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Solution of the problem of the center for cubic differential systems with three affine invariant straight lines of total algebraic multiplicity four

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Abstract. In this article, we study the planar cubic differential systems with a monodromic non-degenerate critical point and three affine invariant straight lines of total multiplicity four. We classify these systems and prove that monodromic point is of the center type if and only if the first Lyapunov quantity vanishes.

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1 Introduction and statement of main results

We consider the cubic differential systems

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \\ \gcd(P,Q) = 1, \end{cases} \tag{1}$$

and the vector fields $\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ associated to systems (1).

A straight line $\alpha x + \beta y + \gamma = 0,\ \alpha, \beta, \gamma \in \mathbb{C}$, is *invariant* for (1) if there exists a polynomial $K_l \in \mathbb{C}[x,y]$ such that the identity $\alpha P(x,y) + \beta Q(x,y) \equiv (\alpha x + \beta y + \gamma)K_l(x,y),\ (x,y) \in \mathbb{R}^2$, holds. If m is the greatest natural number such that $(\alpha x + \beta y + \gamma)^m$ divides $E(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P)$ then we say that $\alpha x + \beta y + \gamma = 0$ has *multiplicity* m [2]. Denote by $\mathsf{m}(\alpha x + \beta y + \gamma)$ the multiplicity of the straight line $\alpha x + \beta y + \gamma = 0$.

For (1) the origin $(0,0)$ is either a focus or a center, i.e. the trajectories in some neighborhood of $(0,0)$ can be spirals or closed. The problem of distinguishing between a center and a focus (the *problem of the center*) arises.

It is known that $(0,0)$ is a center for (1) if and only if in a neighborhood of $(0,0)$ the system has a nonconstant analytic first integral $F(x,y)$ (an analytic integrating factor of the form $\mu(x,y) = 1 + \sum \mu_j(x,y)$) [1]. Also it is known that there exists a formal power series $F(x,y) = x^2 + y^2 + \sum_{j \geq 3} F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (1) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$, i.e. $\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j$. The quantities $L_j,\ j = \overline{1,\infty}$, are polynomials with respect to the coefficients of system (1) called to be *the Lyapunov quantities*. For example, the first Lyapunov quantity looks as

$$L_1 = (bd - ac + 2bf - 2ag + dg - cf + 3k - 3l + p - q)/4.$$

The origin $(0,0)$ is a center for (1) if and only if $L_j = 0,\ j = \overline{1,\infty}$.

There are a great number of papers dedicates to the investigation of the problem of the center for cubic differential systems with invariant straight lines (see, for example, [3–13].

In this work we investigate the problem of center for (1) with three invariant straight lines of total multiplicity four. Our main result is the following one:

Main Theorem. *The cubic system* (1) *with three invariant straight lines of total multiplicity four has at origin* $(0,0)$ *a center if and only if the first Lyapunov quantity vanishes:* $L_1 = 0$.

2 Cubic systems with a monodromic critical point and three real invariant straight lines $l_1, l_2, l_3,\ m(l_1) + m(l_2) + m(l_3) \geq 4$

Let the system (1) have an affine real invariant straight line l_1. By a transformation of the form

$$x \to \nu \cdot (x \cos\varphi + y \sin\varphi),\ y \to \nu \cdot (y \cos\varphi - x \sin\varphi),\ \nu \neq 0$$

we can do l_1 to be described by the equation $x = 1$. Then, $k = -a, m = -c - 1, p = -f, r = 0$ and (1) is reduced to the system

$$\begin{cases} \dot{x} = (1-x)(y + ax^2 + (c+1)xy + fy^2) \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \\ \qquad \max\{deg(P),\ deg(Q)\} = 3,\ \gcd(P,Q) = 1. \end{cases} \tag{2}$$

Lemma 1. *The invariant straight line* $x - 1 = 0$ *of the system* (2) *has multiplicity at least two if and only if one of the following four series of conditions holds:*

$$a = f = 0,\ c = -2; \tag{3}$$

$$a = f = l = 0,\ n = 2 - b + c,\ s = -g - 1,\ c + 2 \neq 0; \tag{4}$$

$$a = 0,\ l = f,\ q = ((c+2)(b + n - c - 2) - df)/f,\ s = -g - 1; \tag{5}$$

$$\begin{matrix} l = f,\ n = (2a + ac - ab + f + fg + fs)/a, \\ q = ((c+2)(1 + g + s) + a^2 - ad)/a. \end{matrix} \tag{6}$$

Remark 1. The system $\{(2), (4),\ c + 1 \neq 1\}$ has the parallel real invariant straight lines $x - 1 = 0$ and $(c+1)x + 1 = 0$.

