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Coincidence and common fixed points theorem with an application in dynamic programming

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Abstract. We prove coincidence and common fixed points theorem for two self-mappings in complete metric spaces. Our theorem generalizes Theorem 1 of [19]. Suitable examples are provided to illustrate the validity of our results. We apply our theorem to establish the existence of common solutions of a system of two functional equations arising in dynamic programming.

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1 Introduction and Preliminaries

Suzuki [40] categorized the theorems which ensure the existence of a fixed point of a mapping T into the following four types.

(T_1) Leader-type [20]: T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$. Such a mapping is called a Picard operator PO, see Picard [30] and Rus [32, 34].

(T_2) Unnamed type: T has a unique fixed point and $\{T^n x\}$ does not necessarily converge to the fixed point.

(T_3) Subrahmanyam-type [39]: T may have more than one fixed point and $\{T^n x\}$ converges to a fixed point for all $x \in X$. Such a mapping is called a weakly Picard operator WPO, see Rus [33, 35].

It is evident that any PO is a WPO, but the converse in not true in general, see Berinde [6].

(T_4) Caristi-type [8, 9]: T may have more than one fixed point and $\{T^n x\}$ does not necessarily converge to a fixed point.

Most of the theorems such as Banach's [3], Cirić's [12] , Kannan's [16], Kirk's [17], Meir and Keeler's [27] and Suzuki's [41] belong to (T_1). Subrahmanyam's theorem [39], Cirić's theorem [13], Theorem 1 of Berinde [6] and Theorems 2.1, 2.2 of Samet et al.[36] appertai to (T_3). Caristi's theorems [8, 9] appertain to (T_4). Furthermore, there are no theorems belonging to (T_2), see Kirk's survey [18].

First of all, we recall the following definitions.

Definition 1 ([10])**.** Let $(E, \leq)$ be a partially ordered set and F a subset of E. F is said to be well ordered if every two elements of F are comparable.

Definition 2. Let $A, S : X \to X$ be two mappings. A point $u \in X$ is said to be

i) a fixed point of A if $Au = u$,

ii) a coincidence point of A and S if $Au = Su$. The point $z = Au = Su$ is called a point of coincidence of A and S.

iii) a common fixed point of A and S if $Au = Su = u$.

iv) A and S are weakly compatible iff they commute at their coincidence points.

We denote by $C(A, S)$ the set of coincidence points of A and S.

Proposition 1 ([1]). *Let $A, S : X \to X$ be two mappings. If A and S have a unique point of coincidence $z = Au = Su$ and A and S are weakly compatible, then z is the unique common fixed point of A and S.*

Definition 3 ([38]). Let X be a Banach space, Y a subset of X and $A, S : Y \to Y$ such that $A(Y) \subset S(Y)$. For $x_0 \in Y$, consider the following iterative scheme:

$$Sx_{n+1} = Ax_n,\ n \in \mathbb{N}.$$

For $Y = X$, this scheme is called Jungck iterative scheme. It was introduced by Jungck [14] in 1976 and it reduces to the Picard iterative scheme when $S = I_X$, where I_X is the identity mapping in X.

Inspired by Rus and Chandok et al.[11], we state the following definitions. In the sequel, (X, d) is a metric space and $A, S : X \to X$ are two mappings such that $A(X) \subset S(X)$.

Definition 4. A and S are called Picard-Jungck operators (brievely PJO) if:

i) A and S have a unique common fixed point z.

ii) The sequence $\{Sx_n\}$ converges to z for each $x \in X$.

If $S = I_X$, where I_X is the identity mapping in X, we obtain the definition of PO.

Most of the operators such as Abbas and Khan's [2], Berinde's (Theorems 3.3, 3.4) [6], Jungck's [14] and Sessa's [37] are PJO.

Further, If the operators T and f in Theorem 2.1 of Chandok and Karapinar [10] are weakly compatible and the set of common fixed points of T and f is well ordered, therefore T and f are PJO, see Theorem 2.2 of Chandok and Karapinar [10].

Definition 5. A and S are said to be weakly Picard-Jungck operators (brievely WPJO) if:

i) A and S have at least one common fixed point.

ii) The sequence $\{Sx_n\}$ converges to a common fixed point for any $x \in X$.

If $S = I_X$, where I_X is the identity mapping in X, we get the definition of WPO.

In addition, if the operators T and f in Theorem 2.1 of Chandok and Karapinar [10] are weakly compatible, so T and f are WPJO.

Definition 6. A and S are called Quasi Picard-Jungck operators (brievely QPJO) if:

i) A and S have a unique point of coincidence z or a coincidence point.

ii) The sequence $\{Sx_n\}$ converges to z for each $x \in X$.

