BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(93), 2020, Pages 88–101
ISSN 1024–7696

Asymmetric Separation of Convex Sets

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Abstract. Based on various types of asymmetric hyperplane separation of a given pair of convex sets K_1 and K_2 in the n-dimensional Euclidean space, we derive a uniform description of existing types of separation. Our argument uses properties of the polar cone $(K_1 - K_2)^{\circ}$. Also, we consider asymmetric separation of convex cones with a common apex.

Mathematics subject classification: 52A20.
Keywords and phrases: Separation, hyperplane, convex, cone.

1 Introduction

Support and separation properties of convex sets are among the most important topics in convexity. Studied by Minkowski [7, 8] on the turn of 20th century, they became useful tools in many mathematical disciplines, especially in convex geometry, linear analysis, convex analysis, and optimization.

We recall that nonempty convex sets K_1 and K_2 in the n-dimensional Euclidean space $\mathbb{R}^n$ are *separated* by a hyperplane $H \subset \mathbb{R}^n$ provided K_1 and K_2 lie in the opposite closed halfspaces determined by H. The concept of separation was gradually refined in the literature, and various types of separation of convex sets are known nowadays. The existing classification and the respective terminology in this regard is mainly due to Klee [5] and Rockafellar [9]. For instance, K_1 and K_2 are called *properly* separated provided $K_1 \cup K_2 \not\subset H$, and they are called *strongly* separated if suitable open ρ-neighborhoods $U_\rho(K_1)$ and $U_\rho(K_2)$ of these sets are separated by H. These two types of separation are the most popular in the literature due to the existence of simple criteria (see [9], § 11), and many other types of separation are often viewed as their derivatives.

In this paper, we deal with an alternative approach to the classification of separating hyperplanes. Namely, we start with various types of asymmetric hyperplane separation of convex sets, and then derive from them existing types of separation. Also, unlike many existing results, which provide conditions for the existence of at least one separating hyperplane, we tend to describe all such hyperplanes.

2 Preliminaries

This section contains necessary definitions, notation, and results on convex sets in $\mathbb{R}^n$ (see, e. g., [9] and [11] for details). The elements of $\mathbb{R}^n$ are called vectors, or points. We denote by $[u,v]$ and (u,v) the closed and open segments with endpoints $u,v \in \mathbb{R}^n$. Also, $u \cdot v$ will mean the dot product of u and v. The zero vector of $\mathbb{R}^n$ is denoted o. A set $L \subset \mathbb{R}^n$ is called an r-dimensional plane if it is translate of a suitable r-dimensional subspace S of $\mathbb{R}^n$: $L = c + S$, where $c \in \mathbb{R}^n$.

In what follows, K stands for a nonempty convex set in $\mathbb{R}^n$. The open ρ-neighborhood of K, denoted $U_\rho(K)$, is the union of all open balls $U_\rho(x)$ of radius $\rho > 0$ centered at $x \in K$. Convex sets K_1 and K_2 are called strongly disjoint provided $U_\rho(K_1) \cap U_\rho(K_2) = \varnothing$ for a suitable $\rho > 0$; the latter occurs if and only if the *inf*-distance $\delta(K_1, K_2)$, defined by

$$\delta(K_1, K_2) = \inf\{\|x_1 - x_2\| : x_1 \in K_1, x_2 \in K_2\},$$

is positive. The notations $\operatorname{cl} K$, $\operatorname{int} K$, $\operatorname{rint} K$, $\operatorname{rbd} K$, and $K^\perp$ stand, respectively, for the closure, interior, relative interior, relative boundary, and the orthogonal complement of K. The linear span of K, denoted $\operatorname{span} K$, is the smallest subspace containing K, affine span of K, denoted $\operatorname{aff} K$, is the intersection of all planes containing K, and $\dim K$ is defined as the dimension of $\operatorname{aff} K$. Also, the direction space and the orthospace of K are defined by $\operatorname{dir} K = \operatorname{span}(K - K)$ and $\operatorname{ort} K = (\operatorname{dir} K)^\perp$, respectively.

A hyperplane in $\mathbb{R}^n$ is a plane which can be described as

$$H = \{x \in \mathbb{R}^n : x \cdot e = \gamma\}, \quad e \neq o, \quad \gamma \in \mathbb{R}. \tag{1}$$

Consequently, a hyperplane of the form (1) is a translate of the hypersubspace

$$S = \{x \in \mathbb{R}^n : x \cdot e = 0\}, \quad e \neq o. \tag{2}$$

Every hyperplane of the form (1) determines a pair of opposite closed halfspaces

$$V_1 = \{x \in \mathbb{R}^n : x \cdot e \le \gamma\} \;\text{ and }\; V_2 = \{x \in \mathbb{R}^n : x \cdot e \ge \gamma\} \tag{3}$$

and a pair of opposite open halfspaces

$$W_1 = \{x \in \mathbb{R}^n : x \cdot e < \gamma\} \;\text{ and }\; W_2 = \{x \in \mathbb{R}^n : x \cdot e > \gamma\}. \tag{4}$$

The (negative) polar cone of a convex set $K \subset \mathbb{R}^n$ is the set

$$K^\circ = \{e \in \mathbb{R}^n : x \cdot e \le 0 \;\text{ for all }\; x \in K\}.$$

The recession cone of K is defined by

$$\operatorname{rec} K = \{e \in \mathbb{R}^n : x + \lambda e \in K \text{ whenever } x \in K \text{ and } \lambda \ge 0\},$$

and the lineality space of K is the subspace given by $\operatorname{lin} K = \operatorname{rec} K \cap (-\operatorname{rec} K)$.