2.1 The cases when the straight lines l_1 and l_2 are parallel

Let $l_1, l_2 \in \mathbb{R}[x,y], l_1 \parallel l_2$. Without loss of generality we can consider $l_1 = x-1$, $m(l_1) \geq 2$. Then $l_2 = x - \alpha$, when $\alpha \in \mathbb{R}$ and $\alpha \neq 0; 1$. The straight line l_2 is invariant for system (2) if

$$a = f = 0,\ \alpha = -1/(c+1),\ c \neq -2. \tag{7}$$

Taking into account Lemma 1 and (7) we obtain that the straight lines $l_{1,2}$ are invariant and $m(l_1) \geq 2$ if and only if the coefficients of cubic system (1) satisfy the series of conditions (4), i.e. (1) has the form

$$\begin{array}{l}\dot{x} = y(1-x)(1+x+cx),\ c \notin \{-2;-1\},\\ \dot{y} = -(x+gx^2+dxy+by^2-(g+1)x^3+qx^2y+(c-b+2)xy^2).\end{array} \tag{8}$$

The straight line l_3 is described by an equation of the form $y - Ax - B = 0,\ A, B \in \mathbb{R},\ B \neq 0$ and is invariant for (8) if and only if at least one of the following four series of conditions holds: **1)** $b = 1,\ q = d(g+1),\ d \neq 0$ ($l_3 = 1-x+dy$); **2)** $d = c+3,\ g = -b-1,\ q = -1,\ b-1 \neq 0$ ($l_3 = 1-bx+(1-b)y$); **3)** $d = -c-3,\ g = -b-1,\ q = 1,\ b-1 \neq 0$ ($l_3 = 1-bx+(b-1)y$); **4)** $b = (2+c)(2+c+dq+q^2)/((2+c-d-q)(2+c+d+q)),\ g = -(8+8c+2c^2-d^2+cdq+q^2+cq^2)/((2+c-d-q)(2+c+d+q)),\ (d+q)(d+3q+cq) \neq 0$ ($l_3 = (2+c-d-q)(2+c+d+q)-(2+c)((2+c+dq+q^2)x+(d+3q+cq)y)$). Under these series of conditions, the system (7) can be expressed, respectively, as follows:

$$\begin{array}{l}\dot{x} = y(1-x)(1+x+cx),\ d(c+1)(c+2) \neq 0,\\ \dot{y} = -(x+gx^2+dxy+y^2-(g+1)x^3+d(g+1)x^2y+(c+1)xy^2);\end{array} \tag{9}$$

$$\begin{array}{l}\dot{x} = y(1-x)(1+x+cx),\ (b-1)(c+1)(c+2) \neq 0,\\ \dot{y} = -(x-(b+1)x^2+(c+3)xy+by^2+bx^3-x^2y+(b+c+2)xy^2);\end{array} \tag{10}$$

$$\begin{array}{l}\dot{x} = y(1-x)(1+x+cx),\ (b-1)(c+1)(c+2) \neq 0,\\ \dot{y} = -(x-(b+1)x^2-(c+3)xy+by^2+bx^3+x^2y+(c-b+2)xy^2);\end{array} \tag{11}$$

$$\begin{array}{l}\dot{x} = y(1-x)(1+x+cx),\quad (c+1)(c+2)(d+q)(d+3q+cq) \neq 0,\\ \dot{y} = -((2+c-d-q)(2+c+d+q)x(1+dy+qxy)+(c+2)(2+c\\ \quad +dq+q^2)(x^3+y^2)+((d+q)(d-q-cq)-2(c+2)^2)x^2\\ \quad +(c+2)(2+3c+c^2-d^2-3dq-2q^2)xy^2)/((c+2)^2-(d+q)^2).\end{array} \tag{12}$$

