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Isohedral tilings for hyperbolic translation group of genus two which admit additional isometries

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Abstract. The paper continues articles [3, 4] where we derived all the Delone classes of isohedral tilings of the hyperbolic plane with disks for the translation group of genus two. The total number of these Delone classes is corrected to 118. The attention is drawn to the parametric diversity of these tilings. The emphasis is made on more symmetric tilings that are promising for further research and application. Isohedral tilings that admit additional isometries are analyzed, their adjacency diagrams are shown in figures.

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1 Introduction

Hyperbolic two-dimensional discrete isometry groups with compact fundamental domains were classified by A. M. Macbeath [1], their number is infinite. Further J. H. Conway [2] proposed for the notation of such groups an orbifold symbol, which is equivalent to Macbeath's symbol and is shorter. Any hyperbolic plane translation group is characterized by its genus and has the orbifold symbol by Conway $\circ\circ\cdots\circ$ where the number of circles is equal to the genus.

In the author's works [3, 4] all Delone classes of isohedral tilings of the hyperbolic plane with disks were derived for the translation group of genus two, with the orbifold symbol $\circ\circ$. Further these results can be used in several directions. Firstly they can be applied for obtaining some body-transitive tilings of the three-dimensional hyperbolic space. Secondly the factorization of the hyperbolic plane by this translation group yields Riemann surfaces of genus two with different metrics. A Riemann surface also can be obtained from a fundamental domain by gluing its boundary with isometries of the group. The most symmetric Riemann surfaces are known as regular maps [5].

Taking in consideration the above mentioned, in the present paper we are going to describe the isohedral tilings for the hyperbolic translation group of genus two in more details. In our investigation we basically adhere to ideas and methods which B. N. Delone (and coauthors) used for obtaining isohedral tilings of the Euclidean plane in [6, 7]. We consider tilings with disks as in the monograph [8], which is a more general setting. The number of the obtained Delone classes is corrected.

Remark that the method of obtaining fundamental isohedral tilings of the hyperbolic plane with compact fundamental domains developed by Z. Lučić and E. Molnár [9, 10] gives only a part of isohedral tilings for a hyperbolic translation group. Here we will compare some results obtained by application of the two methods. Also there exists one more method, based on Delauney–Dress symbols, of obtaining k-isohedral tilings of all three 2-dimensional spaces of constant curvature, which was developed by D. H. Huson (and coauthor) in [11, 12].

We are going to discuss parameters of the polygonal tiles and conditions the tiles must obey. Tilings that admit additional isometries are more appropriate for applications. We will draw attention to such tilings and will describe them more detailed.

2 Basic concepts and methods

A set W of closed topological disks in the plane is called a tiling of the plane if every point of the plane belongs to at least a disk and no two disks have an inner point in common. The disks of a tiling are called tiles.

A non-empty component of the intersection of two or more different tiles is called a vertex of the tiling or an edge of the tiling depending on whether it is a single point or not (then it is a curve). The boundary of a tile is divided by vertices of the tiling into curves that are edges of the tiling, so any tile may be considered as a curvilinear polygon.

Definition 1. Let W be a tiling of the hyperbolic plane with disks, G be a discrete isometry group of the hyperbolic plane with a compact (bounded) fundamental domain. The tiling W is called isohedral with respect to the group G if the group G maps the tiling W onto itself and G acts transitively on the set of the tiles.

Definition 2. Consider all possible pairs (W, G) where W is a tiling of the hyperbolic plane with disks which is isohedral with respect to a discrete hyperbolic isometry group G with a bounded fundamental domain. Two pairs (W, G) and (W', G') are said to belong to the same Delone class if there exists homeomorphic transformation φ of the plane which maps the tiling W onto the tiling W' and the relation $G = \varphi^{-1} G' \varphi$ holds.

Homeomeric types in [8] and equivariantly equivalent tilings in [11] are other terms used with the same meaning.

A Delone class (W, G) is called fundamental if the group G acts simply transitively (one time transitively) on the set of tiles of W. Any translation group admits only fundamental Delone classes (or fundamental tilings).

In articles [3, 4] we solved the problem of finding all fundamental Delone classes of isohedral tilings of the hyperbolic plane with disks for the translation group of genus two.

We followed the scheme of work [6]. First we solved Diophantine equations obtained from Euler theorem and get possible sets of valencies. We used the fact that our isometry group is a group of translations and therefore its orbifold is a manifold without boundary and without singular points. Then we formed ordered cycles of valencies $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, which are the same for each tile in an isohedral tiling.

Now we describe shortly the method of adjacency symbols proposed in [6] (see also [7] or [13]). Let (W, G) be a fundamental isohedral tiling of the plane with disks. Choose a tile and label consecutively all its edges with letters a, b,.... Then apply the group G to the labeled tile resulting in all the tiles being labeled. Now we form an adjacency symbol as follows. The letter which labels the first chosen edge stands first, the letter which labels the adjacent edge of the neighbor tile stands next, then the lower index indicates the valency of the end vertex of the first edge, after that we pass to the second consecutive edge, and so on. If the orientations of the initial and neighbor tiles are opposite, it is indicated with a bar over the second letter. In our case we deal only with translations, they preserve orientation, and no bars are needed.

An adjacency diagram is a polygonal tile where the vertices are labeled with their valencies and the paired edges are connected with arcs.

We generate all possible adjacency symbols for each appropriate equivalence class of ordered cycles. For each candidate in adjacency symbol we check if the condition of transition around a vertex is satisfied, for every vertex equivalence class. Further we must choose one representative among equivalent adjacency simbols. It is adjacency diagrams that help us to determine visually whether two adjacent symbols correspond to the same Delone class.

