

Inequalities of Hermite-Hadamard Type for K -Bounded Modulus Convex Complex Functions

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Abstract. Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. We say that the function $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$ is called K -bounded modulus convex, for the given $K > 0$, if it satisfies the condition

$$|(1 - \lambda)f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y)| \leq \frac{1}{2}K\lambda(1 - \lambda)|x - y|^2$$

for any $x, y \in D$ and $\lambda \in [0, 1]$. In this paper we establish some new Hermite-Hadamard type inequalities for the complex integral on γ , a smooth path from \mathbb{C} , and K -bounded modulus convex functions. Some examples for integrals on segments and circular paths are also given.

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1 Introduction

Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field \mathbb{C} . Let C be a convex set in X . In the recent paper [1] we introduced the following class of functions:

Definition 1. A mapping $f : C \subset X \rightarrow Y$ is called K -bounded norm convex, for some given $K > 0$, if it satisfies the condition

$$\|(1 - \lambda)f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y)\|_Y \leq \frac{1}{2}K\lambda(1 - \lambda)\|x - y\|_X^2 \quad (1)$$

for any $x, y \in C$ and $\lambda \in [0, 1]$. For simplicity, we denote this by $f \in \mathcal{BN}_K(C)$.

We have from (1) for $\lambda = \frac{1}{2}$ the Jensen's inequality

$$\left\| \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right) \right\|_Y \leq \frac{1}{8}K\|x - y\|_X^2$$

for any $x, y \in C$.

We observe that $\mathcal{BN}_K(C)$ is a convex subset in the linear space of all functions defined on C and with values in Y .

In the same paper [1], we obtained the following result which provides a large class of examples of such functions.

Theorem 1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, C an open convex subset of X and $f : C \rightarrow Y$ a twice-differentiable mapping on C . Then for any $x, y \in C$ and $\lambda \in [0, 1]$ we have*

$$\|(1 - \lambda) f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y)\|_Y \leq \frac{1}{2} K \lambda (1 - \lambda) \|y - x\|_X^2, \quad (2)$$

where

$$K_{f''} := \sup_{z \in C} \|f''(z)\|_{\mathcal{L}(X^2; Y)} \quad (3)$$

is assumed to be finite, namely $f \in \mathcal{BN}_{K_{f''}}(C)$.

We have the following inequalities of Hermite-Hadamard type [1]:

Theorem 2. *Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field \mathbb{C} with Y complete. Assume that the mapping $f : C \subset X \rightarrow Y$ is continuous on the convex set C in the norm topology. If $f \in \mathcal{BN}_K(C)$ for some $K > 0$, then we have*

$$\left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1 - \lambda)x + \lambda y) d\lambda \right\|_Y \leq \frac{1}{12} K \|x - y\|_X^2 \quad (4)$$

and

$$\left\| \int_0^1 f((1 - \lambda)x + \lambda y) d\lambda - f\left(\frac{x + y}{2}\right) \right\|_Y \leq \frac{1}{24} K \|x - y\|_X^2 \quad (5)$$

for any $x, y \in C$.

The constants $\frac{1}{12}$ and $\frac{1}{24}$ are best possible.

For a monograph devoted to Hermite-Hadamard type inequalities see [3] and the recent survey paper [2].

Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Following Definition 1, we say that the function $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$ is called K -bounded modulus convex, for the given $K > 0$, if it satisfies the condition

$$|(1 - \lambda) f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y)| \leq \frac{1}{2} K \lambda (1 - \lambda) |x - y|^2 \quad (6)$$

for any $x, y \in D$ and $\lambda \in [0, 1]$. For simplicity, we denote this by $f \in \mathcal{BM}_K(D)$.

All the above results can be translated for complex functions defined on convex subsets $D \subset \mathbb{C}$.

In the following, in order to obtain several inequalities for the complex integral, we need the following facts.