2.2 The cases when the straight lines l_2 and l_3 are parallel

We consider $l_1 = x-1$, $m(l_1) \geq 2$ and the straight line l_2 (l_3) is described by equation $y - Ax - B_1 = 0,\ B_1 \neq 0$ ($y - Ax - B_2 = 0,\ B_2 \neq 0,\ B_2 \neq B_1$). For each of the systems {(2), (3)}, {(2), (5)} and {(2), (6)} we will determine the conditions in order that the straight lines l_2 and l_3 be invariant. Note that straight lines $l_{2,3} = y - Ax - B_{2,3}$ are invariant for system (1) if the identities $\varphi_{2,3}(x) \equiv 0$, where $\varphi_{2,3}(x) = \Big(A \cdot P(x,y) + B_{2,3} \cdot Q(x,y)\Big)\Big|_{y=Ax+B_{2,3}}$, hold.

System {(2), (3)}. In this case

$$\begin{aligned}\varphi_2(x) = &-B_2(A+bB_2+B_2^2 l)-(1+A^2-2AB_2+2AbB_2+B_2d+3AB_2^2 l\\ &+B_2^2 n)x+(2A^2-A^2b-AB_2-Ad-g-3A^2B_2 l-2AB_2 n-B_2 q)x^2\\ &-(A^2+A^3 l+A^2 n+Aq+s)x^3.\end{aligned}$$

The identity $\varphi_2(x)\equiv 0$ gives us

$$\begin{aligned}&A=-B_2(b+lB_2),\\ &d=-(1+2bB_2^2-b^2B_2^2+2B_2^3 l-3bB_2^3 l-2B_2^4 l^2+B_2^2 n)/B_2,\\ &g=-b+bB_2^2-B_2 l+B_2^3 l-b^2B_2^3 l-2bB_2^4 l^2-B_2^5 l^3+bB_2^2 n+B_2^3 ln-B_2 q,\\ &s=B_2(b+B_2 l)(q-B_2(b+B_2 l)(1-bB_2 l-B_2^2 l^2+n)).\end{aligned}$$

In these conditions:

$$\begin{aligned}\varphi_3(x) = &((B_2-B_3)(B_2B_3(b+B_2 l+B_3 l)-((1-B_2B_3 n)+B_2^2(b+B_2 l)(b\\ &+B_2 l+3B_3 l))x+B_2(q+B_2(b+B_2 l)(3B_2 l(b+B_2 l)-2n-1))x^2)/B_2\end{aligned}$$

and $\varphi_3(x)\equiv 0 \Rightarrow \{b=-l(B_2+B_3),\ n=(1-2B_2^2B_3^2 l^2)/(B_2B_3),\ q=l(-2-B_2B_3+B_2^2B_3^2 l^2)\}$. The cubic system takes the form

$$\begin{aligned}&\dot{x}=y(x-1)^2,\\ &\dot{y}=-(B_2B_3x+lB_2B_3(B_2+B_3)x^2+((B_2+B_3)(l^2B_2^2B_3^2-1)\\ &\quad +2lB_2^2B_3^2)xy-lB_2B_3(B_2+B_3)y^2+B_2^2B_3^2 l^2x^3\\ &\quad -lB_2B_3(2+B_2B_3-l^2B_2^2B_3^2)x^2y+(1-2l^2B_2^2B_3^2)xy^2\\ &\quad +lB_2B_3y^3)/(B_2B_3),\ B_2\neq B_3.\end{aligned}\tag{13}$$

System {(2), (5)}. We have

$$\begin{aligned}\varphi_2(x) = &-B_2(A+bB_2+AB_2f+B_2^2 f)-(1+A^2(1+2B_2 f)\\ &+B_2(A(2b+c)+d+B_2(2Af+n)))x-(fg+A^2(A+B_2)f^2\\ &+Af(c(A-B_2)+Ab-B_2+d)+B_2(2b-df-4)-B_2c(4-b+c)\\ &+B_2n(2+c+2Af))x^2/f+(f(g+1)\\ &+A(df+(c+2)(c+2-b-n)+Af(1+c-n)))x^3/f\end{aligned}$$

and $\varphi_2(x)\equiv 0 \Rightarrow$

$$\begin{aligned}&\{b=(-A-AB_2f-B_2^2 f)/B_2,\quad n=(-1+A^2-AB_2c-B_2d)/B_2^2,\\ &g=-((c+2)(Ad-1)-2Af+(A+B_2)(Adf+(c+2)(3A+Ac-d))\\ &-(A+B_2)^2(f(d-A)+(c+2)(c+2-Af))-f(c+2)(A+B_2)^3)/(fB_2),\\ &(A+B_2)(-1+A^2-AB_2-2B_2^2-cAB_2-cB_2^2-dB_2)(2+c+Af+B_2f)=0\}.\end{aligned}$$