If $S = I_X$, where I_X is the identity mapping in X, we get the definition of PO.

Remark 1. i) If A and S commute at z, by Proposition 1, A and S have a unique common fixed point z and so A and S become PJO.

ii) If we remove, the condition of weak compatibility of A and S, in theorems of Abbas and Khan [2], Berinde [6], Jungck [14] and Sessa [37], then A and S become QPJO.

Definition 7. A and S are said to be Quasi weakly Picard-Jungck operators (brievely QWPJO) if:

i) A and S have at least one point of coincidence or a coincidence point.

ii) The sequence $\{Sx_n\}$ converges to a point of coincidence for any $x \in X$.

If $S = I_X$, where I_X is the identity mapping in X, we obtain the definition of WPO.

The operators in Theorem 3.1 of Berinde [7], Theorems 2.4, 2.6 of Nashine and Samet [28] and Theorem 2.1 of Chandok and Karapinar [10] are QWPJO.

In 2014, Khojasteh et al.[19] established the following theorem which appertains to (T_3).

The following theorem was proved by Khojasteh et al.[19].

Theorem 1. *Let (X, d) be a complete metric space and T a mapping from X into itself satisfying the following condition*

$$d(Tx, Ty) \leq \frac{d(y, Tx) + d(x, Ty)}{d(x, Tx) + d(y, Ty) + 1} d(x, y)$$

for all $x, y \in X$. Then

(i) T has at least one fixed point $z \in X$.

(ii) $\{T^n x\}$ converges to a fixed point for all $x \in X$.

(iii) If z and w are distinct fixed points of T, therefore $d(z, w) \geq \frac{1}{2}$.

Rhoades [31] extended the above theorem for two self-mappings by proving the subsequent theorem.

Theorem 2. *Let (X, d) be a complete metric space and S, T self-mappings of X satisfying*

$$d(Sx, Ty) \leq N(x, y)m(x, y)$$

for all $x, y \in X$, where

$$N(x, y) = \frac{\max\{d(x, y), d(x, Sx) + d(y, Ty), d(x, Ty) + d(y, Sx)\}}{d(x, Sx) + d(y, Ty) + 1}$$

and

$$m(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty)+d(y,Sx)}{2}\right\}.$$

Then

(a) S and T have at least one common fixed point $p \in X$.

(b) For n even, $\left\{(ST)^{n/2}x\right\}$ and $\left\{T(ST)^{n/2}x\right\}$ converge to a common fixed point for each $x \in X$.

(c) If p and q are distinct common fixed points of S and T, then $d(p,q) \geq 1/2$.

It is our purpose in this paper to demonstrate coincidence and common fixed points theorem for two self-mappings in complete metric spaces. Our theorem generalizes Theorem 1 of [19]. Examples are furnished to illustrate the validity of our results. We apply our theorem to realize the existence of common solutions of a system of two functional equations arising in dynamic programming.

2 Main Results

The next lemma plays a crucial role in the proof of our main theorem.

Lemma 1. *Let (X,d) be a metric space and $\{x_n\}$ a sequence in X such that*

$$d(x_n, x_{n+1}) \leq \beta_n d(x_{n-1}, x_n) \tag{1}$$

for all $n \in \mathbb{N}^*$, where

$$\beta_n = \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_{n-1},x_n)+d(x_n,x_{n+1})+1}.$$

Then $\{x_n\}$ is a Cauchy sequence.

Proof. As in the proof of Lemma 2.3 of Rhoades [31], assume that $x_{n-1} \neq x_n$ for each $n \geq 1$ and set $t_n = d(x_{n-1}, x_n)$. Therefore

$$\beta_n = \frac{t_n + t_{n+1}}{t_n + t_{n+1} + 1}. \tag{2}$$

Since $0 < \beta_n < 1$, we deduce from (1) that

$$t_{n+1} \leq \beta_n t_n < t_n \text{ for any } n \in \mathbb{N}. \tag{3}$$

We will prove that for all $n \geq 1$, $\beta_n < \beta_{n-1}$. Using (2), we obtain that $\beta_n < \beta_{n-1}$ is equivalent to

$$\frac{t_n + t_{n+1}}{t_n + t_{n+1} + 1} < \frac{t_{n-1} + t_n}{t_{n-1} + t_n + 1}.$$

The above inequality yields $t_{n+1} < t_{n-1}$ which is fulfilled by (3) Consequently

$$t_{n+1} < \beta_1 t_n \text{ for every } n \in \mathbb{N}.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. □

Now, we state and prove our main result.

