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Interior angle sums of geodesic triangles in $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{H}^2\times\mathbf{R}$ geometries

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Abstract. In the present paper we study $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{H}^2\times\mathbf{R}$ geometries, which are homogeneous Thurston 3-geometries. We analyse the interior angle sums of geodesic triangles in both geometries and we prove that in $\mathbf{S}^2\times\mathbf{R}$ space it can be larger than or equal to π and in $\mathbf{H}^2\times\mathbf{R}$ space the angle sums can be less than or equal to π. This proof is a new direct approach to the issue and it is based on the projective model of $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{H}^2\times\mathbf{R}$ geometries described by E. Molnár in [7].

Mathematics subject classification: 53A20, 53A35, 52C35, 53B20.
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1 Introduction

A geodesic triangle in Riemannian geometry and more generally in metric geometry is a figure consisting of three different points together with the pairwise-connecting geodesic curves. The points are known as the vertices, while the geodesic curve segments are known as the sides of the triangle.

In the geometries of constant curvature $\mathbf{E}^3$, $\mathbf{H}^3$, $\mathbf{S}^3$ the well-known sums of the interior angles of geodesic triangles characterize the space. It is related to the Gauss-Bonnet theorem which states that the integral of the Gauss curvature on a compact 2-dimensional Riemannian manifold M is equal to $2\pi\chi(M)$ where $\chi(M)$ denotes the Euler characteristic of M. This theorem has a generalization to any compact even-dimensional Riemannian manifold (see e.g.[2, 5]).

Remark 1. In the Thurston spaces translation curves can be introduced in a natural way (see [7]). These curves are simpler than geodesics and differ from them in **Nil**, $\widetilde{\mathbf{SL_2R}}$ and **Sol** geometries. In $\mathbf{E}^3$, $\mathbf{S}^3$, $\mathbf{H}^3$, $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{H}^2\times\mathbf{R}$ geometries the mentioned curves coincide with each other ([1, 4, 15, 21]).

In [4] we investigated the angle sums of translation and geodesic triangles in $\widetilde{\mathbf{SL_2R}}$ geometry and proved that the possible sum of the interior angles in a translation triangle must be greater than or equal to π. However, in geodesic triangles this sum is less, greater or equal to π.

In [20] we considered the analogous problem for geodesic triangles in **Nil** geometry and proved that the sum of the interior angles of geodesic triangles in **Nil** space is larger than, less than or equal to π. In [1] K. Brodaczewska showed that sum of the interior angles of translation triangles of the **Nil** space is larger than or equal to π.

In [21] we studied the interior angle sums of *translation triangles* in **Sol** geometry and proved that the possible sum of the interior angles in a translation triangle must be greater than or equal to π. Further interesting properties of translation triangles and tetrahedra are described in [15].

However, in $\mathbf{S}^2\times\mathbf{R}$, $\mathbf{H}^2\times\mathbf{R}$ and **Sol** Thurston geometries there are no results concerning the angle sums of *geodesic triangles*. Therefore, it is interesting to study this question in the above three geometries.

In the present paper, we are interested in *geodesic triangles* in $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{H}^2\times\mathbf{R}$ spaces [13, 22].

In Section 2 we describe the projective model and the isometry group of the considered geometries, moreover, we give an overview about its geodesic curves. *In Section 3 we study the* $\mathbf{S}^2\times\mathbf{R}$ *and* $\mathbf{H}^2\times\mathbf{R}$ *geodesic triangles and their properties. We analyse the interior angle sums of geodesic triangles in both geometries and we prove that in* $\mathbf{S}^2\times\mathbf{R}$ *space it can be larger than or equal to* π *and in* $\mathbf{H}^2\times\mathbf{R}$ *space the angle sums can be less than or equal to* π*. This is a consequence of comparison theorems in Riemannian geometry (Toponogov and Alexandrov's theorems, see [3]), since the sectional curvature of* $\mathbf{S}^2\times\mathbf{R}$ *is non-negative and the sectional curvature of* $\mathbf{H}^2\times\mathbf{R}$ *is non-positive.*

Our new proof gives a new direct approach to the issue and it is based on the projective model of $\mathbf{S}^2\times\mathbf{R}$ *and* $\mathbf{H}^2\times\mathbf{R}$ *geometries described by E. Molnár in [7].*

2 Projective models of $\mathbf{H}^2\times\mathbf{R}$ and $\mathbf{S}^2\times\mathbf{R}$ spaces

E. Molnár has shown in [7] that the homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4, \boldsymbol{V}_4, \mathbf{R})$. In our work we shall use this projective model of $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{H}^2\times\mathbf{R}$ geometries. The Cartesian homogeneous coordinate simplex is $E_0(\mathbf{e}_0)$, $E_1^\infty(\mathbf{e}_1)$, $E_2^\infty(\mathbf{e}_2)$, $E_3^\infty(\mathbf{e}_3)$, $(\{\mathbf{e}_i\}\subset\mathbf{V}^4$ with the unit point $E(\mathbf{e}=\mathbf{e}_0+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3))$, which is distinguished by an origin E_0 and by the ideal points of coordinate axes, respectively. Moreover, $\mathbf{y}=c\mathbf{x}$ with $0<c\in\mathbf{R}$ (or $c\in\mathbf{R}\setminus\{0\}$) defines a point $(\mathbf{x})=(\mathbf{y})$ of the projective 3-sphere $\mathcal{PS}^3$ (or that of the projective space $\mathcal{P}^3$ where opposite rays $(\mathbf{x})$ and $(-\mathbf{x})$ are identified). The dual system $\{(\boldsymbol{e}^i)\}\subset\boldsymbol{V}_4$ describes the simplex planes, especially the plane at infinity $(\boldsymbol{e}^0)=E_1^\infty E_2^\infty E_3^\infty$, and generally, $\boldsymbol{v}=\boldsymbol{u}\frac{1}{c}$ defines a plane $(\boldsymbol{u})=(\boldsymbol{v})$ of $\mathcal{PS}^3$ (or that of $\mathcal{P}^3$). Thus $0=\mathbf{x}\boldsymbol{u}=\mathbf{y}\boldsymbol{v}$ defines the incidence of point $(\mathbf{x})=(\mathbf{y})$ and plane $(\boldsymbol{u})=(\boldsymbol{v})$, as $(\mathbf{x})\mathrm{I}(\boldsymbol{u})$ also denotes it. Thus $\mathbf{S}^2\times\mathbf{R}$ can be visualized in the affine 3-space $\mathbf{A}^3$ (so in $\mathbf{E}^3$) as well.

2.1 Geodesic curves in $\mathbf{S}^2\times\mathbf{R}$ space

In this section we recall the important notions and results from the papers [7, 11, 14, 16, 17].

The well-known infinitesimal arc-length square at any point of $\mathbf{S}^2\times\mathbf{R}$ is as follows

$$(ds)^2 = \frac{(dx)^2+(dy)^2+(dz)^2}{x^2+y^2+z^2}. \tag{2.1}$$

We shall apply the usual geographical coordiantes (ϕ,θ), $(-\pi<\phi\le\pi,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2})$ of the sphere with the fibre coordinate $t\in\mathbf{R}$. We describe points in the above coordinate system in our model by the following equations:

$$x^0=1,\quad x^1=e^t\cos\phi\cos\theta,\quad x^2=e^t\sin\phi\cos\theta,\quad x^3=e^t\sin\theta. \tag{2.2}$$

Then we have $x=\frac{x^1}{x^0}=x^1$, $y=\frac{x^2}{x^0}=x^2$, $z=\frac{x^3}{x^0}=x^3$, i.e. the usual Cartesian coordinates. We obtain by [7] that in this parametrization the infinitesimal arc-length square at any point of $\mathbf{S}^2\times\mathbf{R}$ is the following

$$(ds)^2=(dt)^2+(d\phi)^2\cos^2\theta+(d\theta)^2. \tag{2.3}$$

The geodesic curves of $\mathbf{S}^2\times\mathbf{R}$ are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves $\gamma(t(\tau),\phi(\tau),\theta(\tau))$ in our model can be determined by the general theory of Riemann geometry (see [5,17]).