The function $\varphi_3(x)$ looks as

$$\begin{aligned}\varphi_3(x) = &(fB_2B_3(B_3-B_2)(A-B_2B_3f)\\ &+f(B_3-B_2)(2B_3+(1+A^2)(B_2-B_3)+B_2B_3(d+Ac-2AB_2f))x\\ &+(B_2-B_3)((2fA+c+2)(A^2-1)-B_2^2(c+2)^2\\ &-(c+1)(c+2)AB_2+B_2^2 f(-3A-2B_2-cB_2+A^2 f)\\ &-2cfAB_2(A+B_2)-dfB_2(2A+B_2)-dB_2(2+c))x^2\\ &+(A+B_2)(1-A^2+AB_2+2B_2^2+AB_2c+B_2^2c+B_2d)\\ &\times(2+c+Af+B_2f)x^3)/(fB_2^2).\end{aligned}$$

Taking into account that $\gcd(P,Q) = 1$ and $B_2B_3(B_3 - B_2) \neq 0$, the identities $\{\varphi_2(x) \equiv 0, \varphi_3(x) \equiv 0\}$ give us $\{A = f/(c+2),\ B_2 = -1/f,\ B_3 = -f/(c+2),\ b = 1,\ d = ((c+2)^3 + f^2(c+2-f^2))/(f(c+2)^2),\ g = -(c+2+f^2)/(c+2),\ n = ((c+2)^2 - f^2)/(c+2)\}$. In these conditions, together with (5), the system (2) has the form

$$\begin{aligned}
&\dot{x} = y(1-x)(1+x+cx+fy),\\
&\dot{y} = -(f(c+2)^2x - f(c+2)(c+2+f^2)x^2 + ((2+c)^3 + (2+c)f^2 - f^4)xy +\\
&((c+2)^3 + f(c+2)^2y^2 + f^3(c+2)x^3 + f^2(f^2-(c+2)(c+3))x^2y\\
&\quad +f(c+2)((c+2)^2 - f^2)xy^2 + f^2(c+2)^2y^3)/(f(c+2)^2).
\end{aligned} \tag{14}$$

System $\{(2),\ (6)\}$. In this case the functions $\varphi_{2,3}$ have the form:

$$\begin{aligned}
&\varphi_{2,3}(x) = -(aB_{2,3}(bB_{2,3} + A + AB_{2,3}f + B_{2,3}^2f)\\
&\quad +(a(1+A^2+B_{2,3}d) + aAB_{2,3}(2b+c+2Af+2B_{2,3}f)\\
&\quad +B_{2,3}^2(2a-ab+ac+f+fg+fs))x\\
&\quad +(ag + aA(a+Ab+Ac+d+A^2f) + B_{2,3}((c+2)(1+g+s)\\
&\quad +a(a-d)) + AB_{2,3}(3a-2ab+ac+2f+aAf+2fg+2fs))x^2\\
&\quad +(as + A(2+c+Af)(g+1) + aA^2(1-b) + As(c+2)\\
&\quad -Aad + A^2fs)x^3)/a
\end{aligned} \tag{15}$$

and $\{\varphi_2(x) \equiv 0,\ \varphi_3(x) \equiv 0\}$ if and only if at least one of the following set of conditions holds:

i) $a = B_2B_3f\omega,\ b = 1-\omega,\ c = -1 - B_2B_3f^2 - \omega,\ s = B_2B_3f^2(1+B_2B_3\omega),$
$d = -(B_2 + B_3 - 2B_2^2B_3^2f\omega)/(B_2B_3),\ g = f(B_2 + B_3 - B_2^2B_3^2f\omega);$
ii) $a = -(1+B_{2,3}f)/B_{2,3};\ b = -f(B_2+B_3+B_2B_3f),$
$c = (1 - B_2B_3(2 + B_{3,2}f + B_2B_3f^2))/(B_2B_3),$
$d = -(B_{3,2} + B_{2,3}(1+B_{3,2}f)(1 - 2B_{2,3}B_{3,2}^2f - B_2^2B_3^2f^2))/(B_2B_3),$
$g = f(B_{2,3} + 2B_{3,2} + B_2B_3f),\ s = -B_{3,2}f,$