3 Asymmetric Separation

Definition 1. Let convex sets K_1 and K_2 in $\mathbb{R}^n$ be separated by a hyperplane $H \subset \mathbb{R}^n$. We will say that

1) H *nontrivially* separates K_1 *from* K_2 if $K_1 \not\subset H$,

2) H *strictly* separates K_1 *from* K_2 if $K_1 \cap H = \varnothing$,

3) H *strongly* separates K_1 *from* K_2 if there is an open ρ-neighborhood $U_\rho(K_1)$ of K_1 satisfying the condition $U_\rho(K_1) \cap H = \varnothing$.

Remark 1. The term *nontrivially separates from* is, probably, new. The expression *strictly separates from* is equivalent to that *openly separates from* used by Klee [4]; our choice here is due to the direct relation with the well established term *strict separation* (see Definition 2 below). Finally, the expression *strongly separates from* is introduced by Klee [4].

The following obvious lemma reformulates Definition 1 in analytic terms.

Lemma 1. *For convex sets K_1 and K_2 in $\mathbb{R}^n$ and a hyperplane H of the form* (1)*, the assertions below hold.*

1) *H separates K_1 and K_2 if and only if e and γ can be chosen such that*

$$\sup\{x_1 \cdot e : x_1 \in K_1\} \le \gamma \le \inf\{x_2 \cdot e : x_2 \in K_2\}. \tag{5}$$

2) *H nontrivially separates K_1 from K_2 if and only if e and γ can be chosen to satisfy either of the conditions* (6) *and* (7) *below:*

$$\sup\{x_1 \cdot e : x_1 \in K_1\} < \gamma \le \inf\{x_2 \cdot e : x_2 \in K_2\}, \tag{6}$$

$$\begin{aligned} &\inf\{x_1 \cdot e : x_1 \in K_1\} < \sup\{x_1 \cdot e : x_1 \in K_1\} \\ &\quad = \gamma = \inf\{x_2 \cdot e : x_2 \in K_2\}. \end{aligned} \tag{7}$$

3) *H strictly separates K_1 from K_2 if and only if e and γ can be chosen such that*

$$x_1 \cdot e < \gamma \le \inf\{x_2 \cdot e : x_2 \in K_2\} \quad \forall x_1 \in K_1. \tag{8}$$

4) *H strongly separates K_1 from K_2 if and only if e and γ can be chosen to satisfy the inequalities* (6)*.* □

The next proposition describes known results on hyperplane separation related to Definition 1. Clearly, these results provide the existence of at least one such hyperplane; they do not describe all possible separating hyperplanes of a given type.

Proposition 1. Given convex sets K_1 and K_2 in $\mathbb{R}^n$, the assertions below hold.

1. If K_2 is a polyhedron, then there is a hyperplane nontrivially separating K_1 from K_2 if and only if $\operatorname{rint} K_1 \cap K_2 = \varnothing$ (Rockafellar [9], Theorem 20.2).

2. If K_1 and K_2 are disjoint and K_1 contains no halfline in its boundary, then K_1 is strictly separated from K_2 by a hyperplane (Klee [3]).

3. Let K_1 and K_2 be disjoint and closed, at least one of them being compact. Let $z_1 \in K_1$ and $z_2 \in K_2$ be points for which $\delta(K_1, K_2) = \|z_1 - z_2\|$. Then any hyperplane perpendicular to the closed segment $[z_1, z_2]$ and passing through any point of the semi-open segment $(z_1, z_2]$ strongly separates K_1 from K_2 (Minkowski [8], p. 141, for the case when both K_1 and K_2 are compact). $\square$

We observe that the first assertion in Proposition 1 cannot be extended to the case of arbitrary convex sets. Namely, the following example shows that the condition $\operatorname{rint} K_1 \cap \operatorname{cl} K_2 = \varnothing$ is not sufficient for nontrivial separation of K_1 from K_2.

Example 1. Let K_1 and K_2 be planar circular disks in $\mathbb{R}^3$, given by

$$K_1 = \{(x, y, 0) : x^2 + (y-1)^2 \le 1\},$$
$$K_2 = \{(0, y, z) : y^2 + (z-1)^2 \le 1\}.$$

Both K_1 and K_2 are closed convex sets and $\operatorname{rint} K_1 \cap K_2 = \varnothing$. It is easy to see that the coordinate xy-plane (which contains K_1) is the only plane separating K_1 and K_2. Hence K_1 is not nontrivially separated from K_2.

The next two corollaries, which immediately follow from [12], describe all hyperplanes separating a pair of convex bodies or strongly separating a given convex set from another one.

Corollary 1. *For convex sets K_1 and K_2 in $\mathbb{R}^n$, the assertions below hold.*

1) *There is a hyperplane separating K_1 and K_2 if and only if any of the following two conditions is satisfied:*

   (*a*) $o \notin \operatorname{int}(K_1 - K_2)$,
   (*b*) $(K_1 - K_2)^\circ \ne \{o\}$.