Then by (2.2) we get with $c=\sin v$, $\omega=\cos v$ the equation systems of a geodesic curve, visualized in Fig. 3 in our Euclidean model:

$$\begin{gathered} x(\tau)=e^{\tau\sin v}\cos(\tau\cos v),\\ y(\tau)=e^{\tau\sin v}\sin(\tau\cos v)\cos u,\\ z(\tau)=e^{\tau\sin v}\sin(\tau\cos v)\sin u,\\ -\pi<u\le\pi,\quad -\frac{\pi}{2}\le v\le\frac{\pi}{2}. \end{gathered} \tag{2.4}$$

Definition 1. The distance $d(P_1,P_2)$ between the points P_1 and P_2 is defined by the arc length of the shortest geodesic curve from P_1 to P_2.

2.2 Geodesic curves of $\mathbf{H}^2\times\mathbf{R}$ geometry

In this section we recall the important notions and results from the papers [7, 12, 18].

The points of $\mathbf{H}^2\times\mathbf{R}$ space, forming an open cone solid in the projective space $\mathcal{P}^3$, are the following:

$$\mathbf{H}^2\times\mathbf{R}:=\left\{X(\mathbf{x}=x^i\mathbf{e}_i)\in\mathcal{P}^3:-(x^1)^2+(x^2)^2+(x^3)^2<0<x^0,\ x^1\right\}.$$

In this context E. Molnár [7] has derived the infinitesimal arc-length square at any point of $\mathbf{H}^2\times\mathbf{R}$ as follows

$$\begin{gathered} (ds)^2=\frac{1}{(-x^2+y^2+z^2)^2}\cdot[(x)^2+(y)^2+(z)^2](dx)^2+\\ +2dxdy(-2xy)+2dxdz(-2xz)+[(x)^2+(y)^2-(z)^2](dy)^2+\\ +2dydz(2yz)+[(x)^2-(y)^2+(z)^2](dz)^2. \end{gathered} \tag{2.5}$$

This becomes simpler in the following special (cylindrical) coordiantes (t, r, α), $(r \geq 0,\ -\pi < \alpha \leq \pi)$ with the fibre coordinate $t \in \mathbf{R}$. We describe points in our model by the following equations:

$$x^0 = 1, \quad x^1 = e^t \cosh r, \quad x^2 = e^t \sinh r \cos\alpha, \quad x^3 = e^t \sinh r \sin\alpha. \tag{2.6}$$

Then we have $x = \frac{x^1}{x^0} = x^1$, $y = \frac{x^2}{x^0} = x^2$, $z = \frac{x^3}{x^0} = x^3$, i.e. the usual Cartesian coordinates. We obtain by [7] that in this parametrization the infinitesimal arc-length square by (2.1) at any point of $\mathbf{H}^2\!\times\!\mathbf{R}$ is the following

$$(ds)^2 = (dt)^2 + (dr)^2 + \sinh^2 r (d\alpha)^2. \tag{2.7}$$

The geodesic curves of $\mathbf{H}^2\!\times\!\mathbf{R}$ are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves $\gamma(t(\tau), r(\tau), \alpha(\tau))$ in our model can be determined by the general theory of Riemann geometry:

By (2.5) the second order differential equation system of the $\mathbf{H}^2\!\times\!\mathbf{R}$ geodesic curve is the following [18]:

$$\ddot{\alpha} + 2\coth(r)\ \dot{r}\dot{\alpha} = 0,\ \ddot{r} - \sinh(r)\cosh(r)\dot{\alpha}^2 = 0,\ \ddot{t} = 0, \tag{2.8}$$

from which we get first a line as "geodesic hyperbola" on our model of $\mathbf{H}^2$ times a component on $\mathbf{R}$ each running with constant velocity c and ω, respectively:

$$t = c \cdot \tau, \quad \alpha = 0, \quad r = \omega \cdot \tau, \quad c^2 + \omega^2 = 1. \tag{2.9}$$

We can assume that the starting point of a geodesic curve is $(1, 1, 0, 0)$, because we can transform a curve into an arbitrary starting point, moreover, unit velocity with "geographic" coordinates (u, v) can be assumed:

$$\begin{gathered} r(0) = \alpha(0) = t(0) = 0; \quad \dot{t}(0) = \sin v,\ \dot{r}(0) = \cos v \cos u, \dot{\alpha}(0) = \cos v \sin u; \\ -\pi < u \leq \pi,\ -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}. \end{gathered}$$

Then by (2.6) we get with $c = \sin v$, $\omega = \cos v$ the equation systems of a geodesic curve, visualized in Fig. 8 in our Euclidean model [18]:

$$\begin{gathered} x(\tau) = e^{\tau \sin v} \cosh(\tau \cos v), \\ y(\tau) = e^{\tau \sin v} \sinh(\tau \cos v) \cos u, \\ z(\tau) = e^{\tau \sin v} \sinh(\tau \cos v) \sin u, \\ -\pi < u \leq \pi, \quad -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}. \end{gathered} \tag{2.10}$$

Definition 2. The distance $d(P_1, P_2)$ between the points P_1 and P_2 is defined by the arc length of the geodesic curve from P_1 to P_2.

Remark 2. $\mathbf{S}^2\!\times\!\mathbf{R}$ and $\mathbf{H}^2\!\times\!\mathbf{R}$ are affine metric spaces (affine-projective spaces – in the sense of the unified formulation of [7]). Therefore their linear, affine, unimodular, etc. transformations are defined as those of the embedding affine space.

3 Geodesic triangles

We consider 3 points A_1, A_2, A_3 in the projective model of X space (see Section 2) ($X \in \{\mathbf{S}^2\times\mathbf{R}, \mathbf{H}^2\times\mathbf{R}\}$. The *geodesic segments* a_k connecting the points A_i and A_j ($i < j,\ i,j,k \in \{1,2,3\}, k \neq i,j$) are called sides of the *geodesic triangle* with vertices A_1, A_2, A_3 (see Fig. 1, 2).

In Riemannian geometries the infinitesimal arc-lenght square (see (2.1) and (2.5)) is used to define the angle θ between two geodesic curves. If their tangent vectors at their common point are $\mathbf{u}$ and $\mathbf{v}$ and g_{ij} are the components of the metric tensor then

$$\cos(\theta) = \frac{u^i g_{ij} v^j}{\sqrt{u^i g_{ij} u^j \ v^i g_{ij} v^j}} \tag{3.1}$$

It is clear by the above definition of the angles and by the infinitesimal arc-lenght squares that the angles are the same as the Euclidean ones at the starting point of the geodesics.

Considering a geodesic triangle $A_1A_2A_3$ we can assume by the homogeneity of the considered geometries that one of its vertex coincides with the point $A_1 = (1,1,0,0)$ and the other two vertices are $A_2 = (1, x_2, y_2, z_2)$ and $A_3 = (1, x_3, y_3, z_3)$.

We will consider the *interior angles* of geodesic triangles that are denoted at the vertex A_i by ω_i ($i \in \{1,2,3\}$). We note here that the angle of two intersecting geodesic curves depends on the orientation of their tangent vectors.

3.1 Interior angle sums in $\mathbf{S}^2\times\mathbf{R}$ geometry

In order to determine the interior angles of a geodesic triangle $A_1A_2A_3$ and its interior angle sum $\sum_{i=1}^{3}(\omega_i)$, we define *isometric transformations* $\mathbf{T}_{A_i}^{\mathbf{S}^2\times\mathbf{R}}$ ($i \in \{2,3\}$, as elements of the isometry group of $\mathbf{S}^2\times\mathbf{R}$ geometry that maps the A_i onto A_1). Let the isometry $\mathbf{T}_{A_2}^{\mathbf{S}^2\times\mathbf{R}}$ be given by the composition of some special types of $\mathbf{S}^2\times\mathbf{R}$ isometries which transforms a fixed $A_2 = (1, x_2, y_2, z_2)$ point of $\mathbf{S}^2\times\mathbf{R}$ into $(1,1,0,0)$ (up to a positive determinant factor):

$\mathcal{T} = (\mathbf{Id.}, T)$ is a fibre translation,

$$\begin{gathered} A_2 = (1, x_2, y_2, z_2) \to A_2^{\mathcal{T}} = (1, x_2', y_2', z_2') = \\ = A_2^{\mathcal{T}} = \Big(1, \frac{x_2}{\sqrt{x_2^2+y_2^2+z_2^2}}, \frac{y_2}{\sqrt{x_2^2+y_2^2+z_2^2}}, \frac{z_2}{\sqrt{x_2^2+y_2^2+z_2^2}}\Big). \end{gathered} \tag{3.2}$$