Theorem 3. *Let A and S be two mappings of a complete metric space (X,d) into itself verifying*

$$A(X) \subset S(X), \tag{4}$$

$$d(Ax, Ay) \leq N(x,y)M(x,y) \tag{5}$$

for all $x, y \in X$, where

$$N(x,y) = \frac{\max\{d(Sx,Sy), d(Sx,Ax) + d(Sy,Ay), d(Sx,Ay) + d(Sy,Ax)\}}{d(Sx,Ax) + d(Sy,Ay) + 1}, \tag{6}$$

$$M(x,y) = \max\{d(Sx,Sy), d(Sx,Ax), d(Sy,Ay), \frac{d(Sx,Ay) + d(Sy,Ax)}{2}\}. \tag{7}$$

Suppose that $S(X)$ is a closed subspace of X. So

i) A and S have at least one coincidence point $u \in X$ and the Jungck sequence $\{y_n\} = \{Sx_n\}$ converges to $z = Au$ for each $x \in X$. In this case, A and S are QWPJO.

ii) If there exists $u \in C(A,S)$ such that $ASu = SAu$, then $z = Au$ is another coincidence point of A and S. In this case, A and S are QWPJO.

iii) If v is a distinct coincidence point of A and S, therefore $d(Au, Av) \geq \frac{1}{2}$.

iv) If $Au = A^2u$ for some $u \in C(A,S)$ and $ASu = SAu$, then A and S possess at least one common fixed point $z = Au$ and the Jungck sequence $\{y_n\} = \{Sx_n\}$ converges to z for any $x \in X$. In this case, A and S are WPJO.

(v) If z and w are distinct common fixed points of A and S, so $d(z,w) \geq \frac{1}{2}$.

Proof. Let x_0 be an arbitrary point in X. From (4), we can define inductively a sequence $\{y_n\}$ in X such that

$$y_n = Ax_n = Sx_{n+1}, n \in \mathbb{N} \tag{8}$$

Let us show that $\{y_n\}$ is a Cauchy sequence in X. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, so $Ax_n = Sx_{n+1} = Ax_{n+1} = Sx_{n+2}$. Thus, A and S a coincidence point. Therefore, assume that $y_n \neq y_{n+1}$ for each $n \in \mathbb{N}$. Using (5), (6) and (8), we have

$$\begin{aligned} d(y_n, y_{n+1}) &= d(Ax_n, Ax_{n+1}) \\ &\leq N(x_n, x_{n+1})M(x_n, x_{n+1}), \end{aligned} \tag{9}$$

where

$$\begin{aligned} N(x_n, x_{n+1}) &= \frac{\max\{d(Sx_n, Sx_{n+1}), d(Sx_n, Ax_n) + d(Sx_{n+1}, Ax_{n+1}), d(Sx_n, Ax_{n+1})\}}{d(Sx_n, Ax_n) + d(Sx_{n+1}, Ax_{n+1}) + 1} \\ &= \frac{\max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n) + d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1})\}}{d(y_{n-1}, y_n) + d(y_n, y_{n+1}) + 1} \\ &\leq \frac{d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{d(y_{n-1}, y_n) + d(y_n, y_{n+1}) + 1}. \end{aligned}$$

Set $t_n = d(y_{n-1}, y_n)$, $n \geq 1$. So

$$N(x_n, x_{n+1}) \leq \frac{t_n + t_{n+1}}{t_n + t_{n+1} + 1} = \beta_n. \tag{10}$$

By (7) we obtain

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), \\ &\quad d(y_n, y_{n+1}), \frac{d(y_{n-1}, y_{n+1})}{2}\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \end{aligned} \tag{11}$$

Substituting (10) and (11) into (9), we find

$$t_{n+1} \leq \beta_n t_n \text{ for all } n \in \mathbb{N}.$$

According to Lemma 1, we conclude that $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, $\{y_n\}$ converges to $z \in X$. Suppose that $S(X)$ is closed subspace of X. Therefore, $z = Su$ for some $u \in X$. We claim that $Au = z$. If $Au \neq z$, applying (5) we get

$$d(Au, Ax_{n+1}) \leq N(u, x_{n+1})M(u, x_{n+1}), \tag{12}$$

where

$$N(u, x_{n+1}) = \frac{\max\left\{\begin{array}{c} d(Su, Sx_{n+1}), d(Su, Au) + d(Sx_{n+1}, Ax_{n+1}), \\ d(Su, Ax_{n+1}) + d(Sx_{n+1}, Au) \end{array}\right\}}{d(Su, Au) + d(Sx_{n+1}, Ax_{n+1}) + 1},$$

$$M(u, x_{n+1}) = \max\left\{\begin{array}{c} d(Su, Sx_{n+1}), d(Su, Au), d(Sx_{n+1}, Ax_{n+1}), \\ \dfrac{d(Su, Ax_{n+1}) + d(Sx_{n+1}, Au)}{2} \end{array}\right\}.$$

