup for a finite case, establish that the semigroups of the free left k-nilpotent n-tuple semigroup are isomorphic and its automorphism group is isomorphic to the symmetric group. Besides, we characterize the least left k-nilpotent congruence on the free n-tuple semigroup.

Let $w \in F[X]$. Fix $k, n \in \mathbb{N}$. Following [30], if $l_w \geq k$, let $\overset{k}{\overrightarrow{w}}$ denote the initial subword with the length k of w, and if $l_w < k$, let $\overset{k}{\overrightarrow{w}} = w$. It is clear that

$$\overset{k}{\overrightarrow{\overset{k}{\overrightarrow{w_1w_2}}\,w_3}} = \overset{k}{\overrightarrow{w_1w_2w_3}} = \overset{k}{\overrightarrow{w_1\,\overset{k}{\overrightarrow{w_2w_3}}}} \tag{3.1}$$

for all $w_1, w_2, w_3 \in F[X]$. We will also regard that $\overset{0}{\overrightarrow{u}} = \overset{k}{\overrightarrow{\theta}} = \theta$ for all $u \in F^{\theta}[Y]$. Assume that $Y^{(k)} = \{u \in F^{\theta}[Y] \mid l_u + 1 \leq k\}$ and define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on $Y^{(k)}$ by

$$u_1 \boxed{i} u_2 = \overset{k-1}{\overrightarrow{u_1 y_i u_2}}$$

for all $u_1, u_2 \in Y^{(k)}$ and $i \in \overline{n}$. The algebra obtained in this way will be denoted by $Y_n^{(k)}$.

Theorem 3.1. *$Y_n^{(k)}$ is the free left k-nilpotent n-tuple semigroup of rank* 1*.*

Proof. For $u_1, u_2, u_3 \in Y_n^{(k)}$ and $i, j \in \overline{n}$, we have

$$(u_1 \boxed{i} u_2) \boxed{j} u_3 = \overset{k-1}{\overrightarrow{u_1 y_i u_2}} \boxed{j} u_3 = \overset{k-1}{\overrightarrow{\overset{k-1}{\overrightarrow{u_1 y_i u_2}}\, y_j u_3}} = \overset{k-1}{\overrightarrow{u_1 y_i u_2 y_j u_3}}$$

$$= \overset{k-1}{\overrightarrow{u_1 y_i \,\overset{k-1}{\overrightarrow{u_2 y_j u_3}}}} = u_1 \boxed{i} \overset{k-1}{\overrightarrow{u_2 y_j u_3}} = u_1 \boxed{i} (u_2 \boxed{j} u_3)$$

and so, $Y_n^{(k)}$ satisfies all axioms of an n-tuple semigroup.

Show that $Y_n^{(k)}$ is left k-nilpotent. Obviously, $Y_n^{(1)}$ is left 1-nilpotent. Let $k > 1$ and let $u_1 *_1 \ldots *_{k-1} u_k = u$ for arbitrary elements $u_1, \ldots, u_k \in Y_n^{(k)}$, where $*_1, \ldots, *_{k-1} \in \{\boxed{1}, \boxed{2}, \ldots, \boxed{n}\}$. Since $l_u = k - 1$,

$$u \boxed{i} g = \overset{k-1}{\overrightarrow{u y_i g}} = u$$

for any $g \in Y_n^{(k)}$ and $i \in \overline{n}$. Thus, by definition, $Y_n^{(k)}$ is a left nilpotent n-tuple semigroup. Further, if $k = 2$ and $i \in \overline{n}$, then

$$\theta \boxed{i} \theta = \overset{1}{\overrightarrow{y_i}} = y_i \neq \theta,$$

and we conclude that $Y_n^{(2)}$ has left nilpotency index 2. Withal, for $k > 2$ and any $i \in \overline{n}$, we get

$$(\theta *_1 \ldots *_{k-2} \theta) \boxed{i} \theta = y_i^{k-2} \boxed{i} \theta = \overset{k-1}{\overrightarrow{y_i^{k-1}}} = y_i^{k-1} \neq \theta *_1 \ldots *_{k-2} \theta.$$

Therefore $Y_n^{(k)}$ has left nilpotency index k.

Finally, prove that $Y_n^{(k)}$ is the singly generated free object in the variety of left k-nilpotent n-tuple semigroups.