3 Correction of the number of all Delone classes and comparison of results obtained by two methods

Examine two Delone classes of isohedral tilings of the hyperbolic plane with 16-gons denoted in [4] as $16A14, 1$ and $16A14, 2$. Their adjacency symbols are $(ae_3bl_3cg_3dm_3ea_4fn_3gc_3hk_3io_4jp_3kh_3lb_3md_3nf_4oi_3pj_4)$ and, respectively, $(ai_3bk_3cm_3dg_3eo_4fp_3gd_3hl_3ia_4jn_3kb_3lh_3mc_3nj_4oe_3pf_4)$. Their adjacency diagrams are shown in Fig. 1. A given adjacency symbol can be obtained from the respective adjacency diagram as follows: the letter a labels the right bottom edge and we go round counter-clockwise.

In the right diagram a straight line is drawn connecting the left bottom vertex of valency 3 with the opposite vertex. Applying the reflection in this line to the right diagram, which corresponds to Delone class $16A14, 2$, we obtain just the left diagram, which corresponds to Delone class $16A14, 1$. It implies that two diagrams as well as two adjacency symbols determine the same Delone class and we denote this Delone class by $16A14$.

Thus we have shown that Delone classes $16A14, 1$ and $16A14, 2$ from the article [4] coincide. Now we can give the corrected formulation of Theorem 1 as follows.

Theorem 1. *For translation group of genus two there exist 118 Delone classes of*

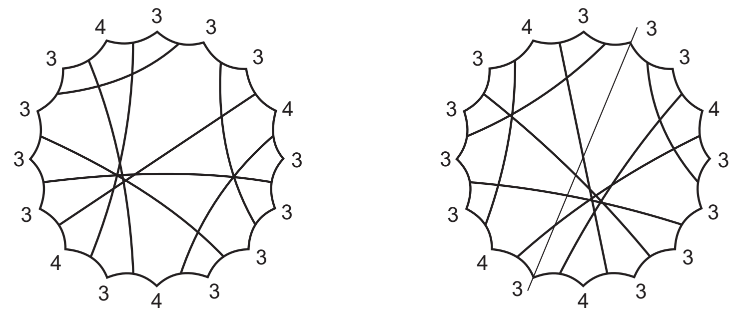


Figure 1. Two adjacency diagrams corresponding to the Delone class of isohedral tilings $16A14$

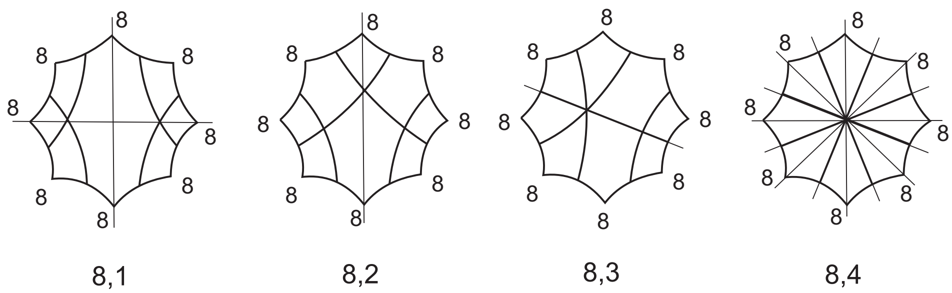


Figure 2. Adjacency diagrams for the Delone classes of isohedral tilings $8,1$, $8,2$, $8,3$, and $8,4$, with reflection axes

isohedral tilings of the hyperbolic plane with disks, and namely 4 classes with 8-gons, 18 classes with 10-gons, 31 classes with 12-gons, 39 classes with 14-gons, 20 classes with 16-gons, and 6 classes with 18-gons.

The results for 8-, 10- and 12-gons have been described in [3], the results for 14-, 16- and 18-gons have been described in [4] and the present article. The adjacency symbols have been given for all 118 Delone classes, the adjacency diagrams have been depicted for a part of them. An adjacency symbol contains information both on generators and relations of the group, it fully determines a Delone class and allows us to restore a tiling.

Now compare our results with results which can be obtained using the method developed in [9, 10]. Examine the case of the minimal number of edges (and vertices) of a fundamental domain for group $\circ\circ$, then the minimal number is 8. In this case the only fundamental domain can be obtained by cutting along the canonical graph with two loops, which gives an 8-gon with canonical pairings. It determines a Delone class that we denoted in [3] by $8,1$ (Fig. 2). However using our method we obtained 3 more Delone classes denoted by $8,2$, $8,3$ and $8,4$ (Fig. 2), the last one being well known (see [5]). Thus the methods [9, 10] yield only a part of possible Delone classes of isohedral tilings of the hyperbolic plane with disks for the translation group of genus two.

4 Parameters and conditions for polygonal tiles

We discuss the 'freedom' of a hyperbolic isometry group and its fundamental domains. The parameter space of a hyperbolic isometry group corresponds to the Teichmüller space of the group, which has the dimension $n = 6g-6+t+2r$, where g is the genus of the orbifold, t is the number of boundary components, r is the number of rotation centers [14]. According to the formula, the hyperbolic translation group of genus two $\circ\circ$ has the parameter space of dimension 6.

We state our results in the most general form, however tilings with convex polygons seem to be more interestig for further research and application. Assuming that isohedral tilings of the hyperbolic plane with convex polygons are considered, we determine which metrical conditions tiles and tilings must satisfy. These conditions are obtained from the facts that adjacent edges have the same length and the sum of angles at a vertex in the hyperbolic plane is equal to 2π. The pairs of adjacent edges appear directly in adjacent symbols. Equivalent vertices can be obtained using adjacency diagrams.