Suppose γ is a smooth path from \mathbb{C} parametrized by $z(t)$, $t \in [a, b]$ and f is a complex function which is continuous on γ . Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z) dz = \int_{\gamma_{u,w}} f(z) dz := \int_a^b f(z(t)) z'(t) dt.$$

We observe that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a, b]$, which is differentiable on the intervals $[a, c]$ and $[c, b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z) dz := \int_{\gamma_{u,v}} f(z) dz + \int_{\gamma_{v,w}} f(z) dz$$

where $v := z(s)$ for some $s \in (a, b)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z) |dz| := \int_a^b f(z(t)) |z'(t)| dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)| dt.$$

Let f and g be holomorphic in D , an open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z) g'(z) dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u,w}} f'(z) g(z) dz. \quad (7)$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| |dz| \leq \|f\|_{\gamma, \infty} \ell(\gamma) \quad (8)$$

where $\|f\|_{\gamma, \infty} := \sup_{z \in \gamma} |f(z)|$.

We also define the p -norm with $p \geq 1$ by

$$\|f\|_{\gamma, p} := \left(\int_{\gamma} |f(z)|^p |dz| \right)^{1/p}.$$

For $p = 1$ we have

$$\|f\|_{\gamma, 1} := \int_{\gamma} |f(z)| |dz|.$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\|f\|_{\gamma, 1} \leq [\ell(\gamma)]^{1/q} \|f\|_{\gamma, p}.$$

Motivated by the above results, in this paper we establish some new Hermite-Hadamard type inequalities for the complex integral on γ , a smooth path from \mathbb{C} , and K -bounded modulus convex functions. Some examples for integrals on segments and circular paths are also given.

2 Integral Inequalities

We have:

Theorem 3. *Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Assume that f is holomorphic on D and $f \in \mathcal{BM}_K(D)$. If $\gamma \subset D$ parametrized by $z(t)$, $t \in [a, b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v \in D$, then*

$$\left| \int_{\gamma} f(z) dz - \left[f(v) + f'(v) \left(\frac{w+u}{2} - v \right) \right] (w-u) \right| \leq \frac{1}{2} K \int_{\gamma} |z-v|^2 |dz|. \quad (9)$$

In particular, we have for $v = \frac{w+u}{2}$ that

$$\left| \int_{\gamma} f(z) dz - f\left(\frac{w+u}{2}\right) (w-u) \right| \leq \frac{1}{2} K \int_{\gamma} \left| z - \frac{w+u}{2} \right|^2 |dz|. \quad (10)$$

Proof. Let $x, y \in D$. Since $f \in \mathcal{BM}_K(D)$, then we have

$$|f((1-\lambda)x + \lambda y) - f(x) + \lambda[f(x) - f(y)]| \leq \frac{1}{2} K \lambda (1-\lambda) |x-y|^2$$

that implies that

$$\left| \frac{f(x + \lambda(y-x)) - f(x)}{\lambda} + f(x) - f(y) \right| \leq \frac{1}{2} K (1-\lambda) |x-y|^2$$

for $\lambda \in (0, 1)$.

Since f is holomorphic on D , then by letting $\lambda \rightarrow 0+$, we get

$$|f'(x)(y-x) + f(x) - f(y)| \leq \frac{1}{2} K |x-y|^2$$

that is equivalent to

$$|f(y) - f(x) - f'(x)(y-x)| \leq \frac{1}{2} K |y-x|^2 \quad (11)$$

for all $x, y \in D$.

We have

$$\begin{aligned} & \int_{\gamma} [f(z) - f(v) - f'(v)(z-v)] dz \\ &= \int_{\gamma} f(z) dz - f(v) \int_{\gamma} dz - f'(v) \left(\int_{\gamma} z dz - v \int_{\gamma} dz \right) \\ &= \int_{\gamma} f(z) dz - f(v)(w-u) - f'(v) \left[\frac{1}{2} (w^2 - u^2) - v(w-u) \right] \\ &= \int_{\gamma} f(z) dz - \left[f(v) + f'(v) \left(\frac{w+u}{2} - v \right) \right] (w-u) \end{aligned} \quad (12)$$

for any $v \in D$.

By using (11) we get

$$\begin{aligned} & \left| \int_{\gamma} f(z) dz - \left[f(v) + f'(v) \left(\frac{w+u}{2} - v \right) \right] (w-u) \right| \\ & \leq \int_{\gamma} |f(z) - f(v) - f'(v)(z-v)| |dz| \leq \frac{1}{2}K \int_{\gamma} |z-v|^2 |dz| \end{aligned}$$

for any $v \in D$, which proves the inequality (9). \square

If the path γ is a segment $[u, w] \subset G$ connecting two distinct points u and w in G then we write $\int_{\gamma} f(z) dz$ as $\int_u^w f(z) dz$.