Obviously, $Y_n^{(k)}$ is generated by θ. Let $(S, \boxed{1}', \boxed{2}', \ldots, \boxed{n}')$ be an arbitrary left k-nilpotent n-tuple semigroup, and let $\pi : \{\theta\} \to S$ be an arbitrary map. Suppose that $\theta\pi = \varepsilon \in S$. Define a map

$$\psi : Y_n^{(k)} \to (S, \boxed{1}', \boxed{2}', \ldots, \boxed{n}') : \omega \mapsto \omega\psi$$

by the rule

$$\omega\psi = \begin{cases} \varepsilon_1 \widetilde{y}_{i_1} \varepsilon_2 \widetilde{y}_{i_2} \ldots \widetilde{y}_{i_{s-1}} \varepsilon_s & \text{if } \omega = y_{i_1} y_{i_2} \ldots y_{i_{s-1}},\ y_{i_p} \in Y, \\ & \quad 1 \le p \le s-1,\ s > 1, \\ \varepsilon & \text{if } \omega = \theta, \end{cases}$$

where $\varepsilon_r = \varepsilon$ for $1 \le r \le s$ and

$$\widetilde{y}_{i_p} = \boxed{b}' \quad \text{for some} \quad b \in \overline{n} \ \Leftrightarrow\ y_{i_p} = y_b \ (1 \le p \le s-1,\ s > 1). \tag{3.2}$$

According to Lemma 2.1, the map ψ is well-defined. Similar to the proof of Lemma 3.7 in [19] we can prove that ψ is a homomorphism. For this, we use Lemma 2.1 and (2.1).

Clearly, $\theta\psi = \theta\pi$. Since θ generates $Y_n^{(k)}$, the uniqueness of ψ is obvious. Thus, $Y_n^{(k)}$ is the free left k-nilpotent n-tuple semigroup of rank 1.

$\square$

Now we present the free left k-nilpotent semigroup.

Let $U_k = \{w \in F[X] \,|\, l_w \le k\}$. A binary operation $\cdot$ is defined on U_k by the rule

$$w_1 \cdot w_2 = \overset{k}{\overrightarrow{w_1 w_2}}$$

for all $w_1, w_2 \in U_k$. With respect to this operation U_k is a semigroup generated by X. It will be denoted by $FLNS_k(X)$.

Lemma 3.2. *$FLNS_k(X)$ is the free left k-nilpotent semigroup.*

Proof. By (3.1), $FLNS_k(X)$ is a semigroup. It is immediate to check that $FLNS_k(X)$ is left k-nilpotent. Let us show that $FLNS_k(X)$ is free left k-nilpotent.

Let T be an arbitrary left k-nilpotent semigroup, and let $\gamma : X \to T$ be an arbitrary map. Define the map

$$\phi : FLNS_k(X) \to T : x_1 \ldots x_h \mapsto (x_1 \ldots x_h)\phi = x_1\gamma \ldots x_h\gamma, \quad x_1, \ldots, x_h \in X.$$

One can show that ϕ is a homomorphism. $\square$

We are ready to construct the free left k-nilpotent n-tuple semigroup of an arbitrary rank.

Define n binary operations $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$ on

$$W_k = \left\{(w,u) \in FLNS_k(X) \times Y_n^{(k)} \ \mid l_w - l_u = 1\right\}$$

by

$$(w_1,u_1)\boxed{i}(w_2,u_2) = (\overset{k}{\overrightarrow{w_1w_2}}, \overset{k-1}{\overrightarrow{u_1y_iu_2}})$$

for all $(w_1,u_1),(w_2,u_2) \in W_k$ and $i \in \overline{n}$. These operations are well-defined, since $l_{\overset{k}{\overrightarrow{w_1w_2}}} - l_{\overset{k-1}{\overrightarrow{u_1y_iu_2}}} = 1$ for all $i \in \overline{n}$. The obtained algebra will be denoted by $F_{(l)}^{k,n}NS(X)$.

Theorem 3.3. *$F_{(l)}^{k,n}NS(X)$ is the free left k-nilpotent n-tuple semigroup.*

Proof. It follows from Theorem 3.1 and Lemma 3.2 that $F_{(l)}^{k,n}NS(X)$ is a left k-nilpotent n-tuple semigroup generated by $X \times \{\theta\}$. We state that this n-tuple semigroup is free left k-nilpotent.