We begin with tilings with 8-gons, which belong to 4 Delone classes (Fig. 2). The sum of all 8 angles of a tile is equal to 2π, for each of 4 classes. For the Delone class $8,1$ with the adjacency symbol $(ac_8bd_8ca_8db_8eg_8fh_8ge_8hf_8)$, the conditions on edges are $a=c$, $b=d$, $e=g$, $f=h$. For the Delone class $8,2$ with the adjacency symbol $(ac_8be_8ca_8dg_8eb_8fh_8gd_8hf_8)$, the conditions on edges are $a=c$, $b=e$, $d=g$, $f=h$. For the Delone class $8,3$ with the adjacency symbol $(ac_8bf_8ca_8dg_8eh_8fb_8gd_8he_8)$, the conditions on edges are $a=c$, $b=f$, $d=g$, $e=h$. For the Delone class $8,4$ with the adjacency symbol $(ae_8bf_8cg_8dh_8ea_8fb_8gc_8hd_8)$, the conditions on edges are $a=e$, $b=f$, $c=g$, $d=h$. Particular cases of $8,1$ and $8,4$ are known with regular 8-gons, i.e. with each angle being equal to $\pi/4$ and the conditions on edges being $a=b=c=d=e=f=g=h$. However some less symmetric variants of these Delone classes can also be realized due to different choices of parameters of the group. Besides, with parameters of the group being fixed, there exists the choice of a fundamental domain.

For tilings by 10-gons with the set of valencies $10A : 3337777777$, all 6 Delone classes of isohedral tilings of the hyperbolic plane with disks are given both with their adjacency symbols and adjacency diagrams in [3]. For each of the Delone classes $10A1$, $10A2$, $10A3$, $10A4,1$, $10A4,2$, and $10A4,3$, the sum of 3 angles of valency 3 is equal to 2π, as well as the sum of 7 angles of valency 7 is equal to 2π. The pairs of edges with equal length can easily be read from their adjacency symbols.

For the set of valencies $10B : 4444666666$, the 6 Delone classes $10B1$, $10B2$, $10B3$, $10B4,1$, $10B4,2$, and $10B5$ are given with adjacency symbols in [3]. Adjacency diagrams can easily be obtained from the corresponding adjacency symbols. For each of these Delone classes, both the sum of 4 angles of valency 4 and the sum of 6 angles of valency 6 are equal to 2π. All the pairs of edges with equal length can be taken from their adjacency symbols.

For the set of valencies $10C : 5555555555$, the 6 Delone classes $10C,1$, $10C,2$, $10C,3$, $10C,4$, $10C,5$, and $10C,6$ are given with adjacency symbols in [3]. For each of these Delone classes, 10 vertices fall into 2 equivalence classes, and the sum of 5 angles of valency 5 at vertices of the same equivalence class is equal to 2π. All the pairs of edges with equal length can be taken from the adjacency symbols.

For tilings by 12-gons with the set of valencies $12A$: 333333666666, 12 Delone classes of isohedral tilings are given with their adjacency symbols in [3]. For each Delone class, 6 vertices of valency 3 fall into 2 equivalence classes and all 6 vertices of valency 6 belong to one equivalence class. So the sums of 2 triples of angles at equivalent vertices of valency 3 and the sum of all 6 angles of valency 6 are equal to 2π. All the pairs of edges with equal length can be seen in adjacency symbols.

For the set of valencies $12B$: 333444455555, 13 Delone classes are given with their adjacency symbols in [3]. For each Delone class, the sum of 3 angles of valency 3 is equal to 2π, the sum of 4 angles of valency 4 is equal to 2π and the sum of 5 angles of valency 5 is equal to 2π. All the pairs of edges with equal length can be seen in adjacency symbols.

For the set of valencies $12C$: 444444444444, 6 Delone classes are given with their adjacency symbols in [3]. For each Delone class, all 12 vertices of valency 4 fall into 3 equivalence classes. So the 3 sums of 4 angles at equivalent vertices are equal to 2π. All the pairs of edges with equal length can be seen in adjacency symbols.

For tilings by 14-gons with the set of valencies $14A$: 33333333355555, 13 Delone classes of isohedral tilings are given with their adjacency symbols in [4]. For each Delone class, 9 vertices of valency 3 fall into 3 equivalence classes and all 5 vertices of valency 5 belong to one equivalence class. So the sums of 3 triples of angles at equivalent vertices of valency 3 and the sum of all 5 angles of valency 5 are equal to 2π. All the pairs of edges with equal length can be taken from adjacency symbols.

For the set of valencies $14B$: 33333344444444, 26 Delone classes are given with their adjacency symbols in [4]. For each Delone class, 6 vertices of valency 3 fall into 2 equivalence classes, as well as 8 vertices of valency 4 fall into 2 equivalence classes. So the two sums of 3 angles at equivalent vertices of valency 3 and the two sums of 4 angles at equivalent vertices of valency 4 are equal to 2π. All the pairs of edges with equal length can be taken from adjacency symbols.

For tilings by 16-gons with the set of valencies $16A$: 3333333333334444, where the letter A is used for convenience, 20 Delone classes of isohedral tilings are given with their adjacency symbols in [4]. For each Delone class, 12 vertices of valency 3 fall into 4 equivalence classes and all 4 vertices of valency 4 belong to one equivalence class. So the 4 sums of 3 angles at equivalent vertices of valency 3 and the sum of all 4 angles of valency 4 are equal to 2π. All the pairs of edges with equal length can be taken from adjacency symbols.

For tilings by 18-gons with the set of valencies 18 : 333333333333333333, 6 Delone classes of isohedral tilings are given with their adjacency symbols in [4]. For each Delone class, all 18 vertices of valency 3 fall into 6 equivalence classes. So the sums of 6 triples of angles at equivalent vertices are equal to 2π. All the pairs of edges with equal length can be taken from adjacency symbols.

5 Tiles admitting additional isometries

More symmetric tilings are of special interest for applications. Here we analyze adjacency diagrams and determine which tiles can admit additional isometries with taking into account their pairing isometries. It will be the first step in the direction of finding symmetric tilings among our list of tilings.

A thorough examination has been done yielding the following results. First we enumerate tilings where a tile can admit an isometry group (in prospect a stabilizer) of order greater than 2, with indication of adjacency diagrams. Also we list tilings where a tile can admit an isometry group of order 2. Each isohedral tiling is given with its adjacency symbol.

