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Quasi-Kählerian manifolds and quasi-Sasakian hypersurfaces axiom

Ahmad Abu-Saleem, Mihail B. Banaru,
Galina A. Banaru, Lidia V. Stepanova

Abstract. It is proved that if a quasi-Kählerian manifold satisfies the quasi-Sasakian hypersurfaces axiom, then it is an almost Kahlerian manifold

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1 Introduction

The geometry of almost Hermitian manifolds and geometry of almost contact metric manifolds belong to the most intensively developing areas of modern mathematics. We mark out their profound inner content as well as their diverse applications in many domains of mathematics and theoretical physics [12]. It is known that almost contact metric structures are induced on oriented hypersurfaces of almost Hermitian manifolds. Almost contact metric structures on hypersurfaces of almost Hermitian manifolds were studied since 1960s by such outstanding mathematicians as S. Sasaki [13], S. Goldberg [9] and H. Yanamoto [17]. In the present paper, we consider the case when the almost Hermitian manifold is quasi-Kählerian (i.e. it belongs to the class $W_1 \oplus W_2$ in Gray–Hervella notation [10]). We remark that the class of quasi-Kählerian manifolds contains all Kählerian, nearly Kählerian and almost Kählerian manifolds that are the best studied types of almost Hermitian manifolds. The main result of our note is the following:

Theorem 1. *If a quasi-Kählerian manifold satisfies the quasi-Sasakian hypersurfaces axiom, then it is an almost Kählerian manifold.*

This short article is a continuation of the authors' researches in the area of interconnection of almost Hermitian and almost contact metric structures (see [1, 3, 4, 6, 7, 14–16] and others).

2 Preliminaries

An almost Hermitian manifold is an even-dimensional manifold M^{2n} with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J if the following condition holds

$$\langle JX,\ JY\rangle = \langle X,\ Y\rangle\,, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}[10]. The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(n)$ [7,12]. Its elements are the frames adapted to the structure (A-frames) that look as follows:

$$(p,\, \varepsilon_1,\, \ldots, \varepsilon_n,\, \varepsilon_{\hat{1}},\, \ldots,\, \varepsilon_{\hat{n}}\,),$$

where ε_a are the eigenvectors corresponded to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponded to the eigenvalue $-i$. Here the index a ranges from 1 to n, and we state $\hat{a} = a + n$. Therefore, the matrixes of the operator of the almost complex structure and of the Riemannian metric written in an A-frame look as follows, respectively:

$$\left(J_j^k\right) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right); \ (g_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right),$$

where I_n is the identity matrix; $k, j = 1, \ldots, 2n$.

We recall that the fundamental form (or Kählerian form [10]) of an almost Hermitian manifold is determined by the relation

$$F(X,\ Y) = \langle X,\ JY\rangle\,, \quad X, Y \in \aleph(M^{2n}).$$

By direct computing it is easy to obtain that in an A-frame the fundamental form matrix looks as follows:

$$(F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right).$$

An almost Hermitian manifold is quasi-Kählerian if the following identity holds [10,12]:

$$\nabla_X(F)(Y, Z) + \nabla_{JX}(F)(JY, Z) = 0,$$

where $X, Y, Z \in \aleph(M^{2n})$. We also remind that the following three identities determine almost Kählerian, nearly Kählerian and Kählerian manifolds, respectively:

$$\nabla F = 0, \quad \nabla_X(F)(X, Y) = 0, \quad dF = 0.$$

We recall also that an almost contact metric structure on an odd-dimensional manifold N is defined by the system of tensor fields $\{\Phi,\, \xi,\, \eta,\, g\}$ on this manifold, where ξ is a vector field, η is a covector field, Φ is a tensor of the type (1, 1) and $g = \langle\cdot,\, \cdot\rangle$ is the Riemannian metric [12]. Moreover, the following conditions are fulfilled:

$$\eta(\xi) = 1,\ \Phi(\xi) = 0,\ \eta \circ \Phi = 0,\ \Phi^2 = -id + \xi \otimes \eta,$$
$$\langle \Phi X, \Phi Y\rangle = \langle X, Y\rangle - \eta\,(X)\,\eta\,(Y)\,,\ X, Y \in \aleph(N),$$

where $\aleph(N)$ is the module of smooth vector fields on N. As the most important examples of almost contact metric structures we can mark out the cosymplectic structure, the nearly cosymplectic structure, the Sasakian structure and the Kenmotsu structure.