Letting $n \to \infty$ in (12) we obtain

$$\begin{aligned} d(Au, z) &\leq \frac{d(Au, z)}{d(Au, z) + 1} d(Au, z) \\ &< d(Au, z). \end{aligned}$$

Hence, $Au = Su = z$. Thus u is a coincidence point of A and S and the Jungck sequence $\{y_n\} = \{Sx_n\}$ converges to $z = Au$ for each $x \in X$.

ii) If there exists $u \in C(A, S)$ such that $ASu = SAu$, then $Az = Sz$, i.e, z is a coincidence point of A and S.

iii) If $Au \neq Av$, applying (5) we find

$$d(Au, Av) \leq N(u, v)M(u, v),$$

where

$$N(u, v) = \frac{\max\left\{\begin{array}{c} d(Su, Sv), d(Su, Au) + d(Sv, Av), \\ d(Su, Av) + d(Sv, Au) \end{array}\right\}}{d(Su, Au) + d(Sv, Av) + 1}$$

$$= 2d(Au, Av),$$

$$\begin{aligned} M(u,v) &= \max\left\{ \begin{array}{c} d(Su,Sv), d(Su,Au), d(Sv,Av), \\ \dfrac{d(Su,Av)+d(Sv,Au)}{2} \end{array} \right\} \\ &= d(Au,Av). \end{aligned}$$

So, $d(Au,Av) \leq 2d^2(Au,Av)$, that is $d(Au,Av) \geq \dfrac{1}{2}$.

iv) If $Au = A^2u$ for each $u \in C(A,S)$ and $ASu = SAu$, therefore $z = Az = Sz$, i.e., A and S possess at least one common fixed point $z \in X$ and the Jungck sequence $\{y_n\} = \{Sx_n\}$ converges to $z = Au$ for any $x \in X$.

(v) If z and w are distinct common fixed points of A and S, employing (5) we have

$$d(z,w) = d(Az,Aw) \leq N(z,w)M(z,w),$$

where $N(z,w) = 2d(z,w)$ and $M(z,w) = d(z,w)$. Thus, $d(z,w) \geq \dfrac{1}{2}$. □

The following examples support our Theorem 3.

Example 1. Let $X = \{0,1,2,3\}$ be endowed with the usual metric. Define $A, S : X \to X$ by:

$$\begin{aligned} A(0) &= 1, A(1) = 2, A(2) = 1, A(3) = 2, \\ S(0) &= 1, S(1) = 3, S(2) = 3, S(3) = 2. \end{aligned}$$

We have $A(X) = \{1,2\} \subset S(X) = \{1,2,3\}$. $C(A,S) = \{0,3\}$. It is easy to see that $A^2(0) = 2 \neq 0$, $A^2(3) = 1 \neq 3$ and A and S do not commute at their coincidence points.

The cases $x = y$ and $(x,y) \in \{(0,2),(0,3),(1,3)\}$ are clear.

1) For the case $(x,y) = (0,1)$ we get

$$\begin{aligned} d(A(0),A(1)) &= d(1,2) = 1 \\ &< N(0,1)M(0,1) = 3, \end{aligned}$$

where

$$\begin{aligned} N(0,1) &= \frac{\max\left\{ \begin{array}{c} d(S(0),S(1)), d(A(0),S(0)) + d(A(1),S(1)), \\ d(A(0),S(1)) + d(S(0),A(1)) \end{array} \right\}}{d(A(0),S(0)) + d(A(1),S(1)) + 1} \\ &= \frac{\max\{d(1,3), d(2,3), d(1,3) + d(1,2)\}}{d(2,3)+1} \\ &= \frac{3}{2}, \\ M(0,1) &= 2. \end{aligned}$$

2) For the case $(x, y) = (1, 2)$ we obtain

$$\begin{aligned} d(A(1), A(2)) &= d(2,1) = 1 \\ &< N(1,2)M(1,2) = \frac{3}{2}, \end{aligned}$$

where

$$\begin{aligned} N(1,2) &= \frac{\max\left\{ \begin{array}{c} d(S(1),S(2)), d(A(1),S(1)) + d(A(2),S(2)), \\ d(A(1),S(2)) + d(S(1),A(2)) \end{array} \right\}}{d(A(1),S(1)) + d(A(2),S(2)) + 1} \\ &= \frac{\max\{d(3,3), d(2,3) + d(1,3), d(2,3) + d(3,1)\}}{d(2,3) + d(1,3) + 1} \\ &= \frac{3}{4}, \\ M(0,1) &= 2. \end{aligned}$$