For tilings with 8-gons, their adjacency symbols have been given in Section 4. Each of the 4 Delone classes admits additional isometries if suitable parameters are chosen. For the Delone class $8,1$ a tile admits the isometry group of order 4 generated by a rotation of order 2 and a reflection, with suitable choice of parameters. For the Delone class $8,4$ a tile admits the isometry group of order 16 generated by a rotation of order 8 and a reflection if the tile is a regular 8-gon (with angle of $\pi/4$). For both Delone classes $8,2$ and $8,3$ tiles admit the isometry group of order 2 generated by respective reflections, with suitable choice of parameters. Adjacency diagrams for all the 4 Delone classes are given in Fig. 2, with the indication of admissible reflection axes.

Among tilings by 10-gons with the set $10A$, the 5 Delone classes of isohedal tilings $10A1$ with the adjacency symbol $(ad_3be_7cf_3da_7eb_3fc_7gi_7hj_7ig_7jh_7)$, $10A3$ with symbol $(ad_3bg_7ch_3da_7ei_7fj_7gb_3hc_7ie_7jf_7)$, $10A4,1$ with the adjacency symbol $(ae_3bg_7cj_7dh_3ea_7fi_7gb_3hd_7if_7jc_7)$, $10A4,2$ with the adjacency symbol $(ah_3bd_7ci_7db_3eg_7fj_7ge_3ha_7ic_7jf_7)$, and $10A4,3$ with the adjacency symbol $(ah_3bd_7cj_7db_3eg_7fi_7ge_3ha_7if_7jc_7)$ admit tiles with the isometry group of order 2 generated by respective reflection, for suitable choice of parameters.

Among tilings with the set $10B$, the Delone class $10B3$ with the adjacency symbol $(ac_4bg_4ca_6di_6ej_6fh_4gb_4hf_6id_6je_6)$ admits tiles with the isometry group of order 4 generated by a rotation of order 2 and a reflection if suitable parameters are chosen (Fig. 3). For the 4 Delone classes $10B1$ with the adjacency symbol $(ac_4be_4ca_6df_4eb_4fd_6gi_6hj_6ig_6jh_6)$, $10B2$, which has the adjacency symbol $(ac_4bf_4ca_6di_6eg_4fb_4ge_6hj_6id_6jh_6)$, $10B4,2$ with the adjacency symbol $(ad_4bh_6cf_4da_6ei_4fc_6gj_6hb_4ie_6jg_6)$, and $10B5$ with the adjacency symbol $(ag_4bh_6ci_4df_6ej_6fd_4ga_6hb_4ic_6je_6)$, tiles admit the isometry group of order 2 generated by respective reflection, for suitable choice of parameters.

Among tilings with the set $10C$, the 2 Delone classes $10C,1$ with the adjacency symbol $(ac_5bd_5ca_5db_5ej_5fh_5gi_5hf_5ig_5je_5)$ and $10C,4$ with the adjacency symbol $(ad_5be_5ch_5da_5eb_5fi_5gj_5hc_5if_5jg_5)$ admit tiles with the isometry group of order 4 generated by a rotation of order 2 and a reflecton, with suitable choice of parameters (Fig. 3). For the Delone class $10C,5$ with symbol $(ad_5bi_5cf_5da_5eh_5fc_5gj_5he_5ib_5jg_5)$ a tile admits the isometry group of order 10 generated by a rotation of order 5 and a reflection if the tile is a regular 10-gon (with angle of $2\pi/5$) (Fig. 3). For the Delone

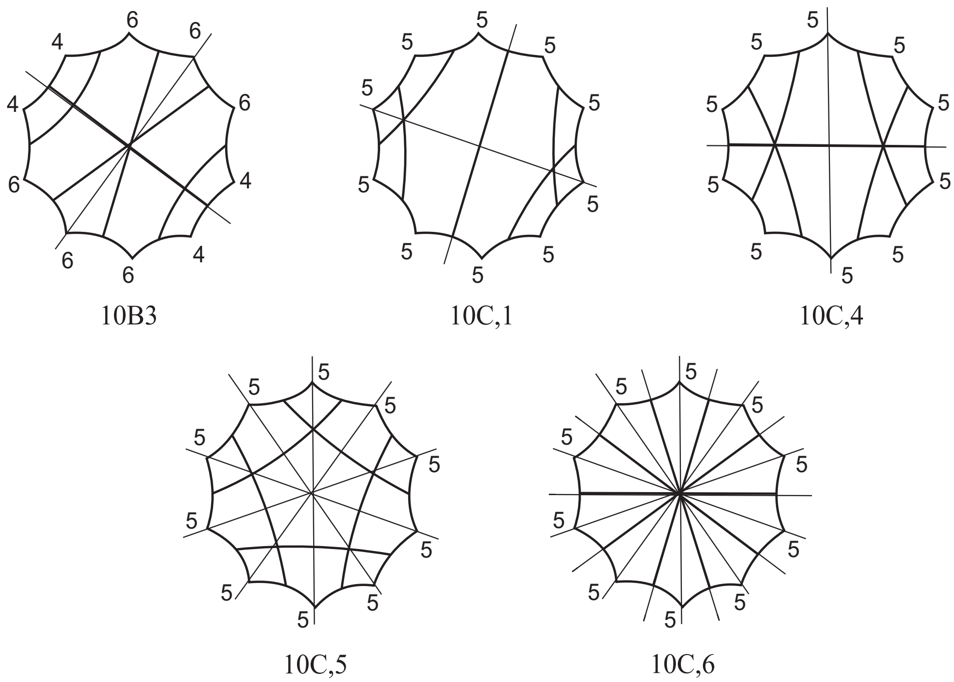


Figure 3. Adjacency diagrams for the Delone classes of isohedral tilings $10B3$, $10C,1$, $10C,4$, $10C,5$, and $10C,6$, with reflection axes

class $10C,6$ with symbol $(af_5bg_5ch_5di_5ej_5fa_5gb_5hc_5id_5je_5)$ a tile admits the isometry group of order 20 generated by a rotation of order 10 and a reflection if the tile also is a regular 10-gon (with angle of $2\pi/5$) (Fig. 3). The Delone class $10C,3$ with symbol $(ac_5bf_5ca_5dh_5ej_5fb_5gi_5hd_5ig_5je_5)$ admits a tile with the isometry group of order 2 generated by a reflection, for suitable choice of parameters.