Let $(S, \boxed{1}', \boxed{2}', \ldots, \boxed{n}')$ be an arbitrary left k-nilpotent n-tuple semigroup, and let $\beta : X \to S$ be an arbitrary map. Define a map

$$\alpha : F_{(l)}^{k,n}NS(X) \to (S, \boxed{1}', \boxed{2}', \ldots, \boxed{n}') : v \mapsto v\alpha$$

by the rule

$$v\alpha = \begin{cases} x_1\beta\widetilde{y}_{i_1}x_2\beta\widetilde{y}_{i_2}\ldots\widetilde{y}_{i_{s-1}}x_s\beta & \text{if } v = (x_1x_2\ldots x_s, y_{i_1}y_{i_2}\ldots y_{i_{s-1}}), \\ & \quad x_j \in X, 1 \le j \le s,\ y_{i_p} \in Y, \\ & \quad 1 \le p \le s-1,\ s > 1, \\ x_1\beta & \text{if } v = (x_1,\theta),\ x_1 \in X, \end{cases}$$

where every $\widetilde{y}_{i_p}$, $1 \le p \le s-1$, $s > 1$, is defined by (3.2). According to Lemma 2.1, α is well-defined.

Applying Lemma 2.1 and (2.1), similar to the proof of Lemma 3.7 in [19] one can show that α is a unique homomorphism extending β. Thus, $F_{(l)}^{k,n}NS(X)$ is the free left k-nilpotent n-tuple semigroup. □

Corollary 3.4. *The free left k-nilpotent n-tuple semigroup $F_{(l)}^{k,n}NS(X)$ generated by a finite set $X \times \{\theta\}$ is finite. Specifically, $|F_{(l)}^{k,n}NS(X)| = \sum_{i=1}^{k} n^{i-1} \cdot |X|^i$.*

Corollary 3.5. *Every free left k-nilpotent n-tuple semigroup of rank $m > 1$ is a subdirect product of the free left k-nilpotent semigroup with m generators and the free left k-nilpotent n-tuple semigroup of rank 1.*

Proof. The rank of $F_{(l)}^{k,n}NS(X)$ is $|X|$. Let $|X| = m > 1$. Then, by the above construction, $F_{(l)}^{k,n}NS(X)$ is a subdirect product of the free left k-nilpotent semigroup $FLNS_k(X)$ with m generators and the free left k-nilpotent n-tuple semigroup $Y_n^{(k)}$ of rank 1. □

Corollary 3.6. *$F^{k,1}_{(l)}NS(X)$ is the free left k-nilpotent semigroup.*

Theorems 3.1 and 3.3 imply the following statement.

Corollary 3.7. *If $|X| = 1$, then $Y_n^{(k)} \cong F^{k,n}_{(l)}NS(X)$.*

Note that, for $n = 2$, Theorem 3.3 yields Theorem 3.1 in [19].

Corollary 3.9 in [19] implies the following statement establishing a relationship between the semigroups of the free left k-nilpotent n-tuple semigroup.

Proposition 3.8. *For any $i, j \in \overline{n}$, the semigroups $(W_k, \boxed{i})$ and $(W_k, \boxed{j})$ of $F^{k,n}_{(l)}NS(X)$ are isomorphic.*

Since the set $X \times \{\theta\}$ is generating for $F^{k,n}_{(l)}NS(X)$, we have the following isomorphism: $Aut\, F^{k,n}_{(l)}NS(X) \cong \Im[X]$.

Congruences play an important role when investigating different universal algebras (see, e.g., [21]). At the end of the paper, we characterize the least left k-nilpotent congruence on the free n-tuple semigroup.