($A_2^{\mathcal{T}}$ has 0 fibre coordinate). $\mathcal{R}_x = (\mathbf{R_x}, 0)$ is a special rotation about x axis with 0 fibre translation which moves the point $(1, x_2', y_2', z_2')$ into the $[x,y]$ plane.

$$\begin{gathered} A_2^{\mathcal{T}} = (1, x_2', y_2', z_2') \to A_2^{\mathcal{T}\mathcal{R}_x} = (1, x_2'', y_2'', 0) = \\ = A_2^{\mathcal{T}\mathcal{R}_x} = (1, x_2', \sqrt{y_2'^2 + z_2'^2}, 0). \end{gathered} \tag{3.3}$$

Similarly, $\mathcal{R}_z = (\mathbf{R_z}, 0)$ is a special rotation about z axis with 0 fibre translation which moves the point $(1, x_2'', y_2'', 0)$ into the $(1,1,0,0)$ point.

$$A_2^{\mathcal{T}\mathcal{R}_x} = (1, x_2'', y_2'', 0) \to A_2^{\mathcal{T}\mathcal{R}_x\mathcal{R}_z} = (1,1,0,0). \tag{3.4}$$

Finally we apply the inverse transformation $\mathcal{R}_x^{-1}$ of rotation $\mathcal{R}_x$ because the geodesic curve $g(A_1, A_2)$ between the points A_1 and A_2 and its image $g(A_1^2, A_1)$ under the transformation $\mathcal{T}\mathcal{R}_x\mathcal{R}_z\mathcal{R}_x^{-1}$ lie in the same plane in Euclidean sense. The matrix of the above transformation $\mathbf{T}_{A_2}^{\mathbf{S}^2\times\mathbf{R}} = \mathcal{T}\mathcal{R}_x\mathcal{R}_z\mathcal{R}_x^{-1}$ is the following:

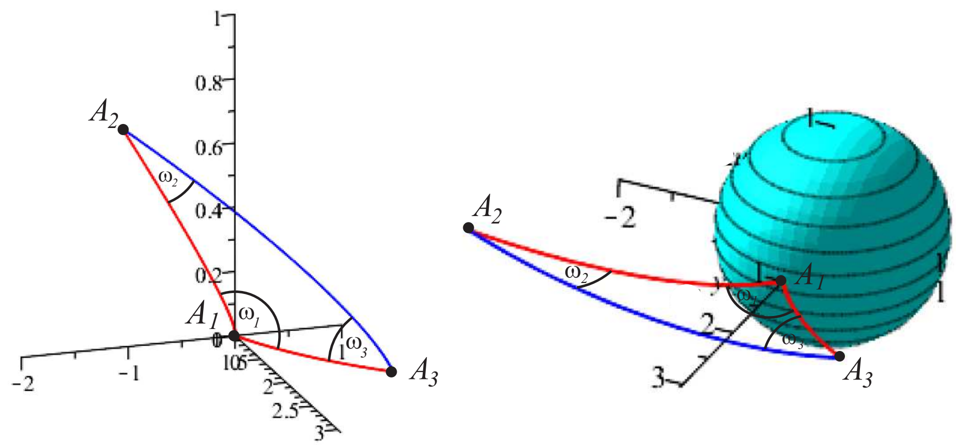

Figure 1. Geodesic triangle with vertices $A_1 = (1, 1, 0, 0)$, $A_2 = (1, 3, -2, 1)$, $A_3 = (1, 2, 1, 0)$ in $\mathbf{S}^2\times\mathbf{R}$ geometry.

$$\mathbf{T}_{A_2}^{\mathbf{S}^2\times\mathbf{R}} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x_2}{(x_2)^2+(y_2)^2+(z_2)^2} & \frac{-y_2}{(x_2)^2+(y_2)^2+(z_2)^2} & \frac{-z_2}{(x_2)^2+(y_2)^2+(z_2)^2} \\ 0 & \frac{y_2}{(x_2)^2+(y_2)^2+(z_2)^2} & \frac{(y_2)^2x_2+(z_2)^2\sqrt{(x_2)^2+(y_2)^2+(z_2)^2}}{((x_2)^2+(y_2)^2+(z_2)^2)((y_2)^2+(z_2)^2)} & \frac{-y_2z_2(-x_2+\sqrt{(x_2)^2+(y_2)^2+(z_2)^2})}{((x_2)^2+(y_2)^2+(z_2)^2)((y_2)^2+(z_2)^2)} \\ 0 & \frac{z_2}{(x_2)^2+(y_2)^2+(z_2)^2} & \frac{(y_2)z_2(-x_2+\sqrt{(x_2)^2+(y_2)^2+(z_2)^2})}{((x_2)^2+(y_2)^2+(z_2)^2)((y_2)^2+(z_2)^2)} & \frac{(z_2)^2x_2+(y_2)^2\sqrt{(x_2)^2+(y_2)^2+(z_2)^2}}{((x_2)^2+(y_2)^2+(z_2)^2)((y_2)^2+(z_2)^2)} \end{pmatrix}, \tag{3.5}$$

and the images $\mathbf{T}_{A_2}^{\mathbf{S}^2\times\mathbf{R}}(A_i)$ of the vertices A_i $(i \in \{1, 2, 3\})$ are the following (see also Fig. 2):

$$\mathbf{T}_{A_2}^{\mathbf{S}^2\times\mathbf{R}}(A_1) = A_1^2 =$$
$$= \Big(1, \frac{x_2}{(x_2)^2+(y_2)^2+(z_2)^2}, \frac{-y_2}{(x_2)^2+(y_2)^2+(z_2)^2}, \frac{-z_2}{(x_2)^2+(y_2)^2+(z_2)^2}\Big),$$
$$\mathbf{T}_{A_2}^{\mathbf{S}^2\times\mathbf{R}}(A_2) = A_2^2 = (1, 1, 0, 0),$$
$$\mathbf{T}_{A_2}^{\mathbf{S}^2\times\mathbf{R}}(A_3) = A_3^2 = \Big(1, \frac{x_2x_3 + y_2y_3}{(x_2)^2+(y_2)^2+(z_2)^2}, \tag{3.6}$$
$$\frac{y_3(z_2)^2\sqrt{(x_2)^2+(y_2)^2+(z_2)^2} + x_2(y_2)^2y_3 - x_3(y_2)^3 - x_3y_2(z_2)^2}{((y_2)^2+(z_2)^2)((x_2)^2+(y_2)^2+(z_2)^2)},$$
$$-\frac{z_2(y_3y_2(\sqrt{(x_2)^2+(y_2)^2+(z_2)^2} - x_2) + x_3(y_2)^2 + x_3(z_2)^2)}{((y_2)^2+(z_2)^2)((x_2)^2+(y_2)^2+(z_2)^2)}\Big).$$

Remark 3. More information about the isometry group of $\mathbf{S}^2\times\mathbf{R}$ and about its discrete subgroups can be found in [16] and [17].