Among tilings by 12-gons with the set $12A$, the 2 Delone classes $12A3$ with the adjacency symbol $(ad_3bh_3ci_3da_6ek_6fl_6gj_3hb_3ic_3jg_6ke_6lf_6)$ and $12A9,1$ with symbol $(ad_3be_6cf_3da_6eb_3fc_6gj_3hk_6il_3jg_6kh_3li_6)$ admit tiles with the isometry group of order 4 generated by a rotation of order 2 and a reflection, with suitable choice of parameters (Fig. 4). For the Delone class $12A9,4$ with the adjacency symbol $(aj_3be_6cl_3dg_6eb_3fi_6gd_3hk_6if_3ja_6kh_3lc_6)$ a tile admits the isometry group of order 12 generated by a rotation of order 6 and a reflection if the tile is a semiregular 12-gon with alternating angles of $\pi/3$ and $2\pi/3$ (Fig. 4). For the 6 Delone classes $12A1$ with symbol $(ad_3bf_3cg_3da_6eh_3fb_3gc_3he_6ik_6jl_6ki_6lj_6)$, $12A2$ with the adjacency symbol $(ad_3bg_3ch_3da_6ek_6fi_3gb_3hc_3if_6jl_6ke_6lj_6)$, $12A6$ with the adjacency symbol $(ae_3bg_3cj_6dh_3ea_6fk_3gb_3hd_6il_6jc_3kf_6li_6)$, $12A8$ with the adjacency symbol $(ae_3bh_3ck_6di_3ea_6fj_6gl_3hb_3id_6jf_6kc_3lg_6)$, $12A9,2$ with the adjacency symbol $(ad_3bg_6ch_3da_6ej_3fk_6gb_3hc_6il_3je_6kf_3li_6)$, and $12A9,3$ with the adjacency symbol $(aj_3be_6ch_3dk_6eb_3fi_6gl_3hc_6if_3ja_6kd_3lg_6)$, tiles admit the isometry group of order 2 generated by respective reflection, with suitable choice of parameters.

Among tilings with the set $12B$, the 7 Delone classes $12B1$ with the adja-

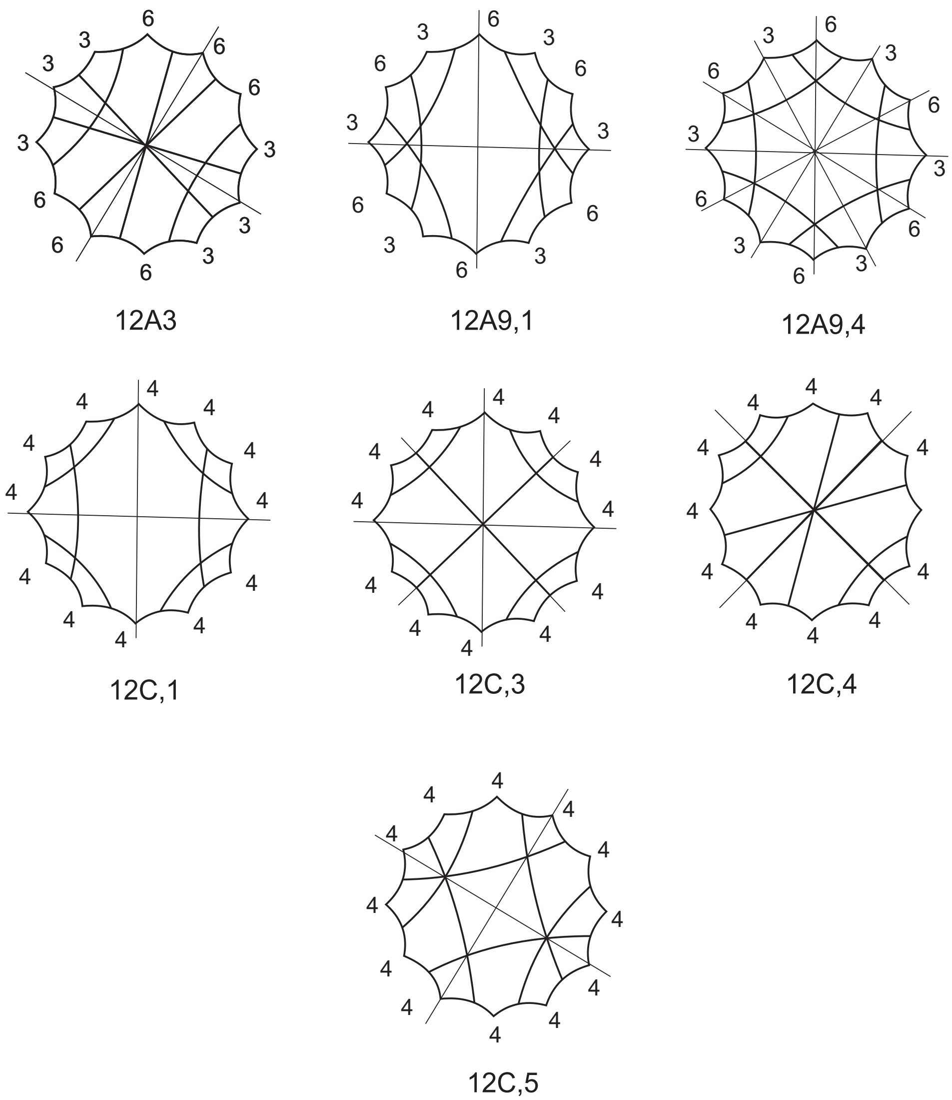


Figure 4. Adjacency diagrams for the Delone classes of isohedral tilings $12A3$, $12A9,1$, $12A9,4$, $12C,1$, $12C,3$, $12C,4$, and $12C,5$, with reflection axes