2) *There is a translate of a hypersubspace* (2) *separating K_1 and K_2 if and only if one of the vectors e and $-e$ belongs to $(K_1 - K_2)^\circ \setminus \{o\}$.*

3) *For any vector $e \in (K_1 - K_2)^\circ \setminus \{o\}$, one has*

$$\sup\{x_1 \cdot e : x_1 \in K_1\} \le \inf\{x_2 \cdot e : x_2 \in K_2\}. \tag{9}$$

Consequently, any scalar γ satisfying the inequalities (5) *can be used in the description* (1) *of a hyperplane which separates K_1 and K_2.* $\square$

Corollary 2. *For convex sets K_1 and K_2 in $\mathbb{R}^n$, the assertions below hold.*

1) *There is a hyperplane strongly separating K_1 from K_2 if and only if $o \notin \operatorname{cl}(K_1 - K_2)$.*

2) *If* $o \notin \mathrm{cl}\,(K_1 - K_2)$ *and* e *is a vector in* $\mathbb{R}^n$ *such that one of the vectors* e *and* $-e$ *belongs to* $\mathrm{rint}\,(K_1 - K_2)^\circ$*, then a suitable translate of the hypersubspace* (2) *strongly separates* K_1 *from* K_2*.*

3) *If there is a translate of a hypersubspace* (2) *strongly separating* K_1 *from* K_2*, then one of the vectors* e *and* $-e$ *belongs to* $(K_1 - K_2)^\circ \setminus \mathrm{lin}\,(K_1 - K_2)^\circ$*. If, additionally, both* K_1 *and* K_2 *are bounded sets, then one of these vectors belongs to* $\mathrm{rint}\,(K_1 - K_2)^\circ$*.*

4) *If* $o \notin \mathrm{cl}\,(K_1 - K_2)$*, then for any vector* $e \in \mathrm{rint}\,(K_1 - K_2)^\circ$*, the inequality*

$$\sup\{x_1 \cdot e : x_1 \in K_1\} < \inf\{x_2 \cdot e : x_2 \in K_2\} \tag{10}$$

holds. Consequently, any scalar γ *satisfying the inequalities* (6) *can be used in the description* (1) *of a hyperplane which strongly separates* K_1 *from* K_2*.* □

The next theorem describes hyperplanes nontrivially separating a given convex set from another one.

Theorem 1. *For convex sets* K_1 *and* K_2 *in* $\mathbb{R}^n$*, the assertions below hold.*

1) *There is a hyperplane nontrivially separating* K_1 *from* K_2 *if and only if any of the following two conditions is satisfied:*

   (*a*) $o \notin \mathrm{cl}\,(K_1 - K_2)$,

   (*b*) $o \in \mathrm{cl}\,(K_1 - K_2)$ *and* $(K_1 - K_2)^\circ \setminus \mathrm{ort}\,K_1 \neq \varnothing$.

2) *There is a translate of a hypersubspace* (2) *nontrivially separating* K_1 *from* K_2 *if and only if any of the following two conditions is satisfied:*

   (*c*) $o \notin \mathrm{cl}\,(K_1 - K_2)$ *and one of the vectors* e *and* $-e$ *belongs to the set*

$$\mathrm{rint}\,(K_1 - K_2)^\circ \cup (\mathrm{rbd}\,(K_1 - K_2)^\circ \setminus \mathrm{ort}\,K_1), \tag{11}$$

   (*d*) $o \in \mathrm{cl}\,(K_1 - K_2)$ *and one of the vectors* e *and* $-e$ *belongs to the set*

$$(K_1 - K_2)^\circ \setminus \mathrm{ort}\,K_1.$$

3) *If* $o \notin \mathrm{cl}\,(K_1 - K_2)$ *and* e *belongs to the set* (11)*, then one of the relations* (10) *and*

$$\inf\{x_1 \cdot e : x_1 \in K_1\} < \sup\{x_1 \cdot e : x_1 \in K_1\} = \inf\{x_2 \cdot e : x_2 \in K_2\} \tag{12}$$

holds and any scalar γ *satisfying the respective conditions* (6) *and* (7) *can be used in the description* (1) *of a hyperplane which nontrivially separates* K_1 *from* K_2*.*

4) *If* $o \in \mathrm{cl}\,(K_1 - K_2)$ *and* $e \in (K_1 - K_2)^\circ \setminus \mathrm{ort}\,K_1$*, then the conditions* (12) *hold and any scalar* γ *satisfying* (7) *can be used in the description* (1) *of a hyperplane which nontrivially separates* K_1 *from* K_2*.*

Proof. 1) Let a hyperplane $H \subset \mathbb{R}^n$ nontrivially separate K_1 from K_2. We will assume that H is described by (1) and that K_1 and K_2 are contained, respectively, in the closed halfspaces V_1 and V_2 given by (3). By Lemma 1, one of the relations (6) and (7) holds. The obvious equality

$$\sup\{x\cdot e : x \in K_1 - K_2\} = \sup\{x_1\cdot e : x_1 \in K_1\} - \inf\{x_2\cdot e : x_2 \in K_2\}, \qquad (13)$$

combined with (6) and (7), gives

$$\sup\{x\cdot e : x \in K_1 - K_2\} \le 0. \qquad (14)$$

Hence the set $K_1 - K_2$ is contained in the homogeneous closed halfspace

$$V = \{x \in \mathbb{R}^n : x\cdot e \le 0\}.$$

Consequently, $e \in V^\circ \subset (K_1 - K_2)^\circ$.

Suppose that $o \in \operatorname{cl}(K_1 - K_2)$. Then (14) implies

$$\sup\{x\cdot e : x \in K_1 - K_2\} = o\cdot e = 0,$$

and (13) gives

$$\sup\{x_1\cdot e : x_1 \in K_1\} = \inf\{x_2\cdot e : x_2 \in K_2\}.$$

The latter equality shows that γ should satisfy the conditions (7).