Similarly to the above computation we get that the images $\mathbf{T}^{\mathbf{S}^2\times\mathbf{R}}_{A_3}(A_i)$ of the vertices A_i $(i\in\{1,2,3\})$ are the following (see also Fig. 2):

$$\begin{gathered}
\mathbf{T}^{\mathbf{S}^2\times\mathbf{R}}_{A_3}(A_1)=A_1^3=\Big(1,\frac{x_3}{(x_3)^2+(y_3)^2},\frac{-y_3}{(x_3)^2+(y_3)^2},0\Big),\\
\mathbf{T}^{\mathbf{S}^2\times\mathbf{R}}_{A_3}(A_3)=A_3^3=A_1=(1,1,0,0),\\
\mathbf{T}^{\mathbf{S}^2\times\mathbf{R}}_{A_3}(A_2)=A_2^3=\Big(1,\frac{x_2x_3+y_2y_3}{(x_3)^2+(y_3)^2},\frac{x_3y_2-x_2y_3}{(x_3)^2+(y_3)^2},\frac{z_2}{\sqrt{(x_3)^2+(y_3)^2}}\Big).
\end{gathered}\tag{3.7}$$

Our aim is to determine angle sum $\sum_{i=1}^{3}(\omega_i)$ of the interior angles of geodesic

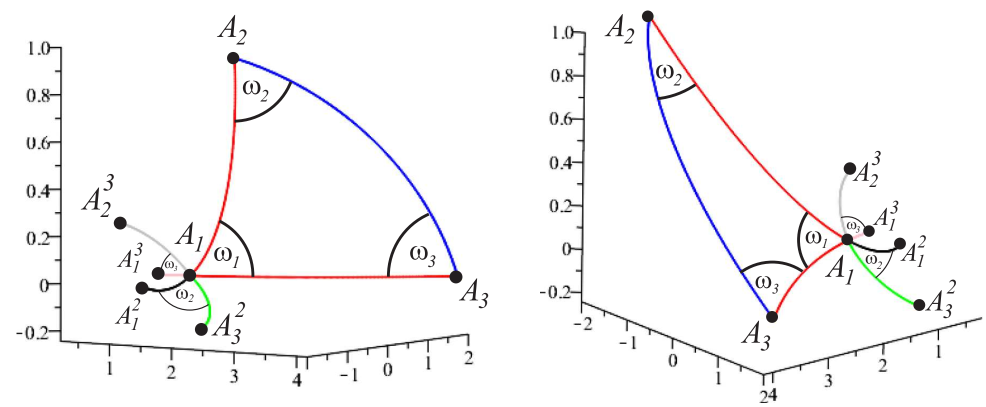

Figure 2. Geodesic triangle with vertices $A_1=(1,1,0,0)$, $A_2=(1,3,-2,1)$, $A_3=(1,2,1,0)$ in $\mathbf{S}^2\times\mathbf{R}$ geometry, and transformed images of its geodesic side segments.

triangles $A_1A_2A_3$ (see Fig. 1, 2). We have seen that ω_1, the angle of geodesic curves with common point at the vertex A_1, is the same as the Euclidean one therefore it can be determined by usual Euclidean sense.

The $\mathbf{T}^{\mathbf{S}^2\times\mathbf{R}}_{A_i}$ $(i=2,3)$ are isometries in $\mathbf{S}^2\times\mathbf{R}$ geometry thus ω_i is equal to the angle $(g(A_i^i,A_1^i),g(A_i^i,A_j^i))\angle$ $(i,j=2,3,\ i\neq j)$ (see Fig. 2) where $g(A_i^i,A_1^i)$, $g(A_i^i,A_j^i)$ are oriented geodesic curves ($A_1=A_2^2=A_3^3$) and ω_1 is equal to the angle $(g(A_1,A_2),g(A_1,A_3))\angle$ where $g(A_1,A_2)$, $g(A_1,A_3)$ are also oriented geodesic curves.

We denote the oriented unit tangent vectors of the oriented geodesic curves $g(A_1,A_i^j)$ with $\mathbf{t}_i^j$ where $(i,j)\in\{(1,3),(1,2),(2,3),(3,2),(3,0),(2,0)\}$ and $A_3^0=A_3$, $A_2^0=A_2$. The Euclidean coordinates of $\mathbf{t}_i^j$ (see Section 2.1) are :

$$\mathbf{t}_i^j=(\sin(v_i^j),\cos(v_i^j)\cos(u_i^j),\cos(v_i^j)\sin(u_i^j)).\tag{3.8}$$

In order to obtain the angle of two geodesic curves $g(A_1,A_i^j)$ and $g(A_1,A_k^l)$ $((i,j)\neq(k,l);\ (i,j),(k,l)\in\{(1,3),(1,2),(2,3),(3,2),(3,0),(2,0)\})$ intersected at the vertex A_1 we need to determine their tangent vectors $\mathbf{t}_s^r$ $((s,r)\in\{(1,3),(1,2),$

$(2,3),(3,2),(3,0),(2,0)\})$ (see (3.8)) at their starting point A_1. From (3.8) it follows that a tangent vector at the origin is given by the parameters u and v of the corresponding geodesic curve (see (2.10)), which can be determined from the homogeneous coordinates of the endpoint of the geodesic curve as the following Lemma shows:

Lemma 1. *Let* $(1,x,y,z)$ $(x,y,z\in\mathbf{R}, x^2+y^2+z^2\neq 0)$ *be the homogeneous coordinates of the point* $P\in\mathbf{S}^2\times\mathbf{R}$. *The paramerters of the corresponding geodesic curve* $g(A_1,P)$ *are the following:*

1. $y,z\in\mathbf{R}\setminus\{0\}$ *and* $x^2+y^2+z^2\neq 1$*;*

$$v=\arctan\Big(\frac{\log\sqrt{x^2+y^2+z^2}}{\arccos\frac{x}{\sqrt{x^2+y^2+z^2}}}\Big),\ u=\arctan\Big(\frac{z}{y}\Big),$$
$$\tau=\frac{\log\sqrt{x^2+y^2+z^2}}{\sin v},\ \textit{where}\ -\pi<u\leq\pi,\ -\pi/2\leq v\leq\pi/2,\ \tau\in\mathbf{R}^+. \tag{3.9}$$

2. $y=0,\ z\neq 0$ *and* $x^2+z^2\neq 1$*;*

$$u=\frac{\pi}{2},\ v=\arctan\Big(\frac{\log\sqrt{x^2+z^2}}{\arccos\frac{x}{\sqrt{x^2+z^2}}}\Big),$$
$$\tau=\frac{\log\sqrt{x^2+z^2}}{\sin v},\ \textit{where}\ -\pi/2\leq v\leq\pi/2,\ \tau\in\mathbf{R}^+. \tag{3.10}$$

3. $y=0,\ z\neq 0$ *and* $x^2+z^2=1$*;*

$$u=\frac{\pi}{2},\ v=0,\ \tau=\arccos(x),\ \tau\in\mathbf{R}^+. \tag{3.11}$$

4. $y,z=0$*;*

$$u=0,\ v=\frac{\pi}{2},\ \tau=\log\sqrt{x^2+y^2+z^2},\ \tau\in\mathbf{R}^+. \tag{3.12}$$

5. $x=0,\ y=0$ *and* $z\neq 1$*;*

$$u=\frac{\pi}{2},\ v=\arctan\frac{2\log|z|}{\pi},\ \tau=\frac{\log|z|}{\sin v},$$
$$-\pi/2\leq v\leq\pi/2,\ \tau\in\mathbf{R}^+. \tag{3.13}$$

□

We obtain directly from the (2.4) equations of the geodesic curves the following

Lemma 2. *Let* P *be an arbitrary point and* $g(A_1,P)$ $(A_1=(1,1,0,0))$ *is a geodesic curve in the considered model of* $\mathbf{S}^2\times\mathbf{R}$ *geometry. The points of the geodesic curve* $g(A_1,P)$ *and the centre of the model* E_0 *lie in a plane in Euclidean sense (see Fig. 3).*
□

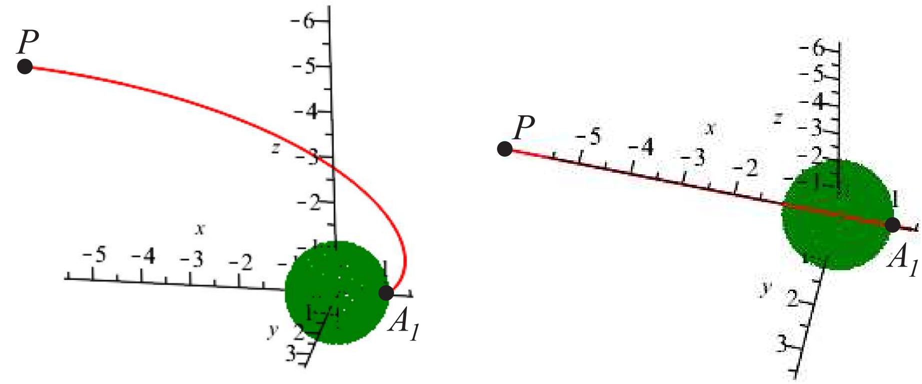

Figure 3. Geodesic curve $g(A_1, P)$ ($A_1 = (1, 1, 0, 0)$ and $P \in \mathbf{S}^2 \times \mathbf{R}$) with "base plane", the plane of a geodesic curve contains the origin $E_0 = (1, 0, 0, 0)$ of the model.