As it was mentioned above, the almost contact metric structures are closely connected to the almost Hermitian structures. For instance, if $(N, \{\Phi, \xi, \eta, g\})$ is an almost contact metric manifold, then an almost Hermitian structure is induced on the product $N \times R$ [12]. If this almost Hermitian structure is integrable, then the input almost contact metric structure is called normal. As it is known, a normal contact metric structure is called Sasakian. On the other hand, we can characterize the Sasakian structure by the following condition [7, 12]:

$$\nabla_X(\Phi)Y = \langle X, Y\rangle\, \xi - \eta(Y)X, \quad X, Y \in \aleph(N).$$

For example, Sasakian structures are induced on totally umbilical hypersurfaces in a Kählerian manifold [13]. As it is well known, the Sasakian structures have many remarkable properties and play a fundamental role in contact geometry. A natural generalization of the Sasakian structure is the quasi-Sasakian structure [8]. An almost contact metric structure $\{\Phi, \xi, \eta, g\}$ is called quasi-Sasakian (qS-) if its fundamental form $\Omega(X, Y) = \langle X, \Phi Y\rangle$ is closed and the following condition holds:

$$N_\Phi + \frac{1}{2}\, d\eta \otimes \xi = 0,$$

where N_Φ is the Nijenhuis tensor of Φ . The theory of quasi-Sasakian structures was created by the outstanding American geometer D. E. Blair [8]. He has established some sufficient conditions under which a qS-manifold is a product of a Sasakian and a Kählerian manifolds.

We remind that an almost Hermitian manifold M^{2n} satisfies the quasi-Sasakian hypersurfaces axiom if a qS-hypersurface passes through every point of this manifold. This terminology was introduced by V. F. Kirichenko [11].

At the end of this section, note that all considered manifolds, tensor fields and similar objects are assumed to be smooth of the class C^∞ .

3 Proof of the theorem

Let us consider a quasi-Kählerian manifold M^{2n} , let N^{2n-1} be its oriented hypersurface. The first group of the Cartan structural equations of a quasi-Kählerian structure written in an A-frame looks as follows [5, 14, 15]:

$$d\,\omega^a = \omega^a_b \wedge \omega^b + B^{abc}\, \omega_b \wedge \omega_c;$$

$$d\,\omega_a = -\omega^b_a \wedge \omega_b + B_{abc}\, \omega^b \wedge \omega^c,$$

where

$$B^{abc} = \frac{i}{2}\, J^a_{[\hat{b}, \hat{c}]};\; B_{abc} = -\frac{i}{2}\, J^{\hat{a}}_{[b, c]}.$$

The systems of functions $\{\, B^{abc} \,\}$, $\{\, B_{abc} \,\}$ are components of the Kirichenko tensors of the almost Hermitian manifold M^{2n} [2]; $\{J^j_{k,m}\}$ are components of ∇J;

here and further $a, b, c = 1, ..., n;$, $\hat{a} = a + n$. The similar tensors were introduced by L. V. Stepanova [14]:

$$\tilde{B}^{abc} = -\frac{i}{2} J^{a}_{\hat{b},\hat{c}}; \quad \tilde{B}_{abc} = \frac{i}{2} J^{\hat{a}}_{b,c}.$$

Let us consider the Cartan structural equations of the almost contact metric structure on an oriented hypersurface N^{2n-1} of a quasi-Kählerian manifold M^{2n} [7, 14]:

$$\begin{aligned} d\omega^{\alpha} &= \omega^{\alpha}_{\beta} \wedge \omega^{\beta} + B^{\alpha\beta\gamma} \omega_{\beta} \wedge \omega_{\gamma} + \\ &+ i\sigma^{\alpha}_{\beta} \omega^{\beta} \wedge \omega + \left(-\sqrt{2}\,\tilde{B}^{n\alpha\beta} - \frac{1}{\sqrt{2}}\tilde{B}^{\alpha\beta n} + i\,\sigma^{\alpha\beta}\right) \omega_{\beta} \wedge \omega; \\ d\omega_{\alpha} &= -\omega^{\beta}_{\alpha} \wedge \omega_{\beta} + B_{\alpha\beta\gamma} \omega^{\beta} \wedge \omega^{\gamma} - \\ &- i\sigma^{\beta}_{\alpha} \omega_{\beta} \wedge \omega + \left(-\sqrt{2}\,\tilde{B}_{n\alpha\beta} - \frac{1}{\sqrt{2}}\tilde{B}_{\alpha\beta n} - i\,\sigma_{a\beta}\right) \omega^{\beta} \wedge \omega; \\ d\omega &= \sqrt{2}\,B_{n\alpha\beta}\, \omega^{\alpha} \wedge \omega^{\beta} + \sqrt{2}\,B^{n\alpha\beta}\, \omega_{\alpha} \wedge \omega_{\beta} - \\ &-2i\,\sigma^{\alpha}_{\beta}\, \omega^{\beta} \wedge \omega_{\alpha} + \left(\tilde{B}_{n\beta n} + i\,\sigma_{n\beta}\right) \omega \wedge \omega^{\beta} + \left(\tilde{B}^{n\beta n} - i\,\sigma^{\beta}_{n}\right) \omega \wedge \omega_{\beta}, \end{aligned} \tag{1}$$

where σ is the second fundamental form of the immersion of the hypersurface N^{2n-1} into M^{2n}; here and further $\alpha, \beta, \gamma = 1, ..., n-1$.