Under the assumption $o \in \operatorname{cl}(K_1 - K_2)$, suppose that $e \in \operatorname{ort} K_1$. Then

$$\operatorname{dir} K_1 = (\operatorname{ort} K_1)^\perp \subset \{e\}^\perp := S = \{x \in \mathbb{R}^n : x\cdot e = 0\}.$$

As a translate of $\operatorname{dir} K_1$, the plane $\operatorname{aff} K_1$ is expressible in the form $\operatorname{aff} K_1 = z + \operatorname{dir} K_1$ for a suitable vector $z \in \mathbb{R}^n$. Consequently,

$$\begin{aligned} K_1 \subset \operatorname{aff} K_1 = z + \operatorname{dir} K_1 \subset z + S = z + \{x \in \mathbb{R}^n : x\cdot e = 0\} \\ = \{x \in \mathbb{R}^n : x\cdot e = \mu\}, \text{ where } \mu = z\cdot e. \end{aligned} \qquad (15)$$

Comparing (1) and (15), we conclude that $\gamma = \mu$ and $H = z + S$. Hence $K_1 \subset H$, contrary to the hypothesis that H nontrivially separates K_1 from K_2. Thus $e \notin \operatorname{ort} K_1$. The latter exclusion shows that $(K_1 - K_2)^\circ \setminus \operatorname{ort} K_1 \ne \varnothing$.

Conversely, assume that any of the conditions (a) and (b) is satisfied. If (a) is satisfied, then, by Corollary 2, there is a hyperplane $H \subset \mathbb{R}^n$ strongly separating K_1 from K_2, and thus nontrivially separating K_1 from K_2. Suppose now that the condition (b) is satisfied. Choose a vector $e \in (K_1 - K_2)^\circ \setminus \operatorname{ort} K_1$. Repeating the above argument in the converse order, we conclude that the inclusion $e \in (K_1 - K_2)^\circ$ implies the existence of a hyperplane H of the form (1) separating K_1 and K_2, while the exclusion $e \notin \operatorname{ort} K_1$ guarantees that H does not contain K_1. Summing up, K_1 is nontrivially separated from K_2.

2) Let a translate of a hypersubspace (2) nontrivially separate K_1 from K_2. By Corollary 1, one of the vectors e and $-e$, say e, belongs to $(K_1 - K_2)^\circ$. If

$o \in \mathrm{cl}\,(K_1 - K_2)$, then, repeating the argument from part 1) above, we obtain the inclusion $e \in (K_1 - K_2)^\circ \setminus \mathrm{ort}\, K_1$. Suppose that $o \notin \mathrm{cl}\,(K_1 - K_2)$. If $e \in \mathrm{rint}\,(K_1 - K_2)^\circ$, then, by Corollary 2, the inequality (10) holds, and for any scalar γ satisfying the condition (6), the hyperplane (1) strongly separates K_1 from K_2. Finally, if $e \notin \mathrm{rint}\,(K_1 - K_2)^\circ$, then $e \in \mathrm{rbd}\,(K_1 - K_2)^\circ$, and, as above, e should not be in $\mathrm{ort}\, K_1$. Summing up, $e \in \mathrm{rbd}\,(K_1 - K_2)^\circ \setminus \mathrm{ort}\, K_1$.

Conversely, repeating the above argument in the converse order, we obtain that each of the conditions (c) and (d) implies the existence of a translate of a hypersubspace (2) nontrivially separating K_1 from K_2.

Assertions 3) and 4) follow from Lemma 1 and the above parts 1) and 2). □

Problem 1. Describe, in the spirit of Corollaries 1 and 2 and Theorem 1, all hyperplanes which *strictly separate* a given convex set from another one.

4 Weak Asymmetric Separation

In this section, we consider *weak types* of asymmetric separation of convex sets. Namely, we discuss the conditions under which (at least) one of the convex sets K_1 and K_2 in $\mathbb{R}^n$ is separated from the other.

Remark 2. The following terminology on weak types of asymmetric separation of convex sets K_1 and K_2 by a hyperplane $H \subset \mathbb{R}^n$ is known in the literature:

1. H *properly separates* K_1 and K_2 if one of the sets is nontrivially separated by H from the other (Rockafellar [9, p. 95]),

2. H *nicely separates* K_1 and K_2 if one of the sets is strictly separated by H from the other (Klee [5]).

The corollary below immediately follows from [12].

Corollary 3. *For convex sets K_1 and K_2 in $\mathbb{R}^n$, the assertions below hold.*

1) *There is a hyperplane nontrivially separating one of the sets K_1 and K_2 from the other if and only if any of the following three conditions is satisfied:*

   (a) $\mathrm{rint}\, K_1 \cap \mathrm{rint}\, K_2 = \varnothing$ (see [9, Theorem 11.3]).

   (b) $o \notin \mathrm{rint}\,(K_1 - K_2)$,

   (c) $(K_1 - K_2)^\circ$ *is not a subspace,*

   (d) $(K_1 - K_2)^\circ \setminus \mathrm{lin}\,(K_1 - K_2)^\circ \neq \varnothing$.

2) *There is a translate of a hypersubspace* (2) *nontrivially separating one of the sets K_1 and K_2 from the other if and only if one of the vectors e and $-e$ belongs to $(K_1 - K_2)^\circ \setminus \mathrm{lin}\,(K_1 - K_2)^\circ$.*

3) *For any vector* $e \in (K_1 - K_2)^\circ \setminus \operatorname{lin}(K_1 - K_2)^\circ$, *both inequalities* (9) *and*

$$\inf\{x_1 \cdot e : x_1 \in K_1\} < \sup\{x_1 \cdot e : x_1 \in K_1\}$$

hold. Consequently, any scalar γ *satisfying the inequalities* (5) *can be used in the description* (1) *of a hyperplane which nontrivially separates one of the sets* K_1 *and* K_2 *from the other.*

For the case of strict separation, we consider, following Brøndsted [2], the "polarity" operation K^{Δ} on a convex set $K \subset \mathbb{R}^n$ defined by

$$K^{\Delta} = \{e \in \mathbb{R}^n : x \cdot e < 0 \ \ \forall\, x \in K \setminus \{o\}\}.$$

We observe that, generally, $K^{\Delta} \neq \operatorname{rint} K^\circ$.