3) For the case $(x, y) = (2, 3)$ we find

$$\begin{aligned} d(A(2), A(3)) &= d(1,2) = 1 \\ &< N(2,3)M(2,3) = \frac{4}{3}, \end{aligned}$$

where

$$\begin{aligned} N(2,3) &= \frac{\max\left\{ \begin{array}{c} d(S(2),S(3)), d(A(2),S(2)) + d(A(3),S(3)), \\ d(A(2),S(3)) + d(S(2),A(3)) \end{array} \right\}}{d(A(2),S(2)) + d(A(3),S(3)) + 1} \\ &= \frac{\max\{d(3,2), d(1,3), d(1,2) + d(3,2)\}}{d(1,3) + 1} \\ &= \frac{2}{3}, \\ M(2,3) &= 2. \end{aligned}$$

Hence, all the hypotheses of Theorem 3 hold. Accordingly, A and S have two coincidence points 0 and 3. Moreover, $d(A(0), A(3)) = d(1,2) > \frac{1}{2}$.

Example 2. Let $X = \{0, 1, 2, 3\}$ be equipped with the usual metric. Define $A, S : X \to X$ by:

$$\begin{aligned} A(0) &= 1, A(1) = 2, A(2) = 1, A(3) = 0, \\ S(0) &= 1, S(1) = 2, S(2) = 3, S(3) = 0. \end{aligned}$$

We have $A(X) = \{0, 1, 2\} \subset S(X) = \{0, 1, 2, 3\}$. $C(A, S) = \{0, 1, 3\}$. It is obvious that $A^2(0) = 2 \neq 0$, $A^2(3) = 1 \neq 3$ and A and S commute at their coincidence points 0 and 3.

The cases $x=y$ and $(x,y)\in\{(0,1),(0,2),(0,3),(1,3)\}$ are plain.

1) For the case $(x,y)=(1,2)$ we get

$$\begin{aligned} d(A(1),A(2)) &= d(2,1)=1 \\ &< N(1,2)M(1,2)=\frac{4}{3}, \end{aligned}$$

where

$$\begin{aligned} N(1,2) &= \frac{\max\left\{\begin{array}{c} d(S(1),S(2)), d(A(1),S(1))+d(A(2),S(2)), \\ d(A(1),S(2))+d(S(1),A(2)) \end{array}\right\}}{d(A(1),S(1))+d(A(2),S(2))+1} \\ &= \frac{\max\{d(2,3),d(1,3),d(2,3)+d(2,1)\}}{d(1,3)+1} \\ &= \frac{2}{3}, \\ M(1,2) &= 2. \end{aligned}$$

2) For the case $(x,y)=(2,3)$ we find

$$\begin{aligned} d(A(2),A(3)) &= d(1,0)=1 \\ &< N(2,3)M(2,3)=4, \end{aligned}$$

where

$$\begin{aligned} N(2,3) &= \frac{\max\left\{\begin{array}{c} d(S(2),S(3)), d(A(2),S(2))+d(A(3),S(3)), \\ d(A(2),S(3))+d(S(2),A(3)) \end{array}\right\}}{d(A(2),S(2))+d(A(3),S(3))+1} \\ &= \frac{\max\{d(3,0),d(1,3),d(1,0)+d(3,0)\}}{d(1,3)+1} \\ &= \frac{4}{3}, \\ M(2,3) &= 3. \end{aligned}$$

Hence, all the conditions of Theorem 3 hold. Consequently, A and S have three coincidence points $0,1$ and 3. Furthermore,

$$\begin{aligned} d(A(0),A(1)) &= d(1,2)>\frac{1}{2}, d(A(0),A(3))=d(1,0)>\frac{1}{2}, \\ d(A(1),A(3)) &= d(2,0)>\frac{1}{2}. \end{aligned}$$

Example 3. Let $X=\{0,1,2,3\}$ be endowed with the usual metric. Define $A,S:X\to X$ by:

$$\begin{aligned} A(0) &= 0, A(1)=1, A(2)=1, A(3)=2, \\ S(0) &= 0, S(1)=1, S(2)=3, S(3)=2. \end{aligned}$$

We have $A(X) = \{0,1,2\} \subset S(X) = \{0,1,2,3\}$. $C(A,S) = \{0,1,3\}$. It is evident that $A^2(0) = A(0) = 0$, $A^2(1) = A(1) = 1$, $A^2(3) = A(2) = 1 \neq 3$, A and S commute at their coincidence points 0 and 1 and do not commute at their coincidence point 3.