Corollary 4. *With the assumptions of Theorem 3, suppose $[u, w] \subset D$ is a segment connecting two distinct points u and w in D and $v \in [u, w]$. Then for $v = (1-s)u + sw$ with $s \in [0, 1]$, we have*

$$\begin{aligned} & \left| \int_u^w f(z) dz - f((1-s)u + sw)(w-u) \right. \\ & \quad \left. - f'((1-s)u + sw) \left(\frac{1}{2} - s \right) (w-u)^2 \right| \\ & \leq \frac{1}{6}K |w-u|^3 [(1-s)^3 + s^3]. \quad (13) \end{aligned}$$

In particular, we have, see also (5),

$$\left| \int_u^w f(z) dz - f\left(\frac{w+u}{2}\right)(w-u) \right| \leq \frac{1}{24}K |w-u|^3. \quad (14)$$

Proof. It follows by Theorem 3 by observing that

$$\begin{aligned} \int_u^w |z-v|^2 |dz| &= |w-u| \int_0^1 |(1-t)u + tw - (1-s)u - sw|^2 dt \\ &= |w-u| \int_0^1 |(1-t)u + tw - (1-s)u - sw|^2 dt \\ &= |w-u|^3 \int_0^1 (t-s)^2 dt = \frac{1}{3} |w-u|^3 [(1-s)^3 + s^3] \end{aligned}$$

for $s \in [0, 1]$. \square

Theorem 5. *Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Assume that f is holomorphic on D and $f \in \mathcal{BM}_K(D)$. If $\gamma \subset D$ parametrized by $z(t)$, $t \in [a, b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v \in D$, then*

$$\left| \frac{1}{2} [f(w)(w-v) + f(u)(v-u) + f(v)(w-u)] - \int_{\gamma} f(z) dz \right|$$

$$\leq \frac{1}{4}K \int_{\gamma} |z - v|^2 |dz|. \quad (15)$$

In particular, we have for $v = \frac{w+u}{2}$ that

$$\begin{aligned} & \left| \frac{1}{2} \left[\frac{f(w) + f(u)}{2} + f\left(\frac{w+u}{2}\right) \right] (w-u) - \int_{\gamma} f(z) dz \right| \\ & \leq \frac{1}{4}K \int_{\gamma} \left| z - \frac{w+u}{2} \right|^2 |dz|. \quad (16) \end{aligned}$$

Proof. By using (11) we get

$$\int_{\gamma} |f(v) - f(z) - f'(z)(v-z)| |dz| \leq \frac{1}{2}K \int_{\gamma} |v-z|^2 |dz| \quad (17)$$

for $v \in D$.

By the complex integral properties, we have

$$\begin{aligned} & \left| \int_{\gamma} [f(v) - f(z) - f'(z)(v-z)] dz \right| \\ & \leq \int_{\gamma} |f(v) - f(z) - f'(z)(v-z)| |dz| \quad (18) \end{aligned}$$

for $v \in D$.

Using integration by parts, we get

$$\begin{aligned} & \int_{\gamma} [f(v) - f(z) - f'(z)(v-z)] dz \\ & = f(v) \int_{\gamma} dz - \int_{\gamma} f(z) dz - \int_{\gamma} f'(z)(v-z) dz \\ & = f(v)(w-u) - \int_{\gamma} f(z) dz - \left[f(z)(v-z)|_u^w + \int_{\gamma} f(z) dz \right] \\ & = f(v)(w-u) - \int_{\gamma} f(z) dz - f(w)(v-w) + f(u)(v-u) - \int_{\gamma} f(z) dz \\ & = f(w)(w-v) + f(u)(v-u) + f(v)(w-u) - 2 \int_{\gamma} f(z) dz, \end{aligned}$$

which implies that

$$\begin{aligned} & \frac{1}{2} [f(w)(w-v) + f(u)(v-u) + f(v)(w-u)] - \int_{\gamma} f(z) dz \\ & = \frac{1}{2} \int_{\gamma} [f(v) - f(z) - f'(z)(v-z)] dz \quad (19) \end{aligned}$$

for $v \in D$.