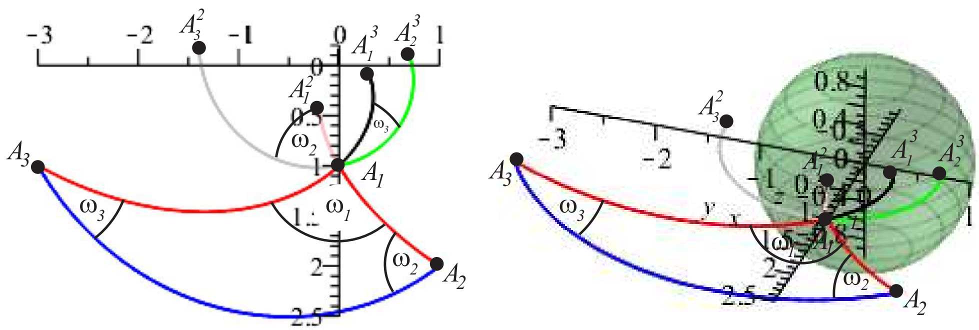

Figure 4. Geodesic triangle with vertices $A_1 = (1, 1, 0, 0)$, $A_2 = (1, 1, -3, 0)$, $A_3 = (1, 2, 1, 0)$ in $\mathbf{S}^2 \times \mathbf{R}$ geometry, and transformed images of its geodesic side segments. The geodesic curve segments $g(A_1, A_2)$, $g(A_2, A_3)$, $g(A_1, A_3)$ lie on the coordinate plane $[x, y]$ and the interior angle sum of this geodesic triangle is $\sum_{i=1}^{3}(\omega_i) = \pi$.

Theorem 1. *If the Euclidean plane of the vertices of an $\mathbf{S}^2 \times \mathbf{R}$ geodesic triangle $A_1A_2A_3$ contains the centre of model E_0 then its interior angle sum is equal to π.*

Proof: We can assume without loss of generality that the vertices A_1, A_2, A_3 of such a geodesic triangle lie in the $[x, y]$ plane of the model. Using Lemma 2 we get that the geodesic segments A_1A_2, A_1A_3 and A_2A_3 are contained by the [x,y] plane, too.

The $\mathbf{S}^2 \times \mathbf{R}$ transformations $\mathbf{T}_{A_2}^{\mathbf{S}^2 \times \mathbf{R}}$ and $\mathbf{T}_{A_3}^{\mathbf{S}^2 \times \mathbf{R}}$ are isometries in $\mathbf{S}^2 \times \mathbf{R}$ geometry, thus ω_2 is equal to the angle $(g(A_2^2, A_1^2), g(A_2^2, A_3^2))\angle$ (see Fig. 2, 4) of the oriented geodesic segments $g(A_2^2, A_1^2)$, $g(A_2^2, A_3^2)$ and ω_3 is equal to the angle $(g(A_3^3, A_1^3), g(A_3^3 A_2^3))\angle$ of the oriented geodesic segments $g(A_3^3, A_1^3)$ and $g(A_3^3, A_2^3)$ $(A_1 = A_2^2 = A_3^3)$.

Substituting the coordinates of the points A_i^j (see (3.5), (3.6) and (3.7)) $((i, j) \in$

$\{(1,3),(1,2),\ (2,3),(3,2),(3,0),(2,0)\}$) into the appropriate equations (3.8-12) of Lemma 1, it is easy to see that

$$\begin{aligned} v_2^0 = -v_1^2,\ u_2^0 - u_1^2 = \pm\pi \Rightarrow \mathbf{t}_2^0 = -\mathbf{t}_1^2, \\ v_3^0 = -v_1^3,\ u_3^0 - u_1^3 = \pm\pi \Rightarrow \mathbf{t}_3^0 = -\mathbf{t}_1^3, \\ v_3^2 = -v_2^3,\ u_3^2 - u_2^3 = \pm\pi \Rightarrow \mathbf{t}_3^2 = -\mathbf{t}_2^3. \end{aligned} \tag{3.13}$$

The endpoints T_i^j of the position vectors $\mathbf{t}_i^j = \overrightarrow{A_1T_i^j}$ lie on the unit sphere centred at the origin. The measure of angle ω_i ($i \in \{1,2,3\}$) of the vectors $\mathbf{t}_i^j$ and $\mathbf{t}_r^s$ is equal to the spherical distance of the corresponding points T_i^j and T_r^s on the unit sphere (see Fig. 4). Moreover, a direct consequence of equations (3.13) is that each point pair (T_2, T_1^2), (T_3,T_1^3), (T_2^3,T_3^2) contains antipodal points related to the unit sphere with centre A_1.

Due to the antipodality $\omega_1 = T_2A_1T_3\angle = T_1^2A_1T_1^3\angle$, therefore their corresponding spherical distances are equal, as well (see Fig. 4). Now, the sum of the interior angles $\sum_{i=1}^{3}(\omega_i)$ can be considered as three consecutive spherical arcs ($T_3^2T_1^2$), ($T_1^2T_1^3$), $T_1^3T_2^3$). Since the points T_2, T_1^2, T_3, T_1^3, T_2^3, T_3^2 lie in the $[x,y]$ plane (see Lemma 2) the sum of these arc lengths is equal to the half of the circumference of the main circle on the unit sphere,i.e. π. □

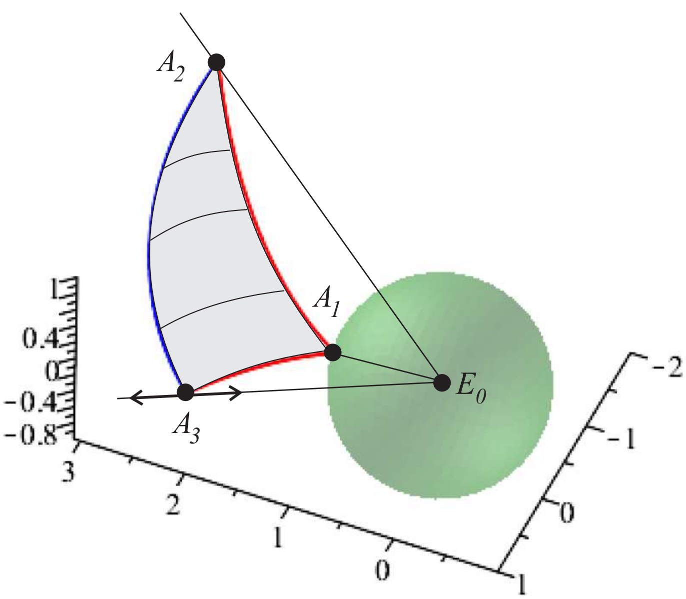

Figure 5. Geodesic triangle with vertices $A_1 = (1,1,0,0)$, $A_2 = (1,3,-2,1)$, $A_3 = (1,2,1,0)$ and the correspondig trihedron with base sphere of $\mathbf{S}^2\times\mathbf{R}$ geometry.

We can determine the interior angle sum of arbitrary geodesic triangle. In the following table we summarize some numerical data of interior angles of given geodesic triangles:

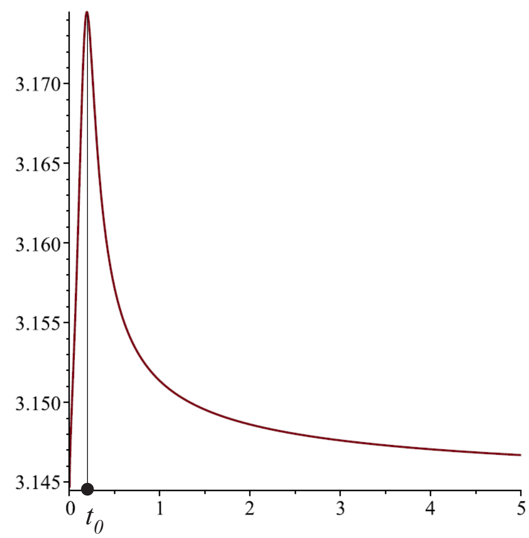

Figure 6. $S(\Delta(t))$ function related to parameters $x_2 = 3, y_2 = -2, z_2 = 1$ $x_3 = 2 \cdot t, y_3 = 1 \cdot t, z_3 = 0$.