The cases $x = y$ and $(x,y) \in \{(0,1), (0,3), (1,2), (1,3)\}$ are obvious.

1) For the case $(x,y) = (0,2)$ we get

$$\begin{aligned} d(A(0),A(2)) &= d(0,1) = 1 \\ &< N(0,2)M(0,2) = 4, \end{aligned}$$

where

$$\begin{aligned} N(0,2) &= \frac{\max\left\{ \begin{array}{c} d(S(0),S(2)), d(A(0),S(0)) + d(A(2),S(2)), \\ d(A(0),S(2)) + d(S(0),A(2)) \end{array} \right\}}{d(A(0),S(0)) + d(A(2),S(2)) + 1} \\ &= \frac{\max\{d(0,3), d(1,3), d(0,3) + d(0,1)\}}{d(1,3)+1} \\ &= \frac{4}{3}, \\ M(0,2) &= 3. \end{aligned}$$

2) For the case $(x,y) = (2,3)$ we obtain

$$\begin{aligned} d(A(2),A(3)) &= d(1,2) = 1 \\ &< N(2,3)M(2,3) = \frac{4}{3}, \end{aligned}$$

where

$$\begin{aligned} N(2,3) &= \frac{\max\left\{ \begin{array}{c} d(S(2),S(3)), d(A(2),S(2)) + d(A(3),S(3)), \\ d(A(2),S(3)) + d(S(2),A(3)) \end{array} \right\}}{d(A(2),S(2)) + d(A(3),S(3)) + 1} \\ &= \frac{\max\{d(3,2), d(1,3), d(1,2) + d(3,2)\}}{d(1,3)+1} \\ &= \frac{2}{3}, \\ M(0,1) &= 2. \end{aligned}$$

Hence, all the assumptions of Theorem 3 hold. Thus, A and S have two common points 0 and 1 and a coincidence point 3. Besides,

$$\begin{aligned} d(A(0),A(1)) &= d(0,1) > \frac{1}{2}, d(A(0),A(3)) = d(0,2) > \frac{1}{2}, \\ d(A(1),A(3)) &= d(1,2) > \frac{1}{2}. \end{aligned}$$

If $S = I_X$ in Theorem 3, where I_X is the identity mapping in X, we have the following corollary.

Corollary 1. *Let A be a mapping of a complete metric space (X,d) into itself satisfying*

$$d(Ax,Ay)\leq N(x,y)M(x,y)$$

for all $x,y\in X$, where

$$N(x,y)=\frac{\max\{d(x,y),d(x,Ax)+d(y,Ay),d(x,Ay)+d(y,Ax)\}}{d(x,Ax)+d(y,Ay)+1}$$

and

$$M(x,y)=\max\{d(x,y),d(x,Ax),d(y,Ay),\frac{d(x,Ay)+d(y,Ax)}{2}\}.$$

Then

i) A is a WPO.

ii) If z and w are distinct fixed points of A, therefore $d(z,w)\geq\frac{1}{2}$.

Example 4. Let $X=\{0,1,2,3\}$ be equipped with the usual metric. Define $A:X\to X$ by:

$$A(0)=0,A(1)=1,A(2)=1,A(3)=0.$$

The cases $x=y$ and $(x,y)\in\{(0,1),(0,3),(1,2)\}$ are clear.

1) For the case $(x,y)=(0,2)$ we get

$$\begin{aligned} d(A(0),A(2)) &= d(0,1)=1\\ &< N(0,2)M(0,2)=3, \end{aligned}$$

where

$$\begin{aligned} N(0,2) &= \frac{\max\left\{\begin{array}{c} d(0,2),d(0,A(0))+d(2,A(2)),\\ d(0,A(2))+d(2,A(0))\end{array}\right\}}{d(0,A(0))+d(2,A(2))+1}\\ &= \frac{\max\{d(0,2),d(2,1),d(0,1)+d(2,0)\}}{d(2,1)+1}\\ &= \frac{3}{2},\\ M(0,2) &= 2. \end{aligned}$$

2) For the case $(x,y)=(1,3)$ we obtain

$$\begin{aligned} d(A(1),A(3)) &= d(1,0)=1\\ &< N(1,3)M(1,3)=\frac{9}{4}, \end{aligned}$$

where

$$N(1,3) = \frac{\max\left\{\begin{array}{c} d(1,3),d(1,A(1))+d(3,A(3)),\\ d(1,A(3))+d(3,A(1))\end{array}\right\}}{d(1,A(1))+d(3,A(3))+1}$$

$$\begin{aligned}
&= \frac{\max\{d(1,3), d(3,0), d(1,0)+d(3,1)\}}{d(3,0)+1} \\
&= \frac{3}{4}, \\
M(1,3) &= 3.
\end{aligned}$$