Comparing the equations (1) with the first group of Cartran structural equations of a qS-structure [12, 14, 16]

$$d\omega^{\alpha} = \omega^{\alpha}_{\beta} \wedge \omega^{\beta} + B^{\alpha}_{\beta} \omega \wedge \omega^{\beta};$$

$$d\omega_{\alpha} = -\omega^{\beta}_{\alpha} \wedge \omega_{\beta} - B^{\beta}_{\alpha} \omega \wedge \omega_{\beta};$$

$$d\omega = 2\, B^{\alpha}_{\beta}\, \omega^{\beta} \wedge \omega_{\alpha},$$

we obtain the conditions that are necessary and sufficient for an almost contact structure on N^{2n-1} to be quasi-Sasakian:

$$1)\ B^{\alpha\beta\gamma} = 0\ ; 2)\ \sigma^{\alpha}_{\beta} = i\,B^{\alpha}_{\beta}\ ; 3)\ B^{n\alpha\beta} = 0; \tag{2}$$

$$4)\ -\sqrt{2}\,\tilde{B}^{n\alpha\beta} - \frac{1}{\sqrt{2}}\tilde{B}^{\alpha\beta n} + i\,\sigma^{\alpha\beta} = 0; 5)\ \tilde{B}^{n\beta n} - i\,\sigma^{\beta}_{n} = 0$$

and the formulae obtained by complex conjugation (no need to write them explicitly). From $(2)_3$ we obtain:

$$B^{n\alpha\beta} = 0 \quad \Rightarrow \tilde{B}^{n[\alpha\beta]} = 0 \quad \Rightarrow \tilde{B}^{n\alpha\beta} = \tilde{B}^{n\beta\alpha}.$$

By alternating $(2)_4$ we get:

$$0 = \sigma^{[\alpha\beta]} = -i\sqrt{2}\tilde{B}^{n[\alpha\beta]} - \frac{i}{\sqrt{2}}\,\tilde{B}^{[\alpha\beta]n} =$$

$$= -\frac{i}{2}\left(\sqrt{2}\tilde{B}^{n\alpha\beta} - \sqrt{2}\tilde{B}^{n\beta\alpha} + \frac{1}{\sqrt{2}}\tilde{B}^{\alpha\beta n} - \frac{1}{\sqrt{2}}\tilde{B}^{\beta\alpha n}\right) = -i\sqrt{2}\tilde{B}^{\alpha\beta n}.$$

We have $\tilde{B}^{\alpha\beta n} = 0$, from this equality we obtain $\sigma^{\alpha\beta} = -i\sqrt{2}\,\tilde{B}^{n\alpha\beta}$. That is why we can rewrite (2) as follows:

$$1)\ B^{\alpha\beta\gamma} = 0;\ \ 2)\ B^{n\alpha\beta} = 0;\ \ 3)\ \tilde{B}^{\alpha\beta n} = 0;\ \ 4)\ \sigma^{\alpha}_{\beta} = i\,B^{\alpha}_{\beta}; \tag{3}$$

$$5)\ \sigma^{\alpha\beta} = -i\sqrt{2}\,\tilde{B}^{n\alpha\beta};\ \ 6)\ \sigma^{\beta}_{n} = i\,\tilde{B}^{n\beta n}$$

and the formulae obtained by complex conjugation.

Now, let us fix a point $p \in M^{2n}$. If the hypersurface N^{2n-1} passes through this point, then the conditions (3) are fulfilled at this point. For $a, b, c = 1, ..., n$ we have:

$$2\,B^{abc} = \tilde{B}^{acb} - \tilde{B}^{abc};$$

$$2\,B^{bca} = \tilde{B}^{bac} - \tilde{B}^{bca};$$

$$2\,B^{cab} = \tilde{B}^{cba} - \tilde{B}^{cab}.$$

Adding the first and second equalities and subtracting the third equality, and also taking into account that the tensor $\tilde{B}^{abc}$ is skew-symmetric with respect to the indices a and b, we obtain

$$\tilde{B}^{abc} = -B^{abc} - B^{bca} + B^{cab}.$$

That is why we get:

$$-\tilde{B}^{\alpha\beta n} = B^{\alpha\beta n} + B^{\beta n\alpha} - B^{n\alpha\beta} = 0;$$

$$B^{nn\beta} + B^{n\beta n} + B^{\beta nn} = B^{nn\beta} - B^{nn\beta} = 0;$$

$$B^{\alpha\beta\gamma} + B^{\beta\gamma\alpha} + B^{\gamma\alpha\beta} = 0.$$

So, we have

$$B^{abc} + B^{bca} + B^{cab} = 0, \quad a, b, c = 1, ..., n. \tag{4}$$

It is not difficult to show that the condition (4) is equivalent to the closure of the fundamental form F , i.e. $dF = 0$, or

$$(dF)_p = 0; \quad p \in M^{2n}.$$

Indeed, it is known [7] that $F = -2i\,\omega^a \wedge \omega_a$. Therefore,

$$d\,F = -2i\,d\omega^a \wedge \omega_a + 2i\,\omega^a \wedge d\,\omega_a;$$

$$d\,F = -2i\,(\omega^a_b \wedge \omega^b \wedge \omega_a + B^{abc}\,\omega_b \wedge \omega_c \wedge \omega_a) +$$
$$+2\,i\,(\omega^a \wedge (-\omega^b_a \wedge \omega_b) + B_{abc}\,\omega^a \wedge \omega^b \wedge \omega^c);$$

$$d\,F = -2i\,(\omega^a_b \wedge \omega^b \wedge \omega_a + B^{[abc]}\,\omega_b \wedge \omega_c \wedge \omega_a) +$$

$$+2\,i\;(\omega_a^b \wedge \omega^a \wedge \omega_b) \;+\; B_{[abc]}\,\omega^a \wedge \omega^b \wedge \omega^c);$$

$$d\,F = -2i\;(B^{[abc]}\,\omega_b \wedge \omega_c \wedge \omega_a - B_{[abc]}\,\omega^a \wedge \omega^b \wedge \omega^c\,).$$

So,

$$d\,F = 0 \quad \Leftrightarrow \quad B^{[abc]} = B_{[abc]} = 0.$$

Taking into account that

$$B^{[abc]} = 0 \quad \Leftrightarrow \quad B^{abc} + B^{bca} + B^{cab} = 0,$$

we conclude

$$d\,F = 0 \quad \Leftrightarrow \quad B^{abc} \;+\; B^{bca} +\; B^{cab} = 0.$$

But the condition $B^{abc} + B^{bca} + B^{cab} = 0$ is the well-known [5, 12] criterion in terms of Kirichenko tensor for an arbitrary quasi-Kählerian manifold to be almost Kählerian (or to be a manifold of class W_2 in Gray–Hervella notation [10]).

We obtain that if a quasi-Sasakian hypersurface N^{2n-1} passes through an arbitrary point of a quasi-Kählerian manifold M^{2n}, then the condition $d\,F = 0$ holds at this point. That is why the manifold M^{2n} is almost Kählerian, Q.E.D.

4 Some comments

Using the well-known facts that the class of quasi-Kählerian manifolds contains the classes of nearly Kählerian (NK-) and almost Kählerian (AK-) manifolds [10], and class of Kählerian manifolds is the intersection of these classes:

$$K = NK \cap AK,$$

we can state the following consequence.

Corollary. If a nearly Kählerian manifold satisfys the quasi-Sasakian hypersurfaces axiom, then it is a Kählerian manifold.

We remark that this result was established by L. V. Stepanova [14] in a different way. Namely, this result was obtained from the fact that the class of Kählerian manifolds is also the intersection of the classes of nearly Kählerian and Hermitian manifolds:

$$K = NK \cap H.$$

We note that the theory of almost contact metric hypersurfaces of Hermitian manifolds (i.e. of almost Hermitian manifolds with integrable almost complex structure) is studied much better than the theory of almost contact metric hypersurfaces of quasi-Kählerian manifolds. The survey [7] contains a large list of papers on this subject.

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Ahmad Abu-Saleem
Department of Mathematics Al al-Bayt University
Marfaq JORDAN
E-mail: *dr_ahmad57@yahoo.com*

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Mihail B. Banaru, Galina A. Banaru
Smolensk State University
4, Przhevalsky Street, Smolensk – 214 000 RUSSIA
E-mail: *mihail.banaru@yahoo.com*

Lidia V. Stepanova
Smolensk Physical and Mathematical Lyceum under MIFI
5, Pamfilov Street, Smolensk – 214 018 RUSSIA
E-mail: *lide@yandex.ru*