By utilising (17)-(19) we get the desired result (15). \square

We have:

Corollary 6. *With the assumptions of Theorem 3, suppose $[u, w] \subset D$ is a segment connecting two distinct points u and w in D and $v \in [u, w]$. Then for $v = (1 - s)u + sw$ with $s \in [0, 1]$, we have*

$$\begin{aligned} & \left| \frac{1}{2} [(1 - s)f(w) + sf(u) + f((1 - s)u + sw)](w - u) - \int_u^w f(z) dz \right| \\ & \leq \frac{1}{12} K |w - u|^3 [(1 - s)^3 + s^3]. \end{aligned} \quad (20)$$

In particular, we have for $v = \frac{w+u}{2}$ that

$$\begin{aligned} & \left| \frac{1}{2} \left[\frac{f(w) + f(u)}{2} + f\left(\frac{w+u}{2}\right) \right] (w - u) - \int_u^w f(z) dz \right| \\ & \leq \frac{1}{48} K |w - u|^3. \end{aligned} \quad (21)$$

We observe that, if f is holomorphic on D and $K = \sup_{z \in D} |f''(z)|$ is finite, then by (9) and (10) we have

$$\begin{aligned} & \left| \int_{\gamma} f(z) dz - \left[f(v) + f'\left(\frac{w+u}{2} - v\right) \right] (w - u) \right| \\ & \leq \frac{1}{2} \sup_{z \in D} |f''(z)| \int_{\gamma} |z - v|^2 |dz| \end{aligned} \quad (22)$$

for all $v \in D$. In particular,

$$\begin{aligned} & \left| \int_{\gamma} f(z) dz - f\left(\frac{w+u}{2}\right) (w - u) \right| \\ & \leq \frac{1}{2} \sup_{z \in D} |f''(z)| \int_{\gamma} \left| z - \frac{w+u}{2} \right|^2 |dz|. \end{aligned} \quad (23)$$

From (15) and (16) we get

$$\begin{aligned} & \left| \frac{1}{2} [f(w)(w - v) + f(u)(v - u) + f(v)(w - u)] - \int_{\gamma} f(z) dz \right| \\ & \leq \frac{1}{4} \sup_{z \in D} |f''(z)| \int_{\gamma} |z - v|^2 |dz|. \end{aligned} \quad (24)$$

for all $v \in D$. In particular,

$$\begin{aligned} & \left| \frac{1}{2} \left[\frac{f(w) + f(u)}{2} + f\left(\frac{w+u}{2}\right) \right] (w - u) \right. \\ & \quad \left. - \int_{\gamma} f(z) dz \right| \leq \frac{1}{4} \sup_{z \in D} |f''(z)| \int_{\gamma} \left| z - \frac{w+u}{2} \right|^2 |dz|. \end{aligned} \quad (25)$$

The inequalities (22)-(25) provide many examples of interest as follows.

If we consider the function $f(z) = \exp z$, $z \in \mathbb{C}$ and $\gamma \subset \mathbb{C}$ parametrized by $z(t)$, $t \in [a, b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ then by (22)-(25) we have by the inequalities

$$\left| \exp w - \exp u - \left(1 + \frac{w+u}{2} - v\right) (w-u) \exp v \right| \leq \frac{1}{2} \sup_{z \in D} |\exp z| \int_{\gamma} |z-v|^2 |dz| \quad (26)$$

for all $v \in \mathbb{C}$. In particular,

$$\left| \exp w - \exp u - \exp\left(\frac{w+u}{2}\right) (w-u) \right| \leq \frac{1}{2} \sup_{z \in D} |\exp z| \int_{\gamma} \left|z - \frac{w+u}{2}\right|^2 |dz|. \quad (27)$$

We also have

$$\left| \frac{1}{2} [(w-v) \exp w + (v-u) \exp u + (w-u) \exp v] - \exp w + \exp u \right| \leq \frac{1}{4} \sup_{z \in D} |\exp z| \int_{\gamma} |z-v|^2 |dz|. \quad (28)$$

for all $v \in \mathbb{C}$. In particular,

$$\left| \frac{1}{2} \left[\frac{\exp w + \exp u}{2} + \exp\left(\frac{w+u}{2}\right) \right] (w-u) - \exp w + \exp u \right| \leq \frac{1}{4} \sup_{z \in D} |\exp z| \int_{\gamma} \left|z - \frac{w+u}{2}\right|^2 |dz|. \quad (29)$$