Theorem 2. *For disjoint convex sets* K_1 *and* K_2 *in* $\mathbb{R}^n$, *the assertions below hold.*

1) *There is a hyperplane strictly separating one of the sets* K_1 *and* K_2 *from the other if and only if* $(K_1 - K_2)^{\Delta} \not\subset \{o\}$.

2) *There is a translate of a hypersubspace* (2) *strictly separating one of the sets* K_1 *and* K_2 *from the other if and only if one of the vectors* e *and* $-e$ *belongs to* $(K_1 - K_2)^{\Delta} \setminus \{o\}$.

3) *For any vector* $e \in (K_1 - K_2)^{\Delta} \setminus \{o\}$, *one of the conditions below is satisfied:*

$$x_1 \cdot e < \inf\{x_2 \cdot e : x_2 \in K_2\} \quad \forall\, x_1 \in K_1, \tag{16}$$

$$\sup\{x_1 \cdot e : x_1 \in K_1\} < x_2 \cdot e \quad \forall\, x_2 \in K_2. \tag{17}$$

Consequently, any scalar γ *satisfying the respective conditions*

$$x_1 \cdot e < \gamma \le \inf\{x_2 \cdot e : x_2 \in K_2\} \quad \forall\, x_1 \in K_1,$$
$$\sup\{x_1 \cdot e : x_1 \in K_1\} \le \gamma < x_2 \cdot e \quad \forall\, x_2 \in K_2$$

can be used in the description (1) *of a hyperplane strictly separating one of the sets* K_1 *and* K_2 *from the other.*

Proof. 1) Let a hyperplane $H \subset \mathbb{R}^n$ strictly separate one of the sets K_1 and K_2 from the other. We will assume that H is described by (1) and that K_1 and K_2 are contained, respectively, in the complementary halfspaces

$$W_1 = \{x \in \mathbb{R}^n : x \cdot e < \gamma\} \quad \text{and} \quad V_2 = \{x \in \mathbb{R}^n : x \cdot e \ge \gamma\}.$$

(The case when K_1 is contained is the closed halfspace V_1 and K_2 is in the complementary open halfspace W_2 is similar.) For any points $x_1 \in K_1$ and $x_2 \in K_2$, one has

$$(x_1 - x_2) \cdot e = x_1 \cdot e - x_2 \cdot e < \gamma - \gamma = 0.$$

Consequently, $K_1 - K_2 = (K_1 - K_2) \setminus \{o\}$ is contained in the homogeneous open halfspace

$$W = \{x \in \mathbb{R}^n : x \cdot e < 0\}, \tag{18}$$

which gives the inclusion $e \in (K_1 - K_2)^{\Delta}$. Hence $(K_1 - K_2)^{\Delta} \not\subset \{o\}$.

Conversely, assume that $(K_1 - K_2)^{\Delta} \not\subset \{o\}$ and choose a vector $e \in (K_1 - K_2)^{\Delta} \setminus \{o\}$. Then $K_1 - K_2$ is contained in the open halfspace (18). So, $(x_1 - x_2) \cdot e < 0$ whenever $x_1 \in K_1$ and $x_2 \in K_2$. Equivalently, $x_1 \cdot e < x_2 \cdot e$ for all $x_1 \in K_1$ and $x_2 \in K_2$. Let

$$\gamma_1 = \sup \{x_1 \cdot e : x_1 = K_1\} \quad \text{and} \quad \gamma_2 = \inf \{x_2 \cdot e : x_2 = K_2\}.$$

Then $\gamma_1 \le \gamma_2$ due to the inclusion $K_1 - K_2 \subset W$ and the inequality

$$\begin{aligned} \gamma_1 - \gamma_2 &= \sup \{x_1 \cdot e : x_1 \in K_1\} - \inf \{x_2 \cdot e : x_2 \in K_2\} \\ &= \sup \{x \cdot e : x \in K_1 - K_2\} \le 0. \end{aligned}$$

If $\gamma_1 < \gamma_2$, then, by Corollary 2, the hyperplane $H' = \{x \in \mathbb{R}^n : x \cdot e = \gamma'\}$ strongly separates K_1 from K_2 for any choice of $\gamma' \in (\gamma_1, \gamma_2]$. Suppose that $\gamma_1 = \gamma_2$ and put $\gamma' = \gamma_1 = \gamma_2$. If there is a point $x_2 \in K_2$ such that $x_2 \cdot e = \gamma'$, then $x_1 \cdot e < \gamma'$ for all $x_1 \in K_1$, implying that H' strictly separates K_1 from K_2. Similarly, if there is a point $x_1 \in K_1$ such that $x_1 \cdot e = \gamma'$, then $\gamma' < x_2 \cdot e$ for all $x_2 \in K_2$, implying that H' strictly separates K_2 from K_1.

2) Let a translate, say H, of a hypersubspace (2) strictly separate one of the sets K_1 and K_2 from the other. Then H is described by (1). By the argument of part 1), one of the vectors e and $-e$ should belong to $(K_1 - K_2)^{\Delta} \setminus \{o\}$. In a similar way, the converse assertion holds.