where $\omega = (1+B_2f)(1+B_3f)$. In these conditions the system $\{(2),(6)\}$ looks as, respectively,

$$\begin{aligned}
&\dot{x} = (1-x)(y + B_2B_3f(1+B_2f)(1+B_3f)x^2\\
&-(1+B_2f+B_3f+2B_2B_3f^2)xy + fy^2),\\
&\dot{y} = -(B_2B_3x + (B_2+B_3)x(B_2B_3fx - y)\\
&-B_2^2B_3^2f(1+B_2f)(1+B_3f)x(B_2B_3fx - 2y)\\
&-B_2B_3f(B_2+B_3+B_2B_3f)y^2\\
&+B_2B_3f(1+B_2B_3(1+B_2f)(1+B_3f))x^2(B_2B_3fx - y)\\
&-B_2B_3fx^2y + (1-B_2B_3f)(1+B_2B_3f)xy^2 + B_2B_3fy^3)/(B_2B_3);
\end{aligned} \tag{16}$$

$$\begin{aligned}
\dot{x} = {}&(x-1)(-B_2B_3y + B_{3,2}(1+B_{2,3}f)x^2 - B_2B_3fy^2\\
&-(1 - B_2B_3 - B_{2,3}B_{3,2}^2f - B_2^2B_3^2f^2)xy)/(B_2B_3),\\
\dot{y} = {}&-(B_2B_3x + B_2B_3f(B_{2,3} + 2B_{3,2} + B_2B_3f)x^2\\
&-(B_2 + B_3 + B_2B_3f - B_2^2B_3^2f(2+B_{2,3}f)(1+B_{3,2}f))xy\\
&-B_2B_3f(B_2+B_3+B_2B_3f)y^2 - B_{2,3}B_{3,2}^2fx^3\\
&-B_2B_3f(1 + B_2B_3 + B_{2,3}B_{3,2}^2f)x^2y\\
&+(1 - B_2^2B_3^2f^2)xy^2 + B_2B_3fy^3)/(B_2B_3).
\end{aligned} \tag{17}$$

2.3 The cases when $l_{1,2,3}$ have different slopes and $m(l_1) \geq 2$, $l_1 \cap l_2 \cap l_3 \neq \emptyset$

We consider $l_1 = x-1$, $l_2 = y-A_2x-B_2$, $l_3 = y-(A_2+B_2-B_3)x-B_3$, $B_2B_3(B_3-B_2) \neq 0$. Denote

$$\psi_j(x) = \Big(A_j \cdot P(x,y) + B_j \cdot Q(x,y)\Big)\Big|_{y=A_jx+B_j}, \; j = 2;3, \tag{18}$$

where $A_3 = A_2 + B_2 - B_3$. In conditions (3) we have

$$\begin{aligned}\psi_{2,3}(x) = &-B_{2,3}(A_{2,3} + bB_{2,3} + B_{2,3}^2 l)\\ &-(B_{2,3}d + 1 + A_{2,3}^2 + B_{2,3}^2 n + A_{2,3}B_{2,3}(2b-2+3B_{2,3}l))x\\ &-(g + A_{2,3}d + B_{2,3}q + A_{2,3}^2(b-2+3B_{2,3}l) + A_{2,3}B_{2,3}(1+2n))x^2\\ &-(s + A_{2,3}(A_{2,3} + A_{2,3}^2 l + A_{2,3}n + q))x^3.\end{aligned}$$

Then $\psi_2(x) \equiv 0 \Rightarrow \{A_2 = -(bB_2 + B_2^2 l)$, $d = -(1 + B_2^2(n + 2b - b^2) - B_2^3 l(3b - 2 + 2B_2 l))/B_2$, $g = -b - B_2(l + q) + B_2^2(b + B_2 l)(1 + n) - B_2^3 l(b + B_2 l)^2$, $s = B_2(b + B_2 l)(q + B_2(b + B_2 l)(-1 + bB_2 l + B_2^2 l^2 - n))\}$ and $\psi_3(x) \equiv 0 \Rightarrow \{b = 1 - B_2 l - B_3 l$, $n = (1 - B_2B_3 + B_2^2B_3 l + B_2B_3^2 l - 2B_2^2B_3^2 l^2)/(B_2B_3)$, $s = (B_2 l - 1)(B_3 l - 1)\} \Rightarrow$

