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Commutative Weakly Tripotent Group Rings

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Abstract. Very recently, Breaz and Cîmpean introduced and examined in Bull. Korean Math. Soc. (2018) the class of so-called *weakly tripotent rings* as those rings R whose elements satisfy at leat one of the equations $x^3 = x$ or $(1-x)^3 = 1-x$. These rings are generally non-commutative. We here obtain a criterion when the commutative group ring RG is weakly tripotent in terms only of a ring R and of a group G plus their sections.
Actually, we also show that these weakly tripotent rings are *strongly invo-clean rings* in the sense of Danchev in Commun. Korean Math. Soc. (2017). Thereby, our established criterion somewhat strengthens previous results on commutative strongly invo-clean group rings, proved by the present author in Univ. J. Math. & Math. Sci. (2018). Moreover, this criterion helps us to construct a commutative strongly invo-clean ring of characteristic 2 which is *not* weakly tripotent, thus showing that these two ring classes are different.

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1 Introduction and Background

Throughout the text of the current article all rings into consideration are assumed to be associative, possessing the identity element 1 which differs from the zero element 0. Our terminology and notation in both ring and group theories are mainly standard and some additional notions will be specified in the sequel. For instance, for a ring R, the symbol $U(R)$ denotes the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R which, in the commutative case, coincides with the nil-radical $N(R)$, and $J(R)$ the Jacobson radical of R. Likewise, for an abelian group G, the letter G^2 stands for the subgroup of G consisting of all elements of the type $\{g^2 \mid g \in G\}$. As usual, RG designates the group ring of G over R with augmentation ideal $I(RG;G)$ generated by elements of the sort $\{g-1 \mid g \in G\}$.

An element t of a ring R is called *tripotent* if the equality $t^3 = t$ holds. If each element of R is with this property, R is said to be *tripotent* as well. The complete description of such rings is well-known as a subdirect product (= a special subring of a direct product) of a family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

On the other side, generalizing the aforementioned concept, in [1] were explored the so-called *weakly tripotent* rings that are rings in which at least one of the elements t or $1-t$ is a tripotent. It is immediate that weakly tripotent rings of characteristic

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3 are themselves tripotent as $(1-t)^3 = 1-t^3 = 1-t$ yields that $t^3 = t$. Interestingly, for any element y from the Jacobson radical $J(R)$ of such a ring R, it must be that $y^2 = 2y$. To look at this, we have that $y^3 = y$ or $(1-y)^3 = 1-y$. In the first version, $y(1-y^2) = 0$ gives that $y = 0$ as $1-y^2 \in U(R)$, so that $y^2 = 0 = 2y$. In the second one, we observe that $(1-y)^2 = 1$ as $1-y \in U(R)$ and hence $y^2 = 2y$, as promised. This substantiates that $y^2 = 2y$ is always true. Replacing $y \to -y$ allows us to get that $4y = 0$.

These rings are, in general, non-commutative and there is no a complete description of their structure yet. However, a complete characterization theorem for a commutative ring R to be weakly tripotent is [1, Theorem 14]. However, since the direct product of an arbitrary family of Boolean rings is still a Boolean ring, we will state below this theorem in an equivalent form, but in a manner which is slightly more transparent and convenient for direct applications, as follows: *A commutative ring R is weakly tripotent if, and only if, $R \cong R_1 \times R_2$, where $R_2 = \{0\}$ or $x^3 = x$ holds $\forall\ x \in R_2$ with $3R_2 = \{0\}$, and $R_1 = \{0\}$ or R_1 is a subdirect product of $R_{11} \times R_{12}$ with $2^k R_1 = \{0\}$ for some $k = 1, 2, 3$ (or, in other words, $3 \in U(R_1)$), where R_{11} and R_{12} are rings such that either $R_{11} = \{0\}$ or $R_{11}/J(R_{11}) \cong \mathbb{Z}_2$ with $y^2 = 2y\ \forall\ y \in J(R_{11})$, and R_{12} is a Boolean ring.* Resultantly, it readily follows that each commutative weakly tripotent ring of even characteristic (i.e., of characteristic 2^k for $k = 1, 2, 3$) is *nil-clean* in the sense that the quotient-ring $R/N(R)$ is Boolean or, equivalently, the factor-ring $R/J(R)$ is Boolean with nil $J(R)$. Besides, an pretty easy consequence is that (compare also with [1, Corollary 9]) a ring R is weakly tripotent such that $3 \in U(R)$ and $Id(R) = \{0, 1\}$ if, and only if, $R/J(R) \cong \mathbb{Z}_2$ with $z^2 = 2z$ for any $z \in J(R)$. A further characterization of arbitrary weakly tripotent rings (possibly non-commutative) is given in [5].