Table 1: $A_1 = (1,0,0,0)$, $A_2 = (1,3,-2,1)$				
A_3/ω_i	ω_1	ω_2	ω_3	$\sum_{i=1}^{3}(\omega_i)$
$(1, 2/\sqrt{5}, 1/\sqrt{5}, 0)$	1.97206	0.26028	0.92635	3.15869
$(1, 2, 1, 0)$	0.94654	0.68775	1.51707	3.15135
$(1, 4, 2, 0)$	0.73193	1.29546	1.12123	3.14862
$(1, 12, 6, 0)$	0.61470	1.99926	0.53246	3.14643
$(1, 2000, 1000, 0)$	0.50628	2.52677	0.11050	3.14355

By the above experiences and computations we obtain the following

Theorem 2. *If the Euclidean plane of the vertices of a* $\mathbf{S}^2 \times \mathbf{R}$ *geodesic triangle* $A_1A_2A_3$ *does not contain the centre of model* E_0 *then its interior angle sum is greater than* π.

Proof: We can assume without loss of generality that the vertices A_1, A_2 of such a geodesic triangle lie in the $[x, y]$ plane of the model. Using Lemma 2 we get that the geodesic segment A_iA_j $((i,j) \in \{(1,2),(1,3),2,3)\})$ is contained in the $A_iA_jE_0$ plane, therefore the sides of triangle $A_1A_2A_3$ lie on the boundary of trihedron given by the points E_0, A_1, A_2, A_3 (see Fig. 2 and 5). It is clear that all types of geodesic triangles can be described by such a triangle. Therefore, it is sufficient to investigate the interior angle sums of geodesic triangles where we fix two of the vertices, e.g. A_1 and A_2, and move the third vertex A_3 on the half straight line E_0A_3 with starting point $E_0 \neq A_3(t)$.

Remark 4. It is well known that if the vertices A_1, A_2, A_3 lie on a sphere of radius $R \in \mathbf{R}^+$ centred at E_0 then the interior angle sum of spherical triangle $A_1A_2A_3$ is greater than π.

Let $\Delta^{\mathbf{S}^2\times\mathbf{R}}(t)$ $(t \in \mathbf{R}^+)$ denote the above geodesic triangle with *interior angles* $\omega_i(t)$ at the vertex A_i $(i \in \{1,2,3\})$.

The interior angle sum function $S(\Delta(t)) = \sum_{i=1}^{3}(\omega_i(t))$ can be determined relative to the parameters $x_2, y_2, z_2, x_3, y_3 \in \mathbf{R}$ by the formulas (2.4), (3.6), (3.7) and by Lemma 1. Analyzing the above complicated continuous functions of single real variable t we get that its maximum is achieved at a point $t_0 \in (0,\infty)$ depending on given parameters. Moreover, $S(\Delta^{\mathbf{S}^2\times\mathbf{R}}(t))$ is stricly increasing on the interval $(0,t_0)$, stricly decreasing on the interval (t_0,∞) and

$$\lim_{t\to 0} S(\Delta^{\mathbf{S}^2\times\mathbf{R}}(t)) = \pi, \qquad \lim_{t\to\infty} S(\Delta^{\mathbf{S}^2\times\mathbf{R}}(t)) = \pi.$$

In Fig. 6 we described the $S(\Delta^{\mathbf{S}^2\times\mathbf{R}}(t))$ function related to geodesic triangle $\Delta^{\mathbf{S}^2\times\mathbf{R}}(t)$ $(t \in (0,5))$ with vertices $A_1 = (1,1,0,0)$, $A_2 = (1,3,-2,1)$, $A_3 = (1, 2\cdot t, 1\cdot t, 0)$. Its maximum is achieved at $t_0 \approx 0.19316$ where $S(\Delta^{\mathbf{S}^2\times\mathbf{R}}(t_0)) \approx 3.17450$. □

Finally we get the following

Theorem 3. *The sum of the interior angles of a geodesic triangle of* $\mathbf{S}^2\times\mathbf{R}$ *space is greater than or equal to* π. □

3.2 Interior angle sums in $\mathbf{H}^2\times\mathbf{R}$ geometry

Similarly to the $\mathbf{S}^2\times\mathbf{R}$ space we investigate the interior angles of a geodesic triangle $A_1A_2A_3$ and its interior angle sum $\sum_{i=1}^{3}(\omega_i)$ in the $\mathbf{H}^2\times\mathbf{R}$ space. Therefore we define *isometric transformations* $\mathbf{T}_{A_i}^{\mathbf{H}^2\times\mathbf{R}}$ $(i \in \{2,3\})$ as elements of the isometry group of $\mathbf{H}^2\times\mathbf{R}$ geometry that maps the A_i onto the vertex A_1. Let the isometry $\mathbf{T}_{A_2}^{\mathbf{H}^2\times\mathbf{R}}$ be given by the composition of some special types of $\mathbf{H}^2\times\mathbf{R}$ isometries, which transforms a fixed $A_2 = (1, x_2, y_2, z_2)$ point of $\mathbf{H}^2\times\mathbf{R}$ into $A_1 = (1,1,0,0)$ (up to a positive determinant factor). The methods, the considered transformations and the determinations of their matrices are similar to the $\mathbf{S}^2\times\mathbf{R}$ case and therefore are not detailed here. The images $\mathbf{T}_{A_2}^{\mathbf{H}^2\times\mathbf{R}}(A_i)$ of the vertices A_i $(i \in \{1,2,3\})$ are the following (see also Fig. 7, 9):

$$\begin{gathered}
\mathbf{T}_{A_2}^{\mathbf{H}^2\times\mathbf{R}}(A_1) = A_1^2 = \\
= \Big(1, \frac{x_2}{(x_2)^2-(y_2)^2-(z_2)^2}, \frac{-y_2}{(x_2)^2-(y_2)^2-(z_2)^2}, \frac{-z_2}{(x_2)^2-(y_2)^2-(z_2)^2}\Big), \\
\mathbf{T}_{A_2}^{\mathbf{H}^2\times\mathbf{R}}(A_2) = A_2^2 = (1,1,0,0), \\
\mathbf{T}_{A_2}^{\mathbf{H}^2\times\mathbf{R}}(A_3) = A_3^2 = \Big(1, \frac{x_2x_3 - y_2y_3}{(x_2)^2-(y_2)^2-(z_2)^2}, \qquad (3.14) \\
\frac{y_3(z_2)^2\sqrt{(x_2)^2-(y_2)^2-(z_2)^2} + x_2(y_2)^2y_3 - x_3(y_2)^3 - x_3y_2(z_2)^2}{((y_2)^2+(z_2)^2)((x_2)^2-(y_2)^2-(z_2)^2)}, \\
-\frac{z_2(y_3y_2(\sqrt{(x_2)^2-(y_2)^2-(z_2)^2} - x_2) + x_3(y_2)^2 + x_3(z_2)^2)}{((y_2)^2+(z_2)^2)((x_2)^2-(y_2)^2-(z_2)^2)}\Big).
\end{gathered}$$

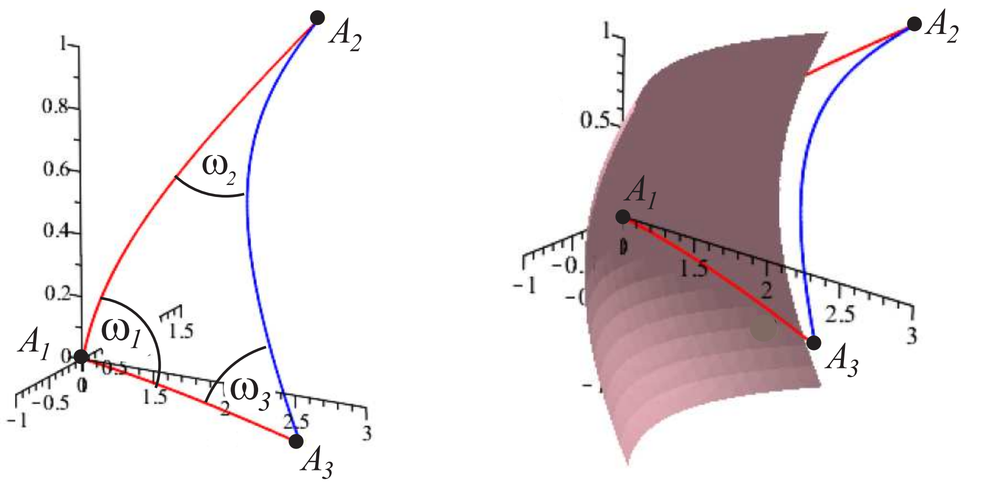

Figure 7. Geodesic triangle with vertices $A_1 = (1,1,0,0)$, $A_2 = (1,2,3/2,1)$, $A_3 = (1,3,-1,0)$ in $\mathbf{H}^2\!\times\!\mathbf{R}$ geometry.