$$\begin{aligned}&\dot{x} = y(x-1)^2, \; B_2B_3(B_3-B_2) \neq 0,\\ &\dot{y} = -(B_2B_3x + B_2B_3(B_2l + B_3l - 2)x^2\\ &\quad +((B_2+B_3)(B_2^2B_3^2l^2 - 1) - B_2^2B_3^2 l)xy\\ &\quad -B_2B_3(B_2l + B_3l - 1)y^2 + B_2B_3(B_2l-1)(B_3l-1)x^3\\ &\quad +((B_2+B_3)(1 - B_2^2B_3^2l^2) + B_2^2B_3^2 l(1 + B_2B_3l^2) - 2B_2B_3 l)x^2y\\ &\quad +(1 - B_2B_3(1 - l(B_2+B_3) + 2B_2B_3l^2))xy^2 + B_2B_3 l y^3)/(B_2B_3).\end{aligned} \tag{19}$$

If conditions (5) hold, then

$$\begin{aligned}&\psi_{2,3}(x) = -B_{2,3}(A_{2,3} + bB_{2,3} + B_{2,3}f(A_{2,3} + B_{2,3}))\\ &-(1 + A_{2,3}^2 + A_{2,3}B_{2,3}(2b+c) + 2A_{2,3}B_{2,3}(A_{2,3}+B_{2,3})f + B_{2,3}(d + B_{2,3}n))x\\ &-(fg + A_{2,3}df - A_{2,3}B_{2,3}f(1+c-2n) + A_{2,3}^2 f(b+c) + A_{2,3}^2 f^2(A_{2,3}+B_{2,3})\\ &+B_{2,3}((c+2)(b+n-c-2) - df))x^2/f\\ &+(f(g+1) + A_{2,3}(df + (c+2)(c+2-b-n)) + A_{2,3}^2 f(1+c-n))x^3/f\end{aligned}$$

and $\{A_3 = A_2 + B_2 - B_3$, $\psi_{2,3} \equiv 0\} \Rightarrow \{A_{2,3} = -B_{2,3}$, $b = 1$, $c = -2$, $d = -(B_2 + B_3)/(B_2B_3)$, $g = -2$, $n = (B_2B_3 - 1)/(B_2B_3)\} \Rightarrow$

$$\begin{aligned}&\dot{x} = y(x-1)(x-fy-1), \; B_2B_3(B_3-B_2) \neq 0,\\ &\dot{y} = -(B_2B_3x - 2B_2B_3x^2 - (B_2+B_3)xy + B_2B_3y^2 + B_2B_3x^3\\ &\qquad +(B_2+B_3)x^2y + (1 - B_2B_3)xy^2 + B_2B_3fy^3)/(B_2B_3).\end{aligned} \tag{20}$$

Let now the conditions (6) hold. Then $\psi_j(x) = \varphi_{2,3}\Big|_{A=A_{2,3}}(x)$, where $\varphi_{2,3}(x)$ are given in (15). It is not difficult to show that the system $\{\psi_2(x) \equiv 0$, $\psi_3(x) \equiv 0$, $A_3 = A_2 + B_2 - B_3$, $B_2B_3(B_3 - B_2) \neq 0\}$ is not consistent.

2.4 The cases when $l_{1,2,3}$ are of generic position and $m(l_1) \geq 2$

Let the straight lines $l_{1,2,3}$ be of generic position, i.e. no pair of the lines is parallel and no more than two lines pass through the same point. We consider $l_1 = x - 1$, $l_{2,3} = y - A_{2,3}x - B_{2,3}$. Then the straight lines $l_{1,2,3}$ are of generic position if $A_3 - A_2 + B_2 - B_3 \neq 0$.