Consider the function $F(z) = \text{Log}(z)$ where $\text{Log}(z) = \ln|z| + i \text{Arg}(z)$ and $\text{Arg}(z)$ is such that $-\pi < \text{Arg}(z) \leq \pi$. Log is called the "principal branch" of the complex logarithmic function. F is analytic on all of $\mathbb{C} \setminus \{x+iy : x \leq 0, y=0\}$ and $F'(z) = \frac{1}{z}$ on this set.

If we consider $f : D \rightarrow \mathbb{C}$, $f(z) = \frac{1}{z}$ where $D \subset \mathbb{C} \setminus \{x+iy : x \leq 0, y=0\}$, then F is a primitive of f on D and if $\gamma \subset D$ parametrized by $z(t)$, $t \in [a, b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$, then

$$\int_{\gamma} f(z) dz = \text{Log}(w) - \text{Log}(u).$$

For $D \subset \mathbb{C} \setminus \{x+iy : x \leq 0, y=0\}$, define $d := \inf_{z \in D} |z|$ and assume that $d \in (0, \infty)$. By the inequalities (22)-(25) we then have

$$\left| \operatorname{Log}(w) - \operatorname{Log}(u) - \left[\frac{1}{v} - \frac{1}{v^2} \left(\frac{w+u}{2} - v \right) \right] (w-u) \right| \leq \frac{1}{d^3} \int_{\gamma} |z-v|^2 |dz| \quad (30)$$

for all $v \in D$. In particular,

$$\left| \operatorname{Log}(w) - \operatorname{Log}(u) - \left(\frac{w+u}{2} \right)^{-1} (w-u) \right| \leq \frac{1}{d^3} \int_{\gamma} \left| z - \frac{w+u}{2} \right|^2 |dz|. \quad (31)$$

We also have

$$\left| \frac{1}{2} \left(\frac{w-v}{w} + \frac{v-u}{u} + \frac{w-u}{v} \right) - \operatorname{Log}(w) + \operatorname{Log}(u) \right| \leq \frac{1}{2d^3} \int_{\gamma} |z-v|^2 |dz|. \quad (32)$$

for all $v \in D$. In particular,

$$\left| \frac{1}{2} \left[\frac{u+w}{2wu} + \left(\frac{w+u}{2} \right)^{-1} \right] (w-u) - \operatorname{Log}(w) + \operatorname{Log}(u) \right| \leq \frac{1}{2d^3} \int_{\gamma} \left| z - \frac{w+u}{2} \right|^2 |dz|. \quad (33)$$

3 Examples for Circular Paths

Let $[a, b] \subseteq [0, 2\pi]$ and the circular path $\gamma_{[a,b],R}$ be centered at 0 and with radius $R > 0$

$$z(t) = R \exp(it) = R(\cos t + i \sin t), \quad t \in [a, b].$$

If $[a, b] = [0, \pi]$ then we get a half circle while for $[a, b] = [0, 2\pi]$ we get the full circle.

Since

$$\begin{aligned} |e^{is} - e^{it}|^2 &= |e^{is}|^2 - 2 \operatorname{Re} \left(e^{i(s-t)} \right) + |e^{it}|^2 \\ &= 2 - 2 \cos(s-t) = 4 \sin^2 \left(\frac{s-t}{2} \right) \end{aligned}$$

for any $t, s \in \mathbb{R}$, then

$$|e^{is} - e^{it}|^r = 2^r \left| \sin \left(\frac{s-t}{2} \right) \right|^r \quad (34)$$

for any $t, s \in \mathbb{R}$ and $r > 0$. In particular,

$$|e^{is} - e^{it}| = 2 \left| \sin \left(\frac{s-t}{2} \right) \right|$$

for any $t, s \in \mathbb{R}$.