cency symbol $(ad_3be_4cf_3da_4eb_3fc_4gl_5hj_5ik_5jh_5ki_5lg_4)$, $12B6$ with the adjacency symbol $(ae_3bg_4ck_5dh_3ea_5fj_4gb_3hd_5il_4jf_5kc_4li_5)$, $12B7$ with the adjacency symbol $(ae_3bi_4cf_5dj_3ea_5fc_4gk_5hl_4ib_3jd_5kg_4lh_5)$, $12B8,2$ with the adjacency symbol $(aj_3be_4ck_5dh_4eb_3fi_5gl_4hd_5if_3ja_5kc_4lg_5)$, $12B9$ with the adjacency symbol $(ae_3bk_4ch_5dl_3ea_5fi_4gj_5hc_4if_5jg_4kb_3ld_5)$, $12B11$ with the adjacency symbol $(ad_3be_5cf_3da_5eb_3fc_5gi_4hk_4ig_5jl_4kh_4lj_5)$, and $12B12$ with the adjacency symbol $(ad_3bh_5ci_3da_5eg_4fk_4ge_5hb_3ic_5jl_4kf_4lj_5)$ admit tiles with the isometry group of order 2 generated by respective reflection if suitable parameters are chosen.

Among tilings with the set $12C$, the 3 Delone classes $12C,1$ with the adjacency symbol $(ac_4be_4ca_4df_4eb_4fd_4gi_4hk_4ig_4jl_4kh_4lj_4)$, $12C,4$ with the adjacency symbol $(ac_4bh_4ca_4dj_4ek_4fl_4gi_4hb_4ig_4jd_4ke_4lf_4)$ and $12C,5$ with the adjacency symbol $(ad_4bf_4ck_4da_4ei_4fb_4gj_4hl_4ie_4jg_4kc_4lh_4)$ admit tiles with the isometry group of order 4 generated by a rotation of order 2 and a reflection if suitable parameters are chosen (Fig. 4). For the class $12C,3$ with symbol $(ac_4bh_4ca_4df_4ek_4fd_4gi_4hb_4ig_4jl_4ke_4lj_4)$ a tile admits the isometry group of order 8 generated by a rotation of order 4 and a reflection if suitable parameters are chosen (Fig. 4). For the 2 Delone classes $12C,2$ with the adjacency symbol $(ac_4bf_4ca_4dj_4eg_4fb_4ge_4hk_4il_4jd_4kh_4li_4)$ and $12C,6$ with symbol $(ad_4bi_4cf_4da_4ej_4fc_4gk_4hl_4ib_4je_4kg_4lh_4)$ a tile admits the isometry group of order 2 generated by respective reflection, with suitable choice of parameters.

Among tilings by 14-gons with the set $14A$, the 7 Delone classes $14A1$ with the adjacency symbol $(aj_3be_3cg_3dh_3eb_3fi_3gc_3hd_3if_3ja_5km_5ln_5mk_5nl_5)$, $14A6$ with symbol $(ad_3bf_3cg_3da_5eh_3fb_3gc_3he_5il_3jm_5kn_3li_5mj_3nk_5)$, $14A7$ with the adjacency symbol $(ad_3bh_3ci_3da_5el_3fm_5gj_3hb_3ic_3jg_5kn_3le_5mf_3nk_5)$, $14A10,1$ with adjacency symbol $(ai_3bm_3cf_3dj_5ek_3fc_3gl_5hn_3ia_5jd_3ke_5lg_3mb_3nh_5)$, $14A10,2$ with adjacency symbol $(ak_3bf_3cm_3dh_5en_3fb_3gj_5hd_3il_5jg_3ka_5li_3mc_3ne_5)$, $14A10,3$ with adjacency symbol $(ak_3bm_3cf_3dh_5ei_3fc_3gl_5hd_3ie_5jn_3ka_5lg_3mb_3nj_5)$, and $14A11$ with adjacency symbol $(af_3bh_3ck_3dm_5ei_3fa_5gl_3hb_3ie_5jn_3kc_3lg_5md_3nj_5)$ admit tiles with the isometry group of order 2 generated by respective reflection if suitable parameters are chosen.

Among tilings with the set $14B$, the 9 Delone classes $14B3,1$ with the adjacency symbol $(ad_3bi_3cj_3da_4eg_4fm_4ge_4hk_3ib_3jc_3kh_4ln_4mf_4nl_4)$, $14B3,2$ with symbol $(ad_3bi_3cj_3da_4el_4fm_4gn_4hk_3ib_3jc_3kh_4le_4mf_4ng_4)$, $14B10,1$ with the adjacency symbol $(af_3bi_3cl_4dg_4ej_3fa_4gd_4hm_3ib_3je_4kn_4lc_3mh_4nk_4)$, $14B10,2$ with the symbol $(af_3bi_3cl_4dn_4ej_3fa_4gk_4hm_3ib_3je_4kg_4lc_3mh_4nd_4)$, $14B10,3$ with adjacency symbol $(am_3bi_3ce_4dg_4ec_3fh_4gd_4hf_3ib_3jl_4kn_4lj_3ma_4nk_4)$, $14B10,4$ with the adjacency symbol $(am_3bi_3ce_4dk_4ec_3fh_4gn_4hf_3ib_3jl_4kd_4lj_3ma_4ng_4)$, $14B10,5$ with the symbol $(am_3bi_3ce_4dn_4ec_3fh_4gk_4hf_3ib_3jl_4kg_4lj_3ma_4nd_4)$, $14B13,1$ with adjacency symbol $(ad_3be_4cf_3da_4eb_3fc_4gn_4hk_3il_4jm_3kh_4li_3mj_4ng_4)$, and $14B13,5$ with the symbol $(ak_3be_4cm_3dh_4eb_3fj_4gn_4hd_3il_4jf_3ka_4li_3mc_4ng_4)$ admit tiles with the isometry group of order 4 generated by a rotation of order 2 and a reflection if suitable parameters are chosen (Fig. 5). For the 5 Delone classes $14B1$ with the adjacency symbol $(ad_3bf_3cg_3da_4eh_3fb_3gc_3he_4ik_4jm_4ki_4ln_4mj_4nl_4)$, $14B2$ with symbol $(ad_3bg_3ch_3da_4el_4fi_3gb_3hc_3if_4jm_4kn_4le_4mj_4nk_4)$, $14B11,2$ with adjacency symbol $(af_3bi_4cm_3dg_4ej_3fa_4gd_3hl_4ib_3je_4kn_4lh_3mc_4nk_4)$, $14B11,4$ with adjacency symbol $(am_3bi_4cf_3dg_4eh_3fc_4gd_3he_4ib_3jl_4kn_4lj_3ma_4nk_4)$, and $14B12$ with adjacency symbol