Our major motivation to write up this paper is to use the cited above theorem in order to deduce a full criterion for a commutative group ring to be weakly tripotent only in terms of the coefficient ring and the former group plus their divisions. This will be successfully done in the next section.

2 Main Results

Let us recall that a ring R is said to be *strongly invo-clean* in [2], provided for any element $r \in R$ the existence of an idempotent $e \in R$ and a unit $v \in R$ of order at most 2 such that $r = e + v$ with $ev = ve$. These ring were completely classified in [2, Corollary 2.17] with the aid of structural results from [6].

The next relationship considerably strengthens [1, Corollary 4].

Proposition 1. *Every weakly tripotent ring is strongly invo-clean.*

Proof. For such a ring R, we have $r^3 = r$ or $(1-r)^3 = 1-r$ whenever $r \in R$. In the first case, one writes that $r = (1-r^2) + (r^2 + r - 1)$. A direct manipulation shows that $1 - r^2 \in Id(R)$ as $r^2 \in Id(R)$ and that $(r^2 + r - 1)^2 = 1$ observing elementarily that these two elements commute, as required.

Dealing with the other equality $(1-r)^3 = 1-r$, the replacement $r \to 1-r$ with the above trick at hand lead to this that $1-r = f+w$ for some commuting idempotent $f = 1-(1-r)^2 = 2r-r^2$ and an involution $w = 1-3r+r^2$. Therefore, $r = (1-f)+(-w)$, where $(1-f)^2 = 1-f$ and $(-w)^2 = w^2 = 1$, as required. □

We will demonstrate now that the converse implication is totally untrue. Before doing that, we need the following technicality.

Lemma 1. *Let B be a boolean ring and let G be an abelian group. Then the group ring BG is weakly tripotent if, and only if, $B \cong \mathbb{Z}_2$ and $G^2 = \{1\}$.*

Proof. "**Necessity.**" Given $g \in G$, it follows that $g^3 = g$ or $(1-g)^3 = 1-g$. The first equality yields that $g^2 = 1$. The second equality can be written as $g^3 - g^2 + g = 0$ since $2 = 0$ in both B and BG. If we assume that $g^2 \neq 1$, then one sees that $g^3 \neq g$, $g^3 \neq g^2$ and $g^2 \neq g$. This assumption, however, leads to a contradiction since then $g^3 - g^2 + g$ is an element in BG written in canonical form. Finally, it must be in the second case that $g^2 = 1$ as well, as expected.

Next, for every $b \in B$, we consider the element $b + g \in BG$, where $1 \neq g \in G$. Therefore, $(b+g)^3 = b+g$ or $[(1-b)-g]^3 = (1-b)-g$. In the first possibility, we derive that $b + bg + b + g = b + g$, i.e., that $b + bg = 0$. This forces at once that $b = 0$. Since $1-b$ is again an idempotent, the second possibility guarantees in a way of similarity that $1-b = 0$, that is, $b = 1$. Hence $B = \{0,1\} \cong \mathbb{Z}_2$, as asserted.

"**Sufficiency.**" Each (possibly non-zero) element x of BG has take the form $x = g_1 + \cdots + g_n$ for some $n \in \mathbb{N}$. It is clear that $x^2 = n$, whence $x^2 = 1$ if $n = 2k+1$ or $x^2 = 0$ if $n = 2k$. In the first situation, one infers that $x^3 = x$, as wanted. In the second situation, $(1-x)^2 = 1 + x^2 = 1$ so that $(1-x)^3 = 1-x$, as desired. □

So, we are in a position to exhibit the concrete construction.