Remark 5. More information about the isometry group of $\mathbf{H}^2\times\mathbf{R}$ and about its discrete subgroups can be found in [18].

Similarly to the above computation we get that the images $\mathbf{T}^{\mathbf{H}^2\times\mathbf{R}}_{A_3}(A_i)$ of the vertices A_i ($i \in \{1,2,3\}$) are the following (see also Fig. 7, 9):

$$\begin{gathered}
\mathbf{T}^{\mathbf{H}^2\times\mathbf{R}}_{A_3}(A_1) = A_1^3 = \Big(1, \frac{x_3}{(x_3)^2-(y_3)^2}, \frac{-y_3}{(x_3)^2-(y_3)^2}, 0\Big), \\
\mathbf{T}^{\mathbf{H}^2\times\mathbf{R}}_{A_3}(A_3) = A_3^3 = A_1 = (1,1,0,0), \\
\mathbf{T}^{\mathbf{H}^2\times\mathbf{R}}_{A_3}(A_2) = A_2^3 = \Big(1, \frac{x_2x_3-y_2y_3}{(x_3)^2-(y_3)^2}, \frac{x_3y_2-x_2y_3}{(x_3)^2-(y_3)^2}, \frac{z_2}{\sqrt{(x_3)^2-(y_3)^2}}\Big).
\end{gathered} \tag{3.15}$$

The method is the same as that used for $\mathbf{S}^2\!\times\!\mathbf{R}$ case to determine angle sum $\sum_{i=1}^3(\omega_i)$ of the interior angles of geodesic triangles $A_1A_2A_3$ (see Fig. 7, 9). We have seen that ω_1, the angle of geodesic curves with common point at the vertex A_1, is the same as the Euclidean one therefore it can be determined in usual Euclidean sense.

ω_i is equal to the angle $(g(A_i^i, A_1^i), g(A_i^i, A_j^i))\angle$ $(i,j = 2,3,\ i \neq j)$ (see Fig. 7, 9) where $g(A_i^i, A_1^i)$, $g(A_i^i, A_j^i)$ are oriented geodesic curves ($A_1 = A_2^2 = A_3^3$) and ω_1 is equal to the angle $(g(A_1,A_2), g(A_1,A_3))\angle$ where $g(A_1,A_2)$, $g(A_1,A_3)$ are also oriented geodesic curves. We denote the oriented unit tangent vectors of the oriented geodesic curves $g(A_1, A_i^j)$ with $\mathbf{t}_i^j$ where $(i,j) \in \{(1,3),(1,2),(2,3),(3,2),(3,0),(2,0)\}$ and $A_3^0 = A_3$, $A_2^0 = A_2$. The Euclidean coordinates of $\mathbf{t}_i^j$ coincide with the coordinates in (3.8) (see Section 2.2). In order to obtain the angle of two geodesic curves $g(A_1, A_i^j)$ and $g(A_1, A_k^l)$ ($(i,j) \neq (k,l)$; $(i,j),(k,l) \in \{(1,3),(1,2),\ (2,3),(3,2),(3,0),(2,0)\}$) intersected at the vertex A_1 we need to determine their tangent vectors $\mathbf{t}_s^r$ ($(s,r) \in \{(1,3),(1,2),\ (2,3),(3,2),(3,0),(2,0)\}$)

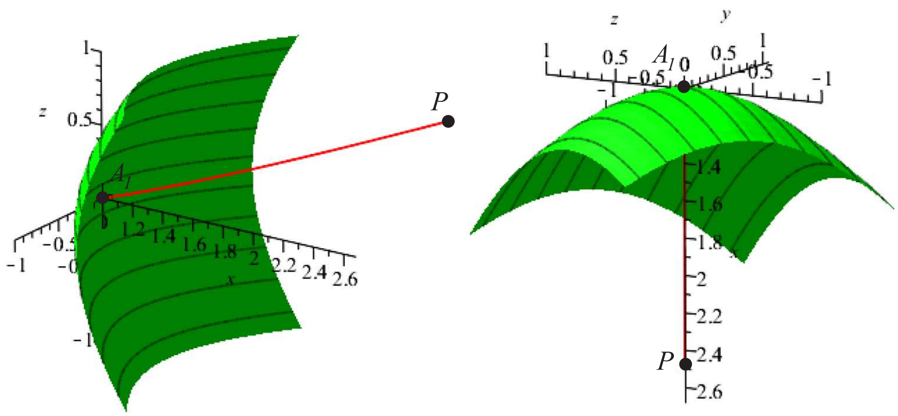

Figure 8. Geodesic curve $g(A_1, P)$ ($A_1 = (1, 1, 0, 0)$ and $P \in \mathbf{H}^2 \times \mathbf{R}$) with "base plane" (the "upper" sheet of the two-sheeted hyperboloid), the plane of a geodesic curve contains the origin $E_0 = (1, 0, 0, 0)$ of the model.

(see (2.10) and (3.8)) at their starting point A_1. From (3.8) it follows that a tangent vector at the origin is given by the parameters u and v of the corresponding geodesic curve (see (2.10)), which can be determined from the homogeneous coordinates of the endpoint of the geodesic curve as the following Lemma shows:

Lemma 3. *Let* $(1, x, y, z)$ $(x, y, z \in \mathbf{R}, x^2 - y^2 - z^2 \geq 0,\ x \geq 0)$ *be the homogeneous coordinates of the point* $P \in \mathbf{H}^2 \times \mathbf{R}$*. The paramerters of the corresponding geodesic curve* $g(A_1, P)$ *are the following:*

1. $y, z \in \mathbf{R} \setminus \{0\}$ *and* $x^2 - y^2 - z^2 \neq 1$*;*

$$v = \arctan\Big(\frac{\log\sqrt{x^2 - y^2 - z^2}}{\operatorname{arccosh}\frac{x}{\sqrt{x^2-y^2-z^2}}}\Big),\ u = \arctan\Big(\frac{z}{y}\Big),$$
$$\tau = \frac{\log\sqrt{x^2 - y^2 - z^2}}{\sin v},\ \textit{where } -\pi < u \leq \pi,\ -\pi/2 \leq v \leq \pi/2,\ \tau \in \mathbf{R}^+. \tag{3.16}$$

2. $y = 0,\ z \neq 0$ *and* $x^2 - z^2 \neq 1$*;*

$$u = \frac{\pi}{2},\ v = \arctan\Big(\frac{\log\sqrt{x^2 - z^2}}{\operatorname{arccosh}\frac{x}{\sqrt{x^2-z^2}}}\Big),$$
$$\tau = \frac{\log\sqrt{x^2 - z^2}}{\sin v},\ \textit{where } -\pi/2 \leq v \leq \pi/2,\ \tau \in \mathbf{R}^+. \tag{3.17}$$

3. $y = 0,\ z \neq 0$ *and* $x^2 - z^2 = 1$*;*

$$u = \frac{\pi}{2},\ v = 0,\ \tau = \operatorname{arccosh}(x),\ \tau \in \mathbf{R}^+. \tag{3.18}$$

4. $y, z = 0$;

$$u = 0,\ v = \frac{\pi}{2},\ \tau = \log(x),\ \tau \in \mathbf{R}^+. \tag{3.19}$$

□

We obtain directly from the (2.10) equations of the geodesic curves the following

Lemma 4. *Let P be an arbitrary point and $g(A_1, P)$ ($A_1 = (1,1,0,0)$) is a geodesic curve in the considered model of $\mathbf{H}^2\times\mathbf{R}$ geometry. The points of the geodesic curve $g(A_1, P)$ and the centre of the model E_0 lie in a plane in Euclidean sense (see Fig. 8).* □

The proof of the next theorem essentially is the same as the proof of Theorem 1.