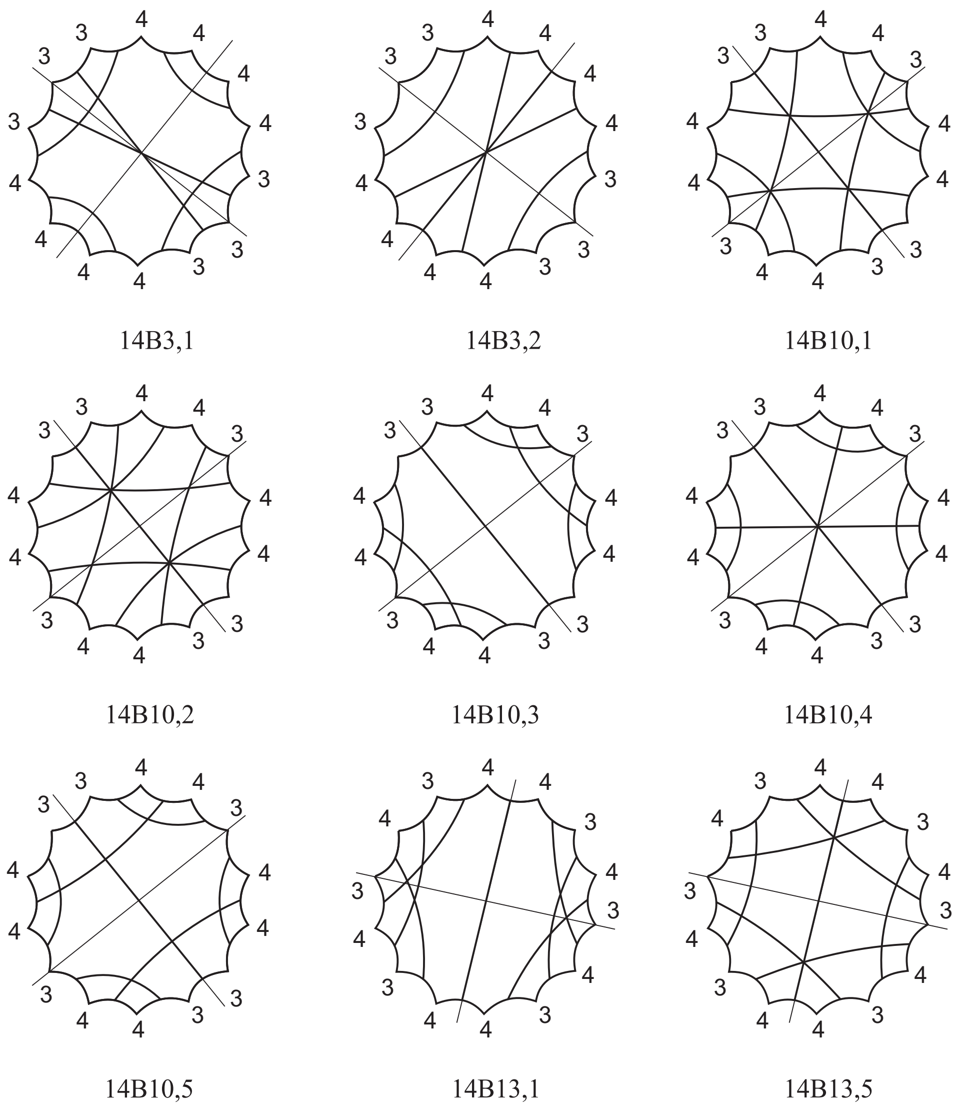
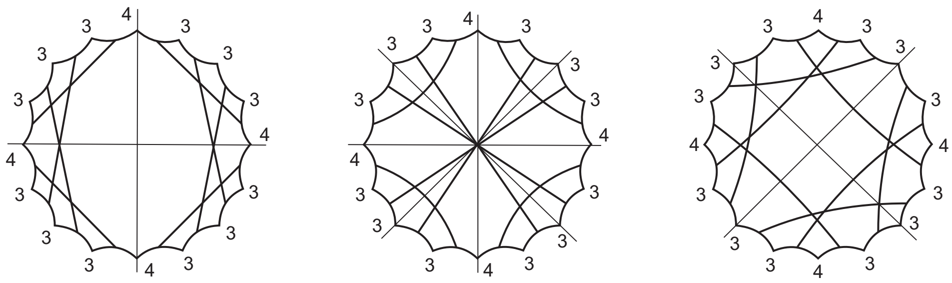


Figure 5. Adjacency diagrams for the Delone classes of isohedral tilings $14B3,1$, $14B3,2$, $14B10,1$, $14D10,2$, $14D10,3$, $14B10,4$, $14B10,5$, $14B13,1$, and $14B13,5$, with reflection axes



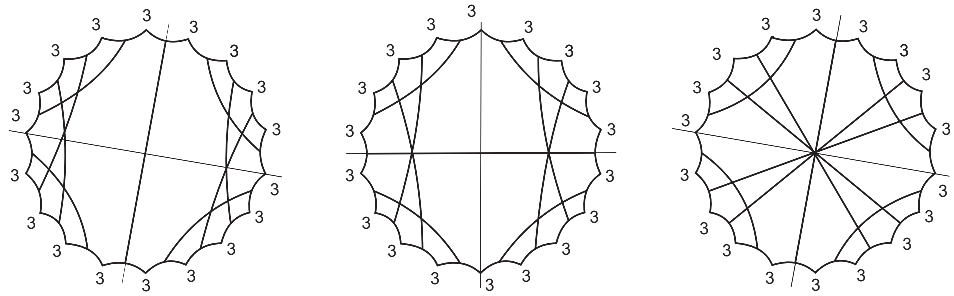
16A16,1 16A16,2 16A16,3

Figure 6. Adjacency diagrams for the Delone classes of isohedral tilings $16A16,1$, $16A16,2$ and $16A16,3$, with reflection axes