3) For the case $(x,y)=(2,3)$ we find

$$\begin{aligned}
d(A(2),A(3)) &= d(1,0)=1 \\
&< N(2,3)M(2,3) = \frac{12}{5},
\end{aligned}$$

where

$$\begin{aligned}
N(2,3) &= \frac{\max\left\{\begin{array}{c} d(2,3), d(2,A(2))+d(3,A(3)), \\ d(2,A(3))+d(3,A(2)) \end{array}\right\}}{d(2,A(2))+d(3,A(3))+1} \\
&= \frac{\max\{d(2,3), d(2,1)+d(3,0), d(2,0)+d(3,1)\}}{d(2,1)+d(3,0)+1} \\
&= \frac{4}{5}, \\
M(2,3) &= 3.
\end{aligned}$$

Hence, A satisfies all the assumptions of Corollary 1 and A has two distinct fixed points 0 and 1. Besides, $d(0,1)=1>\frac{1}{2}$. Since $d(A(2),A(3))=1$ and

$$\frac{d(2,A(3))+d(3,A(2))}{d(2,A(2))+d(3,A(3))+1}d(2,3)=\frac{4}{5},$$

we get $1>\frac{4}{5}$. Therefore, Theorem 1 of [19] cannot be applicable.

Remark 2. It is worth mentioning that Corollary 1 cannot be applicable for the mappings A and S in Example 3, but our Theorem 3 is applicable because for the case $(x,y)=(2,3)$ we get

$$\begin{aligned}
d(A(2),A(3)) &= d(1,2)=1 \\
&> N(2,3)M(2,3) = \frac{2}{3},
\end{aligned}$$

where

$$\begin{aligned}
N(2,3) &= \frac{\max\left\{\begin{array}{c} d(2,3), d(2,A(2))+d(3,A(3)), \\ d(2,A(3))+d(3,A(2)) \end{array}\right\}}{d(2,A(2))+d(3,A(3))+1} \\
&= \frac{\max\{d(2,3), d(2,1)+d(3,2), d(2,2)+d(3,1)\}}{d(2,1)+d(3,2)+1}
\end{aligned}$$

$$\begin{aligned} &= \frac{2}{3}, \\ M(2,3) &= 1. \end{aligned}$$

$$\begin{aligned} d(S(2),S(3)) &= d(3,2) = 1 \\ &> N(2,3)M(2,3) = \frac{2}{3}, \end{aligned}$$

where

$$\begin{aligned} N(2,3) &= \frac{\max\left\{\begin{array}{c} d(2,3), d(2,S(2)) + d(3,S(3)), \\ d(2,S(3)) + d(3,S(2)) \end{array}\right\}}{d(2,S(2)) + d(3,S(3)) + 1} \\ &= \frac{\max\{d(2,3), d(2,3) + d(3,2), d(2,2) + d(3,3)\}}{d(2,3) + d(3,2) + 1} \\ &= \frac{2}{3}, \\ M(2,3) &= 1. \end{aligned}$$

Also, Theorem 1 of [19] cannot be applicable for the mappings A and S. This shows that our Theorem 3 is a genuine generalization of Corollary 1 and Theorem 1 of [19].

3 Application in dynamic programming

Let X and Y be Banach spaces, $S \subset X$ be the state space, $D \subset Y$ be the decision space and I_X be the identity mapping on X. $B(S)$ denotes the set of all bounded real-valued functions on S and

$$d(f,g) = \sup_{x \in S} |f(x) - g(x)|.$$

It is clear that $(B(S), d)$ is a complete metric space.

As proposed in Bellman and Lee [5], the basic form of the functional equation in dynamic programming is

$$f(x) = opt_{y \in D} H(x, y, f(T(x,y))),\ x \in S,$$

where x and y denote the state and decision vectors, respectively. T denotes the transformation of the process, $f(x)$ denotes the optimal return function with the initial state x and *opt* represents sup or inf.

Many authors proved the existence and the uniqueness of solutions or common solutions for several classes of functional equations or systems of functional equations arising in dynamic programming by employing various fixed and common fixed point theorems, see Bhakta and Mitra [4], Kalinde et al.[15], Li et al.[21], Liu [22], Liu et al.[23–26] and Pathak et al.[29].

In this section, applying Theorem 3, we establish the existence of common solutions of the following system of two functional equations arising in dynamic programming.

$$f_i(x) = opt_{y\in D}\{u(x,y) + H_i(x,y,f_i(T(x,y)))\},\ x\in S,\ i=1,2, \tag{13}$$

where $u : S\times D \to S$, $T : S\times D \to S$ and $H_i : S\times D\times\mathbb{R}\to\mathbb{R}$, $i=1,2$.