For $k \in \mathbb{N}$ define a binary relation γ_k on the free n-tuple semigroup $F_nTS(X)$ by

$$(w_1, u_1)\gamma_k(w_2, u_2) \quad \text{if and only if} \quad (\overset{k}{\overrightarrow{w_1}}, \overset{k-1}{\overrightarrow{u_1}}) = (\overset{k}{\overrightarrow{w_2}}, \overset{k-1}{\overrightarrow{u_2}}).$$

Theorem 3.9. *The relation γ_k on the free n-tuple semigroup $F_nTS(X)$ is the least left k-nilpotent congruence.*

Proof. Define a map $\pi_k : F_nTS(X) \to F^{k,n}_{(l)}NS(X)$ by

$$(w, u) \mapsto (w, u)\pi_k = (\overset{k}{\overrightarrow{w}}, \overset{k-1}{\overrightarrow{u}}),\ (w, u) \in F_nTS(X).$$

Let $(w_1, u_1), (w_2, u_2) \in F_nTS(X)$ and $i \in \overline{n}$. It is not difficult to check that

$$\overset{k}{\overrightarrow{w_1 w_2}} = \overset{k}{\overrightarrow{\overset{k}{\overrightarrow{w_1}}\,\overset{k}{\overrightarrow{w_2}}}}, \quad \overset{k-1}{\overrightarrow{u_1 y_i u_2}} = \overset{k-1}{\overrightarrow{\overset{k-1}{\overrightarrow{u_1}}\ y_i\ \overset{k-1}{\overrightarrow{u_2}}}}.$$

Using it, we have

$$((w_1, u_1)\boxed{i}(w_2, u_2))\pi_k = (w_1w_2, u_1y_iu_2)\pi_k = (\overset{k}{\overrightarrow{w_1w_2}}, \overset{k-1}{\overrightarrow{u_1y_iu_2}})$$
$$= (\overset{k}{\overrightarrow{\overset{k}{\overrightarrow{w_1}}\,\overset{k}{\overrightarrow{w_2}}}}, \overset{k-1}{\overrightarrow{\overset{k-1}{\overrightarrow{u_1}}\ y_i\ \overset{k-1}{\overrightarrow{u_2}}}}) = (\overset{k}{\overrightarrow{w_1}}, \overset{k-1}{\overrightarrow{u_1}})\boxed{i}(\overset{k}{\overrightarrow{w_2}}, \overset{k-1}{\overrightarrow{u_2}}) = (w_1, u_1)\pi_k\boxed{i}(w_2, u_2)\pi_k.$$

Thus, π_k is a homomorphism. Evidently, π_k is a surjection. Since by Theorem 3.3 $F^{k,n}_{(l)}NS(X)$ is the free left k-nilpotent n-tuple semigroup, Δ_{π_k} is the least left k-nilpotent congruence on $F_nTS(X)$. From the definition of π_k it follows that $\Delta_{\pi_k} = \gamma_k$. So, γ_k is the least left k-nilpotent congruence on $F_nTS(X)$. □

For $n = 2$, Theorem 3.9 implies Theorem 4.1 in [19].

For $k \in \mathbb{N}$ define a binary relation μ_k on the free semigroup $F[X]$ by

$$w_1 \mu_k w_2 \quad \text{if and only if} \quad \overset{k}{\overrightarrow{w_1}} = \overset{k}{\overrightarrow{w_2}}\,.$$

Corollary 3.10. *The relation μ_k on the free semigroup $F[X]$ is the least left k-nilpotent congruence.*

Remark 3.11. *In [19], the first author of this paper constructed the free left k-nilpotent doppelsemigroup, using the definition of $\overset{k}{\overrightarrow{w}}$ for the case $l_w \geq k$. In this paper, we extend the definition of $\overset{k}{\overrightarrow{w}}$ for the case $l_w < k$ as in [30]. This gives us the opportunity to simplify the definition of operations of free left k-nilpotent n-tuple semigroups in comparison with the definition of operations of free left k-nilpotent doppelsemigroups. These changes imply a significant simplification of proofs for theorems in the present paper.*

Remark 3.12. *In order to construct free right k-nilpotent n-tuple semigroups and characterize the least right k-nilpotent congruence on the free n-tuple semigroup we use the duality principle.*

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Anatolii V. Zhuchok, Yuliia V. Zhuchok
Department of Algebra and System Analysis, Luhansk Taras Shevchenko National University, Gogol square, 1, Starobilsk 92703, Ukraine
E-mail: *zhuchok.av@gmail.com, yulia.mih1984@gmail.com*

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Oksana O. Odintsova
Department of Mathematics, "A. S. Makarenko" Sumy State Pedagogical University, street Romenska, 87, Sumy 40002, Ukraine
E-mail: *oincube@yahoo.com*