Assertion 3) follows from Lemma 1 and the above parts 1) and 2). ☐

Remark 3. A description of hyperplanes strongly separating one of the convex sets K_1 and K_2 from the other repeats Corollary 2 with one variation: in part 4), the scalar γ should be chosen to satisfy (6) or the symmetric conditions

$$\sup \{x_1 \cdot e : x_1 \in K_1\} \le \gamma < \inf \{x_2 \cdot e : x_2 \in K_2\}.$$

5 Symmetric Separation

In this section, we consider *symmetric separation* of convex sets. Namely, we describe the conditions under which each of the convex sets K_1 and K_2 in $\mathbb{R}^n$ is separated from the other.

Definition 2. Let convex sets K_1 and K_2 in $\mathbb{R}^n$ be separated by a hyperplane $H \subset \mathbb{R}^n$. We will say that

1) H *nontrivially* separates K_1 *and* K_2 if $K_1 \not\subset H$ and $K_2 \not\subset H$,

2) H *strictly* separates K_1 *and* K_2 if $K_1 \cap H = K_2 \cap H = \varnothing$,

3) H *strongly* separates K_1 *and* K_2 if there is a scalar $\rho > 0$ such that

$$U_\rho(K_1) \cap H = U_\rho(K_2) \cap H = \varnothing.$$

Remark 4. Nontrivial separation is called *real* separation by Bair and Jongmans [1] and *definite* separation in [11, Definition 10.1]. The terms *strict* and *strong* separation are used in the survey of Klee [5] and in numerous publications afterwards.

The next proposition, proved in [11, Theorem 10.6], relates Definitions 1 and 2.

Proposition 2. Let K_1 and K_2 be convex sets and H_1 and H_2 be hyperplanes in $\mathbb{R}^n$ such that H_i separates (nontrivially, strictly, or strongly) K_i from K_{3-i}, $i = 1, 2$. Then there is a hyperplane containing the set $H_1 \cap H_2$ and separating (nontrivially, strictly, or strongly) K_1 and K_2. □

The corollary below immediately follows from Theorem 1 and Proposition 2.

Corollary 4. *For convex sets K_1 and K_2 in $\mathbb{R}^n$, the assertions below hold.*

1) *There is a hyperplane nontrivially separating K_1 and K_2 if and only if one of the following two conditions is satisfied:*

   (*a*) $o \notin \mathrm{cl}\,(K_1 - K_2)$,

   (*b*) $o \in \mathrm{cl}\,(K_1 - K_2)$ *and* $(K_1 - K_2)^\circ \setminus (\mathrm{ort}\, K_1 \cup \mathrm{ort}\, K_2) \neq \varnothing$.

2) *There is a translate of a hypersubspace* (2) *nontrivially separating K_1 and K_2 if and only if one of the following conditions is satisfied:*

   (*c*) $o \notin \mathrm{cl}\,(K_1 - K_2)$ *and one of the vectors e and $-e$ belongs to the set*

$$\mathrm{rint}\,(K_1 - K_2)^\circ \cup (\mathrm{rbd}\,(K_1 - K_2)^\circ \setminus (\mathrm{ort}\, K_1 \cup \mathrm{ort}\, K_2)), \tag{19}$$

   (*d*) $o \in \mathrm{cl}\,(K_1 - K_2)$ *and one of the vectors e and $-e$ belongs to the set*

$$(K_1 - K_2)^\circ \setminus (\mathrm{ort}\, K_1 \cup \mathrm{ort}\, K_2).$$

3) *If $o \notin \mathrm{cl}\,(K_1 - K_2)$ and e belongs to the set* (19)*, then one of the inequalities* (10)*,* (12)*, and*

$$\sup\{x_1 \cdot e : x_1 \in K_1\} = \inf\{x_2 \cdot e : x_2 \in K_2\} < \sup\{x_2 \cdot e : x_2 \in K_2\}$$

   holds and the respective value of γ satisfying (5) *can be used in the description* (1) *of a hyperplane which nontrivially separates K_1 and K_2.*

4) *If $o \in \mathrm{cl}\,(K_1 - K_2)$ and $e \in (K_1 - K_2)^\circ \setminus (\mathrm{ort}\, K_1 \cup \mathrm{ort}\, K_2)$, then*

$$\begin{aligned}\inf\{x_1 \cdot e : x_1 \in K_1\} &< \sup\{x_1 \cdot e : x_1 \in K_1\} \\ &= \inf\{x_2 \cdot e : x_2 \in K_2\} < \sup\{x_2 \cdot e : x_2 \in K_2\}\end{aligned}$$

   and any scalar γ satisfying (5) *can be used in the description* (1) *of a hyperplane which nontrivially separates K_1 and K_2.* □

Problem 2. Describe all hyperplanes which *strictly separate* a given pair of convex sets K_1 and K_2 in $\mathbb{R}^n$.

Remark 5. A description of hyperplanes strongly separating one of the convex sets K_1 and K_2 from the other repeats Corollary 2 with one variation: in part 4), the scalar γ in the description of separating hyperplane (1) can be chosen to satisfy the conditions

$$\sup\{x_1 \cdot e : x_1 \in K_1\} < \gamma < \inf\{x_2 \cdot e : x_2 \in K_2\}.$$

6 Separation of Convex Cones

This section deals with various types of separation of convex cones which have a common apex. We recall that a convex set $C \subset \mathbb{R}^n$ is called a *cone* with apex $a \in \mathbb{R}^n$ provided $a + \lambda(x - a) \in C$ whenever $x \in C$ and $\lambda \geq 0$. This definition implies (letting $\lambda = 0$) that C contains its apex a, although a stronger condition $\lambda > 0$ can be beneficial; see, e. g., [6]. The set $\operatorname{ap} C = C \cap (2a - C)$ is called the *apex set* of C. It is known that $\operatorname{ap} C$ is the largest plane through a contained in C (see [6] and [11, Theorem 5.17]). Obviously, $C \neq \operatorname{ap} C$ if and only if C is not a plane.