Example 1. There exists a commutative strongly invo-clean ring of characteristic 2 which is *not* weakly tripotent.

In fact, consider the group ring $K = (\mathbb{Z}_2 \times \mathbb{Z}_2)G$ with $G^2 = \{1\}$. Utilizing Lemma 1 this group ring is manifestly non-weakly tripotent. Nevertheless, we claim that such a ring is strongly invo-clean. To show this, we apply the chief result from [9] to deduce that K is nil-clean in the sense that each element $a \in K$ is writable as $a = q + e$, where $q \in Nil(K) = N(K)$ and $e \in Id(K)$. On the other hand, in view of [8] or [7], one finds that $N(K) = I(KG; G)$ because $N(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{0\}$. It is obvious now that $N(K)$ has an index of nilpotence not exceeding 2 as char $(K) = 2$. And since $a = (q+1) + (1+e)$ with $(q+1)^2 = 1$ and $(1+e)^2 = 1+e$, we are set. This concludes our initial claim in the example.

We are now ready to proceed by proving with our necessary and sufficient condition for a commutative group ring to be weakly tripotent, thus generalizing Lemma 1 listed above.

Theorem 1. *Suppose R is a commutative ring and G is an abelian group. Then the group ring RG is weakly tripotent if, and only if, exactly one of the next two points holds:*

(i) $G = \{1\}$ and R is weakly tripotent.

(ii) $G \neq \{1\}$ with $G^2 = \{1\}$, R is weakly tripotent such that $R \cong R_1 \times R_2$, where both R_1 and R_2 are weakly tripotent rings with $R_1 = \{0\}$ or R_1 a subdirect product of $L \times \mathbb{Z}_2$ for either $L = \{0\}$ or $L/J(L) \cong \mathbb{Z}_2$ having $y^2 = 2y$ for all $y \in J(L)$, and $R_2 = \{0\}$ or $\mathrm{char}(R_2) = 3$ with $x^3 = x$ for all $x \in R_2$, and either
(ii.1) $|G| = 2$, $2d^2 = 2d$ for all $d \in L$ (and hence $4L = \{0\}$);
or
(ii.2) $|G| > 2$, $\mathrm{char}(L) = 2$.

Proof. As $RG \cong R$ whenever G is the trivial group, we shall hereafter assume that it is non-trivial.

"**Necessity.**" Each element x in RG satisfies one of the equations $x^3 = x$ or $(1-x)^3 = 1-x$. Thus, for any $1 \neq g \in G$, one has that $g^3 = g$ or $(1-g)^3 = 1-g$. The first equality gives that $g^2 = 1$. The second one assures that $g^3 - 3g^2 + 2g = 0$. Assume in a way of contradiction that $g^2 \neq 1$. Since then $g \neq g^3 \neq g^2 \neq g$ and since the equation $g^3 - 3g^2 + 2g = 0$ is a canonical record, we will obtain a contrary to our assumption. Consequently, one extracts in both cases that $g^2 = 1$. Even something more: in the second equality we have that $3g - 3 = 0$ implying $3 = 0$. Thus the equation $(1-x)^3 = 1-x$ is tantamount to $x^3 = x$, as expected.

Furthermore, as $R \subseteq RG$, or even there is an epimorphism (= a surjective homomorphism) $RG \to R$ defined by the augmentation map, we employ [1, Lemma 1] to get that R is weakly tripotent, too. We, therefore, may write in conjunction with the listed above theorem for commutative weakly tripotent rings that $R \cong R_1 \times R_2$, where both R_1 and R_2 are weakly tripotent rings with $R_1 = \{0\}$ or R_1 a subdirect product of $L \times B$ for some Boolean ring B and either $L = \{0\}$ or $L/J(L) \cong \mathbb{Z}_2$ having $z^2 = 2z$ for all $z \in J(L)$, and $R_2 = \{0\}$ or $\mathrm{char}(R_2) = 3$. It is then an easy technical matter to check that $RG \cong R_1G \times R_2G$, where R_1G, R_2G both remain weakly tripotent. We will study these two direct factors separately:

About R_1G: Here R_1G is a subdirect product of $LG \times BG$, where LG and BG are both weakly tripotent rings with $2 \in J(L)$ for the first ring. In the latter case, Lemma 1 enables us that $B \cong \mathbb{Z}_2$, as stated. Concentrating now on LG, we shall consider two possibilities on the cardinality of the basis group G. In fact, firstly suppose that $|G| = 2$. Then, for every $r \in L$, we consider the element $r(1-g) \in J(LG)$, where $1 \neq g \in G$ with $g^2 = 1$ (see, for instance,[7]). Since $[r(1-g)]^2 = 2r(1-g)$, one inspects by simple manipulations that $(2r^2 - 2r) - (2r^2 - 2r)g = 0$ implying immediately the desired equality $2r^2 = 2r$. Secondly, suppose that $|G| > 2$. Then there are two different elements $g, h \in G \setminus \{1\}$ with $g^2 = h^2 = 1$. Since both $g - 1 \in J(LG)$ and $1 - h \in J(LG)$, it follows directly that $g - h = (g-1) + (1-h) \in J(LG)$ and so $(g-h)^2 = 2(g-h)$ yielding that

$2gh + 2g - 2h - 2 = 0$. But as $gh \neq 1$ because $g \neq h$ as well as $gh \neq g$ and $gh \neq h$ because $h \neq 1$ and $g \neq 1$, we conclude after all that $2 = 0$ in L, as required.

About R_2G: As it was easily observed above, all weakly tripotent rings having characteristic 3 are obviously tripotent. Since then the non-zero ring R_2 of characteristic 3 has to be tripotent, we are done.

"**Sufficiency.**" Writing $RG \cong R_1G \times R_2G$, we must explore both direct factors R_1G and R_2G.

Dealing with the non-zero variant of the first direct factor R_1G, we detect that R_1G is a subring of $LG \times \mathbb{Z}_2G$. The application of Lemma 1 is a guarantor that $\mathbb{Z}_2G$ is weakly tripotent bearing in mind that $G^2 = \{1\}$. We now assert that $J(LG) = J(L)G + I(LG; G)$. Indeed, seeing that $2 \in J(L)$, one verifies in virtue of [7] (see also [8]) that $J(LG) = J(L)G + \langle r(g-1) \mid r \in L, g \in G\rangle \subseteq J(L)G + I(LG; G)$. To derive the converse inclusion, we differ two possible cases for the cardinality of G: if $|G| = 2$, then $g(g-1) = -(g-1)$ as $g^2 = 1$. If now $|G| > 2$ with $G^2 = \{1\}$ and $2 = 0$ in L, we are observing that $I^2(LG; G) = \{0\}$ and so $I(LG; G) \subseteq J(LG)$ since $I(LG; G)$ is a nil ideal, giving the pursued equality, because $J(L)G \subseteq J(LG)$ holds always (cf.[7]). That is why, $LG/J(LG) = LG/[J(L) + I(LG; G)] \cong L/J(L) \cong \mathbb{Z}_2$, as needed. What suffices to prove in order to complete this point is that $z^2 = 2z$ for any $z \in J(LG)$. This, however, follows directly by the same token as in the proof of [3, Theorem 2.3]. Finally, we arrive at the fact that LG is a weakly tripotent ring, whence so does R_1G in accordance with [1, Lemma 1].

Further, concerning the second direct factor R_2G, since R_2 is either zero or a tripotent ring of characteristic 3, one plainly obtains that R_2G is also a tripotent ring by taking into account that $G^2 = \{1\}$. We, finally, appeal to [1, Proposition 6] to conclude that RG is weakly tripotent, as claimed. $\square$

We close our work with the following challenging question.

Proposition 2. *Find a criterion for a non-commutative group ring to be weakly tripotent.*

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