For $s = a$ and $s = b$ we have

$$|e^{ia} - e^{it}| = 2 \left| \sin \left(\frac{a-t}{2} \right) \right| \quad \text{and} \quad |e^{ib} - e^{it}| = 2 \left| \sin \left(\frac{b-t}{2} \right) \right|.$$

If $u = R \exp(ia)$ and $w = R \exp(ib)$ then

$$\begin{aligned} w - u &= R [\exp(ib) - \exp(ia)] = R [\cos b + i \sin b - \cos a - i \sin a] \\ &= R [\cos b - \cos a + i (\sin b - \sin a)]. \end{aligned}$$

Since

$$\cos b - \cos a = -2 \sin \left(\frac{a+b}{2} \right) \sin \left(\frac{b-a}{2} \right)$$

and

$$\sin b - \sin a = 2 \sin \left(\frac{b-a}{2} \right) \cos \left(\frac{a+b}{2} \right),$$

hence

$$\begin{aligned} w - u &= R \left[-2 \sin \left(\frac{a+b}{2} \right) \sin \left(\frac{b-a}{2} \right) + 2i \sin \left(\frac{b-a}{2} \right) \cos \left(\frac{a+b}{2} \right) \right] \\ &= 2R \sin \left(\frac{b-a}{2} \right) \left[-\sin \left(\frac{a+b}{2} \right) + i \cos \left(\frac{a+b}{2} \right) \right] \\ &= 2Ri \sin \left(\frac{b-a}{2} \right) \left[\cos \left(\frac{a+b}{2} \right) + i \sin \left(\frac{a+b}{2} \right) \right] \\ &= 2Ri \sin \left(\frac{b-a}{2} \right) \exp \left[\left(\frac{a+b}{2} \right) i \right]. \end{aligned}$$

We also have

$$z'(t) = Ri \exp(it) \quad \text{and} \quad |z'(t)| = R$$

for $t \in [a, b]$.

In what follows we assume that f is defined on a domain containing the circular path $\gamma_{[a,b],R}$ and that f is holomorphic on that domain.

Consider the circular path $\gamma_{[a,b],R}$ and assume that $v = R \exp(is) \in \gamma_{[a,b],R}$ with $s \in [a, b]$. Then by using the inequality (9) we get

$$\begin{aligned} & \left| Ri \int_a^b f(R \exp(it)) \exp(it) dt \right. \\ & \quad \left. - \left[f(R \exp(is)) + f'(R \exp(is)) \left(\frac{R \exp(ib) + R \exp(ia)}{2} - R \exp(is) \right) \right] \right. \\ & \quad \quad \left. \times 2Ri \sin \left(\frac{b-a}{2} \right) \exp \left[\left(\frac{a+b}{2} \right) i \right] \right| \\ & \leq \frac{1}{2} \sup_{t \in [a,b]} |f''(R \exp(it))| R \int_a^b |R \exp(it) - R \exp(is)|^2 dt \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sup_{t \in [a, b]} |f''(R \exp(it))| R^3 \int_a^b 4 \sin^2 \left(\frac{s-t}{2} \right) dt \\
&= 2 \sup_{t \in [a, b]} |f''(R \exp(it))| R^3 \int_a^b \sin^2 \left(\frac{s-t}{2} \right) dt,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\left| \int_a^b f(R \exp(it)) \exp(it) dt \right. \\
&\quad \left. - 2R \left[f(R \exp(is)) + f'(R \exp(is)) \left(\frac{\exp(ib) + \exp(ia)}{2} - \exp(is) \right) \right] \right. \\
&\quad \quad \left. \times \sin \left(\frac{b-a}{2} \right) \exp \left[\left(\frac{a+b}{2} \right) i \right] \right| \\
&\quad \leq 2 \sup_{t \in [a, b]} |f''(R \exp(it))| R^2 \int_a^b \sin^2 \left(\frac{s-t}{2} \right) dt \quad (35)
\end{aligned}$$

for $s \in [a, b]$.

Since

$$\sin^2 \left(\frac{s-t}{2} \right) = \frac{1 - \cos(s-t)}{2},$$

hence

$$\begin{aligned}
&\int_a^b \sin^2 \left(\frac{s-t}{2} \right) dt \\
&= \int_a^b \frac{1 - \cos(s-t)}{2} dt = \frac{1}{2} [b-a - \sin(b-s) - \sin(s-a)] \\
&= \frac{1}{2} \left[b-a - 2 \sin \left(\frac{b-a}{2} \right) \cos \left(\frac{a+b}{2} - s \right) \right] \\
&= \frac{b-a}{2} - \sin \left(\frac{b-a}{2} \right) \cos \left(\frac{a+b}{2} - s \right)
\end{aligned}$$

for $s \in [a, b]$.