Theorem 4. *If the Euclidean plane of the vertices of a $\mathbf{H}^2\times\mathbf{R}$ geodesic triangle $A_1A_2A_3$ contains the centre of model E_0 then its interior angle sum is equal to π (see Fig. 9).* □

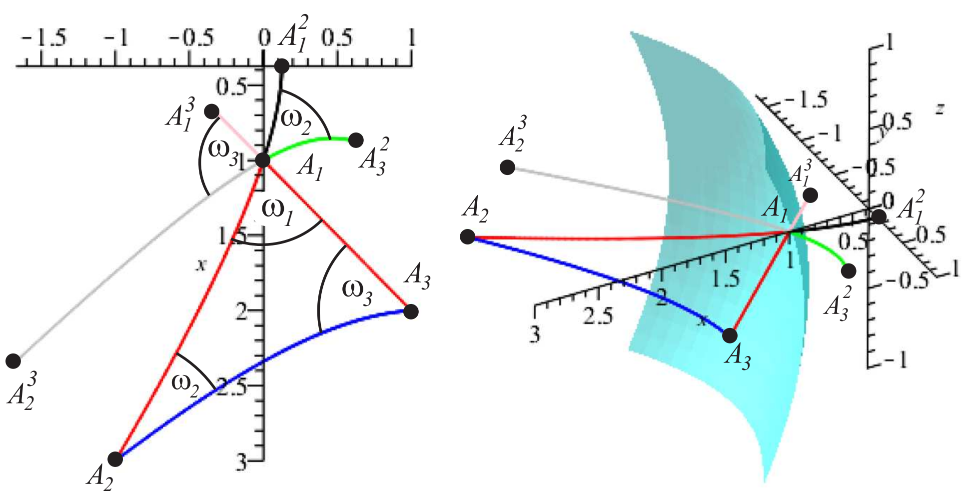

Figure 9. Geodesic triangle with vertices $A_1 = (1,1,0,0)$, $A_2 = (1,2,3/2,1)$, $A_3 = (1,3,-1,0)$ in $\mathbf{H}^2\times\mathbf{R}$ geometry, and transformed images of its geodesic side segments. The geodesic curve segments $g(A_1, A_2)$, $g(A_2, A_3)$, $g(A_3, A_1)$ lie on the coordinate plane $[x, y]$ and the interior angle sum of this geodesic triangle is $\sum_{i=1}^{3}(\omega_i) = \pi$.

We can determine the interior angle sum of arbitrary $\mathbf{H}^2\times\mathbf{R}$ geodesic triangle. In the following table we summarize some numerical data of interior angles of given geodesic triangles:

Table 2: $A_1 = (1,0,0,0)$, $A_2 = (1,2,3/2,1)$				
A_3	ω_1	ω_2	ω_3	$\sum_{i=1}^{3}(\omega_i)$
$(1, 3/\sqrt{8}, -1/\sqrt{8}, 0)$	2.54659	0.06953	0.41780	3.03392
$(1, 3, -1, 0)$	1.93230	0.49280	0.69816	3.12325
$(1, 6, -2, 0)$	1.83102	0.71611	0.58348	3.13061
$(1, 9, -3, 0)$	1.80083	0.81224	0.51964	3.13270
$(1, 3000, -1000, 0)$	1.70394	1.25735	0.17793	3.13922

By the above experiences and computations we obtain the following

Theorem 5. *If the Euclidean plane of the vertices of a* $\mathbf{H}^2\times\mathbf{R}$ *geodesic triangle* $A_1A_2A_3$ *does not contain the centre of model* E_0 *then its interior angle sum is less than* π.

Proof: The proof is similar to the $\mathbf{S}^2\times\mathbf{R}$ case.

We can assume without loss of generality that the vertices A_1, A_2 of such a geodesic triangle lie in the $[x, y]$ plane of the model. Using Lemma 4 we get that the geodesic segment A_iA_j $((i,j) \in \{(1,2),(1,3),2,3)\})$ is contained in the $A_iA_jE_0$ plane, therefore the sides of triangle $A_1A_2A_3$ lie on the boundary of trihedron given by the points E_0, A_1, A_2, A_3. It is clear that all types of geodesic triangles can be described by such a triangle. Therefore, it is sufficient to investigate the interior angle sums of geodesic triangles where we fix two of the vertices, e.g. A_1 and A_2, and move the third vertex A_3 on the half straight line E_0A_3 with starting point $E_0 \neq A_3(t)$.

Remark 6. It is well known that if the vertices A_1, A_2, A_3 lie in an "upper" sheet of the two-sheeted hyperboloid (in the hyperboloid model of the hyperbolic plane geometry where the straight lines of hyperbolic 2-space are modeled by geodesics on the hyperboloid) centred at E_0 then the interior angle sum of hyperbolic triangle $A_1A_2A_3$ is less than π.

Let $\Delta(t)$ $(t \in \mathbf{R}^+)$ denote the above geodesic triangle with *interior angles* $\omega_i(t)$ at the vertex A_i $(i \in \{1,2,3\})$.

The interior angle sum function $S(\Delta^{\mathbf{H}^2\times\mathbf{R}}(t)) = \sum_{i=1}^{3}(\omega_i(t))$ can be determined relative to the parameters $x_2, y_2, z_2, x_3, y_3 \in \mathbf{R}$ by the formulas (2.10), (3.14), (3.15) and by Lemma 3. Analyzing the above complicated continuous functions of single real variable t we get that its maximum is achieved at a point $t_0 \in (0,\infty)$ depending on given parameters. Moreover, $S(\Delta^{\mathbf{H}^2\times\mathbf{R}}(t))$ is stricly increasing on the interval $(0, t_0)$, stricly decreasing on the interval (t_0, ∞) and

$$\lim_{t\to 0} S(\Delta^{\mathbf{H}^2\times\mathbf{R}}(t)) = \pi, \qquad \lim_{t\to\infty} S(\Delta^{\mathbf{H}^2\times\mathbf{R}}(t)) = \pi.$$

In Fig. 10 we described the $S(\Delta(t))$ function related to geodesic triangle $\Delta(t)$ $(t \in (0,5))$ with vertices $A_1 = (1,1,0,0)$, $A_2 = (1,2,3/2,1)$, $A_3 = (1, 3\cdot t, -1\cdot t, 0)$. Its minimum is achieved at $t_0 \approx 0.36392$ where $S(\Delta^{\mathbf{H}^2\times\mathbf{R}}(t_0)) \approx 3.03236$. □

Finally we obtain the following

Theorem 6. *The sum of the interior angles of a geodesic triangle of* $\mathbf{H}^2\times\mathbf{R}$ *space is less than or equal to* π. □

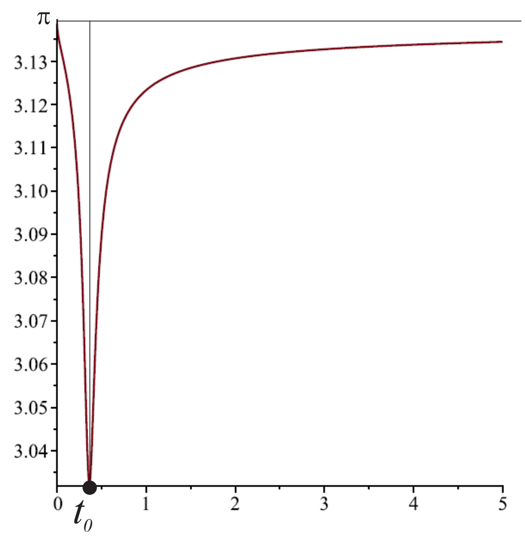

Figure 10. $S(\Delta^{\mathbf{H}^2\times\mathbf{R}}(t))$ function related to parameters $x_2 = 2, y_2 = 3/1, z_2 = 1$ $x_3 = 3 \cdot t, y_3 = -1 \cdot t, z_3 = 0$.

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