The next corollary follows from [12] and Corollary 4.

Corollary 5. *Let C_1 and C_2 be convex cones with a common apex a, and let $D_1 = C_1 - a$ and $D_2 = C_2 - a$. The assertions below hold.*

1) *There is a hyperplane nontrivially separating one of the cones C_1 and C_2 from the other if and only if $(C_1 - C_2)^\circ$ is not a subspace.*

2) *A hyperplane $H \subset \mathbb{R}^n$ of the form*

$$H = \{x \in \mathbb{R}^n : x \cdot e = a \cdot e\}, \quad e \neq o, \tag{20}$$

nontrivially separates one of the cones C_1 and C_2 from the other if and only if one of the vectors e and $-e$ belongs to $(C_1 - C_2)^\circ \setminus \operatorname{lin}(C_1 - C_2)$.

3) *There is a hyperplane nontrivially separating C_1 from C_2 if and only if*

$$(C_1 - C_2)^\circ \setminus \operatorname{ort} C_1 \neq \varnothing.$$

4) *A hyperplane $H \subset \mathbb{R}^n$ of the form (20) nontrivially separates C_1 from C_2 if and only if one of the vectors e and $-e$ belongs to $(C_1 - C_2)^\circ \setminus \operatorname{ort} C_1$.*

5) *There is a hyperplane nontrivially separating C_1 and C_2 if and only if*

$$(C_1 - C_2)^\circ \setminus (\operatorname{ort} C_1 \cup \operatorname{ort} C_2) \neq \varnothing. \tag{21}$$

6) *A hyperplane $H \subset \mathbb{R}^n$ of the form (20) nontrivially separates C_1 and C_2 if and only if one of the vectors e and $-e$ belongs to the set (21).* □

If closed convex cones C_1 and C_2 with a common apex are separated by a hyperplane $H \subset \mathbb{R}^n$, then H supports both C_1 and C_2. Consequently, $\operatorname{ap} C_1 \cup \operatorname{ap} C_2 \subset H$ (see, e. g., [11], Theorem 9.43). In this regard, we recall the definition from [10]: a hyperplane H *sharply separates* C_1 and C_2 provided H separates them and

$$C_1 \cap H = \operatorname{ap} C_1 \quad \text{and} \quad C_2 \cap H = \operatorname{ap} C_2. \tag{22}$$

An asymmetric version of this definition is formulated as follows.

Definition 3. Let C_1 and C_2 be closed convex cones in $\mathbb{R}^n$, with a common apex, and let $H \subset \mathbb{R}^n$ be a hyperplane separating C_1 and C_2. We will say that H *sharply* separates C_1 *from* C_2 if $H \cap C_1 = \operatorname{ap} C_1$.

The next theorem gives a criterion for sharp separation of a convex cone from another one in terms of their polar cones.

Theorem 3. *If C_1 and C_2 are closed convex cones in $\mathbb{R}^n$ with a common apex a, then the following conditions are equivalent.*

1) *C_1 is sharply separated from C_2.*
2) *The set $E = \operatorname{rint}(C_1 - a)^\circ \cap (a - C_2)^\circ$ has positive dimension.*

Proof. Put $F_1 = C_1 - a$ and $F_2 = C_2 - a$. Then both F_1 and F_2 are closed convex cones with common apex o. Furthermore, $\operatorname{ap} F_i = \operatorname{ap} C_i - a$, $i = 1, 2$, and the set E from the condition 2) can be described as

$$E = \operatorname{rint} F_1^\circ \cap (-F_2)^\circ = \operatorname{rint} F_1^\circ \cap (-F_2^\circ). \tag{23}$$

1) $\Rightarrow$ 2) Let C_1 be sharply separated from C_2 by a hyperplane of the form

$$H = \{x \in \mathbb{R}^n : x \cdot e = \gamma\}, \quad e \neq o.$$

Clearly, F_1 and F_2 are separated by the $(n-1)$-dimensional subspace

$$S = H - a = \{x \in \mathbb{R}^n : x \cdot e = o\}.$$

Furthermore, S sharply separates F_1 from F_2 due to

$$S \cap F_1 = (H - a) \cap (C_1 - a) = H \cap C_1 - a = \operatorname{ap} C_1 - a = \operatorname{ap} F_1,$$

Without loss of generality, we may assume that

$$F_1 \subset V_1 = \{x \in \mathbb{R}^n : x \cdot e \le 0\} \quad \text{and} \quad F_2 \subset V_2 = \{x \in \mathbb{R}^n : x \cdot e \ge 0\}.$$

We assert that $e \in E$. To show the inclusion $e \in \operatorname{rint} F_1^\circ$, we will consider separately the cases when F_1 is or is not a subspace.

Assume first that F_1 is a subspace. Then $F_1 = \operatorname{ap} F_1 \subset S$, which gives the inclusion $e \in S^\perp \subset F_1^\perp$. Since $F_1^\perp$ is a subspace, we obtain $e \in F_1^\perp = F_1^\circ = \operatorname{rint} F_1^\circ$.

Suppose now that F_1 is not a subspace. Then $F_1 \neq \operatorname{ap} F_1$, and the condition $S \cap F_1 = \operatorname{ap} F_1$ implies the inclusion $F_1 \setminus \operatorname{ap} F_1 \subset \operatorname{int} V_1$. In this case, Theorem 8.6 from [11] shows that $e \in \operatorname{rint} F_1^\circ$.