Therefore by (35) we get

$$\begin{aligned}
&\left| \int_a^b f(R \exp(it)) \exp(it) dt \right. \\
&\quad \left. - 2R \left[f(R \exp(is)) + f'(R \exp(is)) \left(\frac{\exp(ib) + \exp(ia)}{2} - \exp(is) \right) \right] \right. \\
&\quad \quad \left. \times \sin \left(\frac{b-a}{2} \right) \exp \left[\left(\frac{a+b}{2} \right) i \right] \right| \\
&\quad \leq 2R^2 \sup_{t \in [a, b]} |f''(R \exp(it))| \left[\frac{b-a}{2} - \sin \left(\frac{b-a}{2} \right) \cos \left(\frac{a+b}{2} - s \right) \right] \quad (36)
\end{aligned}$$

for $s \in [a, b]$.

In particular, for $s = \frac{a+b}{2}$, we obtain from (36) the best possible inequality

$$\begin{aligned}
& \left| \int_a^b f(R \exp(it)) \exp(it) dt \right. \\
& \quad - 2R \left[f \left(R \exp \left(\frac{a+b}{2} i \right) \right) + f' \left(R \exp \left(\frac{a+b}{2} i \right) \right) \right. \\
& \quad \quad \times \left. \left. \left(\frac{\exp(ib) + \exp(ia)}{2} - \exp \left(\frac{a+b}{2} i \right) \right) \right] \right. \\
& \quad \quad \times \sin \left(\frac{b-a}{2} \right) \exp \left[\left(\frac{a+b}{2} \right) i \right] \Big| \\
& \leq 2R^2 \sup_{t \in [a, b]} |f''(R \exp(it))| \left[\frac{b-a}{2} - \sin \left(\frac{b-a}{2} \right) \right]. \quad (37)
\end{aligned}$$

By utilising the inequality (24) for the circular path $\gamma_{[a, b], R}$ and $v = R \exp(is) \in \gamma_{[a, b], R}$ with $s \in [a, b]$, we also get

$$\begin{aligned}
& \left| f(R \exp(ib)) \sin \left(\frac{b-s}{2} \right) \exp \left[\left(\frac{s+b}{2} \right) i \right] \right. \\
& \quad + f(R \exp(ia)) \sin \left(\frac{s-a}{2} \right) \exp \left[\left(\frac{a+s}{2} \right) i \right] \\
& \quad + f(R \exp(is)) \sin \left(\frac{b-a}{2} \right) \exp \left[\left(\frac{a+b}{2} \right) i \right] \\
& \quad \quad \left. - \int_a^b f(R \exp(it)) \exp(it) dt \right| \\
& \leq R^2 \sup_{t \in [a, b]} |f''(R \exp(it))| \left[\frac{b-a}{2} - \sin \left(\frac{b-a}{2} \right) \cos \left(\frac{a+b}{2} - s \right) \right]. \quad (38)
\end{aligned}$$

In particular, for $s = \frac{a+b}{2}$, we get from (38) best possible inequality

$$\begin{aligned}
& \left| f(R \exp(bi)) \sin \left(\frac{b-a}{4} \right) \exp \left[\left(\frac{a+3b}{4} \right) i \right] \right. \\
& \quad + f(R \exp(ia)) \sin \left(\frac{b-a}{4} \right) \exp \left[\left(\frac{3a+b}{4} \right) i \right] \\
& \quad + f \left(R \exp \left(\frac{a+b}{2} i \right) \right) \sin \left(\frac{b-a}{2} \right) \exp \left[\left(\frac{a+b}{2} \right) i \right] \\
& \quad \quad \left. - \int_a^b f(R \exp(it)) \exp(it) dt \right| \\
& \leq R^2 \sup_{t \in [a, b]} |f''(R \exp(it))| \left[\frac{b-a}{2} - \sin \left(\frac{b-a}{2} \right) \right]. \quad (39)
\end{aligned}$$

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