$(am_3be_4ck_3dg_4eb_3fl_4gd_3hj_4in_4jh_3kc_4lf_3ma_4ni_4)$, a tile admits the isometry group of order 2 generated by respective reflection, for suitable choice of parameters. For the Delone class $14B13,6$ with symbol $(ak_3bl_4cf_3dh_4ei_3fc_4gn_4hd_3ie_4jm_3ka_4lb_3mj_4ng_4)$ a tile admits the isometry group of order 2 generated by a rotation of order 2, with suitable choice of parameters.

Among tilings by 16-gons, the 2 Delone classes $16A16,1$ with the adjacency symbol $(ad_3bf_3cg_3da_4eh_3fb_3gc_3he_4il_3jn_3ko_3li_4mp_3nj_3ok_3pm_4)$ and $16A16,3$ with symbol $(al_3bf_3co_3di_4ep_3fb_3gk_3hm_4id_3jn_3kg_3la_4mh_3nj_3oc_3pe_4)$ admit tiles with the isometry group of order 4 generated by a rotation of order 2 and a reflection if suitable parameters are chosen (Fig. 6). For the Delone class $16A16,2$ with symbol $(ad_3bj_3ck_3da_4eh_3fn_3go_3he_4il_3jb_3kc_3li_4mp_3nf_3og_3pm_4)$ a tile admits the isometry group of order 8 generated by a rotation of order 4 and a reflection, with suitable choice of parameters (Fig. 6). For the 8 Delone classes $16A1$ with symbol $(ao_3bk_3cf_3dh_3ei_3fc_3gj_3hd_3ie_3jg_3kb_3ln_4mp_4nl_3oa_4pm_4)$, $16A2$ with the adjacency symbol $(ao_3be_3ch_3di_3eb_3fn_3gj_3hc_3id_3jg_3km_4lp_4mk_3nf_3oa_4pl_4)$, $16A3$ with symbol $(ao_3bf_3ch_3dm_3ei_3fb_3gn_3hc_3ie_3jl_4kp_4lj_3md_3ng_3oa_4pk_4)$, $16A4$ with symbol $(ao_3bf_3cl_3dh_3em_3fb_3gn_3hd_3ik_4jp_4ki_3lc_3me_3ng_3oa_4pj_4)$, $16A6$ with adjacency symbol $(aj_3be_3cg_3dh_3eb_3fi_3gc_3hd_3if_3ja_4kn_3lo_4mp_3nk_4ol_3pm_4)$, $16A10$ with symbol $(an_3be_3ci_3dj_3eb_3fm_4gp_3hk_3ic_3jd_3kh_3lo_4mf_3na_4ol_3pg_4)$, $16A12$ with symbol $(af_3bk_3co_3dh_3el_3fa_4gm_3hd_3in_4jp_3kb_3le_3mg_4ni_3oc_3pj_4)$, and $16A15$ with symbol $(am_3bg_3co_3di_3ek_4fp_3gb_3hl_3id_3jn_4ke_3lh_3ma_4nj_3oc_3pf_4)$ tiles admit the isometry group of order 2 generated by respective reflection, for suitable choice of parameters. For the Delone class $16A5$ which has the adjacency symbol $(ao_3be_3ck_3dl_3eb_3fn_3gi_4hp_4ig_3jm_3kc_3ld_3mj_3nf_3oa_4ph_4)$ a tile admits the isometry group of order 2 generated by a rotation of order 2, with suitable choice of parameters.

Among tilings with 18-gons, the 3 Delone classes $18,1$ with the adjacency symbol $(ad_3bf_3cg_3da_3eh_3fb_3gc_3he_3ir_3jm_3ko_3lp_3mj_3nq_3ok_3pl_3qn_3ri_3)$, $18,2$ with symbol $(ad_3bg_3ch_3da_3en_3fi_3gb_3hc_3if_3jm_3kp_3lq_3mj_3ne_3or_3pk_3ql_3ro_3)$ and $18,5$ with symbol $(ad_3bk_3cl_3da_3eh_3fo_3gp_3he_3ir_3jm_3kb_3lc_3mj_3nq_3of_3pg_3qn_3ri_3)$ admit tiles with the isometry group of order 4 generated by a rotation of order 2 and a reflection if



18,1 18,2 18,5

Figure 7. Adjacency diagrams for the Delone classes of isohedral tilings $18,1$, $18,2$ and $18,5$, with reflection axes

suitable parameters are chosen (Fig. 7). For the 3 Delone classes $18,3$ with symbol $(ad_3bh_3ci_3da_3em_3fp_3gj_3hb_3ic_3jg_3ko_3lq_3me_3nr_3ok_3pf_3ql_3rn_3)$, $18,4$ with symbol $(ad_3bi_3cj_3da_3em_3fo_3gq_3hk_3ib_3jc_3kh_3lp_3me_3nr_3of_3pl_3qg_3rn_3)$ and $18,6$ with symbol $(af_3bi_3cn_3dq_3ej_3fa_3gl_3ho_3ib_3je_3kp_3lg_3mr_3nc_3oh_3pk_3qd_3rm_3)$, a tile admits the isometry group of order 2 generated by respective reflection, with suitable choice of parameters.

Altogether there are 78 Delone classes of isohedral tilings of the hyperbolic plane with disks for translation group of genus two that admit tilings with additional isometries. The analysis of adjacency diagrams has yielded that 29 Delone classes admit tiles with the isometry group of order greater than 2, their adjacency diagrams are shown in Fig. 2–7. The adjacency diagrams are given for all the 78 Delone classes.

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