Theorem 4. *Suppose that the following conditions are verified*
(c_1) *u and H_i are bounded for $i=1,2$,*
(c_2) *For all $(x,y)\in S\times D$, $g,h\in B(S)$ and $t\in S$*

$$|H_1(x,y,g(t)) - H_1(x,y,h(t))| \le N(g(t),h(t))M(g(t),h(t)), \tag{14}$$

where

$$N(g(t),h(t)) = \frac{\max\left\{\begin{array}{c}|A_2g(t)-A_2h(t)|, |A_2g(t)-A_1g(t)| + |A_2h(t)-A_1h(t)|,\\ |A_2g(t)-A_1h(t)| + |A_2h(t)-A_1g(t)|\end{array}\right\}}{d(A_2g,A_1g)+d(A_2h,A_1h)+1},$$

$$M(g(t),h(t)) = \max\left\{\begin{array}{c}|A_2g(t)-A_2h(t)|, |A_2g(t)-A_1g(t)|, |A_2h(t)-A_1h(t)|,\\ \dfrac{|A_2g(t)-A_1h(t)| + |A_2h(t)-A_1g(t)|}{2}\end{array}\right\}$$

and

$$A_ig_i(x) = opt_{y\in D}\{u(x,y) + H_i(x,y,g_i(T(x,y)))\}, x\in S, i=1,2.$$

(c_3) : $A_1(B(S))\subset A_2(B(S))$,
(c_4) : $A_1u = A_1^2u$ *for some $u\in C(A_1,A_2)$ and $A_1A_2u = A_2A_1u$, $u\in B(S)$.*

Then, the system of functional equations (13) possesses at least one common solution in $B(S)$. In addition, if z and w are two distinct solutions of (13) therefore $d(z,w)\ge \dfrac{1}{2}$.

Proof. It follows from (c_1) and (c_2) that A_1 and A_2 are self-mappings in $B(S)$. Assume that $opt_{y\in D} = \sup_{y\in D}$. For each $g,h\in B(S)$, $x\in S$ and $\epsilon>0$, there exist $y,z\in D$ such that

$$A_1g(x) < u(x,y) + H_1(x,y,g(T(x,y))) + \epsilon, \tag{15}$$

$$A_1h(x) < u(x,z) + H_1(x,z,h(T(x,z))) + \epsilon. \tag{16}$$

It is easy to see that

$$A_1g(x) \ge u(x,z) + H_1(x,z,g(T(x,z))), \tag{17}$$

$$A_1h(x) \ge u(x,y) + H_1(x,y,h(T(x,y))). \tag{18}$$

By virtue of (15) and (18), we infer that

$$A_1g(x) - A_1h(x) < H_1(x,y,g(T(x,y))) - H_1(x,y,h(T(x,y))) + \epsilon \tag{19}$$

$$\begin{aligned} &\leq |H_1(x,y,g(T(x,y))) - H_1(x,y,h(T(x,y)))| + \epsilon \\ &\leq N(g(t),h(t))M(g(t),h(t)) + \epsilon. \end{aligned}$$

From (16) and (17) we conclude that

$$\begin{aligned} A_1g(x) - A_1h(x) &> H_1(x,z,g(T(x,z))) - H_1(x,z,h(T(x,z))) - \epsilon \\ &\geq -|H_1(x,z,g(T(x,z))) - H_1(x,z,h(T(x,z)))| - \epsilon \\ &\geq -N(g(t),h(t))M(g(t),h(t)) - \epsilon. \end{aligned} \tag{20}$$

It follows from (19) and (20) that

$$|A_1g(x) - A_1h(x)| \leq N(g(t),h(t))M(g(t),h(t)) + \epsilon.$$

Using (14) and the above inequality we obtain

$$|A_1g(x) - A_2h(x)| \leq N(g,h)M(g,h) + \epsilon.$$

Hence

$$d(A_1g, A_1h) \leq N(g,h)M(g,h) + \epsilon. \tag{21}$$

Similarly, the inequality (21) also holds for $opt_{y\in D} = \inf_{y\in D}$. Letting $\epsilon \to 0$ in (21) we deduce that

$$d(A_1g, A_1h) \leq N(g,h)M(g,h).$$

Due to Theorem 3, A_1 and A_2 have at least one common fixed point $z \in B(S)$, i.e., z is a common solution of the system of functional equations (13). In addition, if z and w are two distinct solutions of (13), therefore $d(z,w) \geq \dfrac{1}{2}$. □

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