For the inclusion $e \in (-F_2^\circ)$, we observe first that V_2 can be expressed as

$$V_2 = \{x \in \mathbb{R}^n : x \cdot (-e) \leq 0\}.$$

Consequently, the inclusion $F_2 \subset V_2$ gives $-e \in V_2^\circ \subset F_2^\circ$, or $e \in -F_2^\circ$.

Summing up, $e \in \operatorname{rint} F_1^\circ \cap (-F_2^\circ) = E$, implying that $\dim E > 0$.

2) $\Rightarrow$ 1) Suppose that $\dim E > 0$, and choose a nonzero vector $e \in E$. By the above argument, $F_1 \subset V_1$ and $F_2 \subset V_2$ such that $F_1 \setminus \operatorname{ap} F_1 \subset \operatorname{int} V_1$ if F_1 is not a plane, and $F_1 \setminus \operatorname{ap} F_1 = \varnothing$ if F_1 is a plane. Hence $S \cap F_1 = \operatorname{ap} F_1$, implying that S sharply separates F_1 from F_2. Consequently, H sharply separates C_1 from C_2. □

Analysis of the proof of Theorem 3 reveals the following corollary.

Corollary 6. *Let C_1 and C_2 be closed convex cones in $\mathbb{R}^n$, with a common apex. If C_1 is not a plane and is sharply separated from C_2, then C_1 is nontrivially separated from C_2.* □

Remark 6. The converse to Corollary 6 assertion is not true. For instance, in $\mathbb{R}^2$, the cone $C_1 = \{(x, 0) : x \in \mathbb{R}\}$ is separated sharply but not properly from the cone $C_2 = \{(x, y) : 0 \leq x, 0 \leq y \leq x\}$, while C_2 is separated properly but not sharply from C_1.

Theorem 4. *Let C_1 and C_2 be closed convex cones in $\mathbb{R}^n$, with common apex a. The following conditions are equivalent.*

1) *C_1 and C_2 are sharply separated.*
2) *Each of the cones C_1 and C_2 is sharply separated from the other.*
3) *The set $D = \operatorname{rint}(C_1 - a)^\circ \cap \operatorname{rint}(a - C_2)^\circ$ has positive dimension.*

Proof. The equivalence of conditions 1) and 3) is proved in [11, Theorem 10.16] (initially given in [10] for the case when neither C_1 nor C_2 is a plane). Since 1) obviously implies 2), it suffices to show that 2) $\Rightarrow$ 3).

So, assume that each of the cones C_1 and C_2 is sharply separated from the other. By Theorem 3, there are nonzero vectors

$$e_1 \in \operatorname{rint}(C_1 - a)^\circ \cap (a - C_2)^\circ \quad \text{and} \quad e_2 \in \operatorname{rint}(C_2 - a)^\circ \cap (a - C_1)^\circ.$$

Obviously, the second inclusion can be rewritten as

$$-e_2 \in (C_1 - a)^\circ \cap \operatorname{rint}(a - C_2)^\circ.$$

If $e_1 = -e_2$, then $e_1 \in \operatorname{rint}(C_1 - a)^\circ \cap \operatorname{rint}(a - C_2)^\circ = D$. Because D is a convex cone with improper apex o, one has $(o, e_1] \subset D$, which implies the inequality $\dim D > 0$.

Let $e_1 \neq -e_2$. Then the open segment $I = (e_1, -e_2)$ is one-dimensional. Because $(C_1 - a)^\circ$ is a closed convex cone, the inclusions $e_1 \in \operatorname{rint}(C_1 - a)^\circ$ and $-e_2 \in (C_1 - a)^\circ$ imply that $I \subset \operatorname{rint}(C_1 - a)^\circ$ (see [9, Theorem 6.1]). By a similar argument, $I \subset \operatorname{rint}(a - C_2)^\circ$. So,

$$I \subset \operatorname{rint}(C_1 - a)^\circ \cap \operatorname{rint}(a - C_2)^\circ = D,$$

again resulting in the inequality $\dim D > 0$. □

References

[1] J. Bair, F. Jongmans, *La séparation vraie dans un espace vectoriel.* Bull. Soc. Roy. Sci. Liège 41 (1972), 163–170.

[2] A. Brøndsted, *The inner aperture of a convex set.* Pacific J. Math. 72 (1977), 335–340.

[3] V. L. Klee, *Strict separation of convex sets.* Proc. Amer. Math. Soc. 7 (1956), 735–737.

[4] V. L. Klee, *Maximal separation theorems for convex sets*, Trans. Amer. Math. Soc. 134 (1968), 133–147.

[5] V. L. Klee, *Separation and support properties of convex sets – a survey*, Control Theory and the Calculus of Variations (University of California, LA, 1968), pp. 235–303, Academic Press, New York, 1969.

[6] J. Lawrence, V. Soltan, *On unions and intersections of nested families of cones*, Beitr. Algebra Geom. 57 (2016), 655–665.

[7] H. Minkowski, *Geometrie der Zahlen.* I, Teubner, Leipzig, 1896; II. Teubner, Leipzig, 1910.

[8] H. Minkowski, *Gesammelte Abhandlungen.* Bd 2, Teubner, Leipzig, 1911.

[9] R. T. Rockafellar, *Convex Analysis.* Princeton Universty Press, Princeton, NJ, 1970.

[10] V. Soltan, *Polarity and separation of cones*, Linear Algebra Appl. 538 (2018), 212–224.

[11] V. Soltan, *Lectures on convex sets. Second edition*, World Scientific, Hackensack, NJ, 2020.

[12] V. Soltan, *Separating hyperplanes of convex sets*, J. Convex Anal. 28 (2021), accepted.

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Received September 1, 2020