In conditions (3) we have $\psi_2(x) \equiv 0$ and $\psi_3(x) \equiv 0$ if $\{A_2 = -bB_2,\ d = -(B_2 + B_3)/(B_2B_3),\ g = -b(2 + (b-1)^2B_2B_3),\ s = b^2(1 + (b-1)^2B_2B_3),\ A_3 = -bB_3,\ l = 0,\ n = (1 - 2bB_2B_3 + b^2B_2B_3)/(B_2B_3),\ q = b(B_2 + B_3)(1 + B_2B_3 - 2bB_2B_3 + b^2B_2B_3)/(B_2B_3)\} \Rightarrow$

$$\begin{aligned} \dot{x} &= (x-1)^2y, \\ \dot{y} &= -(B_2B_3x(bx-1)(b(1+(b-1)^2B_2B_3)x - 1) \\ &\quad +(B_2+B_3)x(b(1+(b-1)^2B_2B_3)x-1)y \\ &\quad +(x + bB_2B_3(1+(b-2)x))y^2)/(B_2B_3). \end{aligned} \tag{21}$$

In conditions (5) we have $\psi_2(x) \equiv 0$ and $\psi_3(x) \equiv 0$ if:

1) $\{b = 1,\ c = g = -2,\ d = (-B_2 - B_3)/(B_2B_3),\ n = (1 - B_2B_3)/(B_2B_3),\ A_2 = -B_2,\ A_3 = -B_3\}$;

2)$\{b = 1,\ c = -(2B_{2,3} + f)/B_{2,3},\ g = -1 + B_{2,3}f,\ d = -(1 + B_{2,3}^2 + B_{2,3}^3f)/B_{2,3},\ n = (B_{2,3}^2 - 1)f/B_{2,3},\ A_2 = A_3 = -B_{2,3},\ B_{3,2} = -1/f\}$;

3) $\{b = 1,\ c = g = -2,\ d = (1 + A_3^2 + A_3f - A_3^3f)/A_3,\ n = (1 - A_3^2)f/A_3,\ A_2 = -B_2 = 1/(A_3 + f - A_3^2f),\ B_3 = -1/f\}$.

From these four conditions the following four systems result, respectively:

$$\begin{aligned} \dot{x} &= (1-x)y(1-x+fy),\ f \neq 0, \\ \dot{y} &= -(B_2B_3x - 2B_2B_3x^2 + B_2B_3x^3 \\ &\quad -(B_2+B_3)xy + (B_2+B_3)x^2y \\ &\quad +B_2B_3y^2 + (1 - B_2B_3)xy^2 + B_2B_3fy^3)/(B_2B_3). \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{x} &= (1-x)y(B_{2,3} - B_{2,3}x - fx + B_{2,3}fy)/B_{2,3},\ f \neq 0, \\ \dot{y} &= (-B_{2,3}x + B_{2,3}(1 - B_{2,3}f)x^2 + B_{2,3}^2fx^3 + (1 + B_{2,3}^2 \\ &\quad +B_{2,3}^3f)xy - B_{2,3}(B_{2,3} - f + B_{2,3}^2f)x^2y - B_{2,3}y^2 \\ &\quad +f(1 - B_{2,3}^2)xy^2 - B_{2,3}fy^3)/B_{2,3}. \end{aligned} \tag{23}$$

$$\begin{aligned} \dot{x} &= (1-x)y(1-x+fy),\ f \neq 0, \\ \dot{y} &= -(A_3(x-1)^2x + (-1 - A_3^2 \\ &\quad +A_3(A_3^2 - 1)f)(x-1)xy + (A_3 + fx \\ &\quad -A_3^2fx)y^2 + A_3fy^3)/A_3. \end{aligned} \tag{24}$$

In conditions (6) we have $\psi_2(x) \equiv 0$ and $\psi_3(x) \equiv 0$ if:

1) $\{a = B_2(A_3 + B_3)^2f(1 + B_3f)/(B_3 + B_2B_3f),\ b = -A_3/B_3 - (A_3 + B_3)f,\ c = (-A_3B_2f + B_3(-2 - f(A_3 + 3B_2 + B_3 + 2B_2(A_3 + B_3)f)))/(B_3 + B_2B_3f),\ d = (-B_3 + B_2(-1 - B_3f + B_2f(-1 + 2(A_3 + B_3)^2(1 + B_3f))))/(B_2B_3(1 + B_2f)),\ g = (A_3(2B_3 + B_2(A_3 + B_3)^2) + B_3(A_3B_2(3 + A_3^2 - A_3B_2) + (A_3 - B_2)(1 + 2A_3B_2)B_3 + (1 + (A_3 - B_2)B_2)B_3^2)f + B_2B_3(-A_3^3B_2 - 4A_3^2B_2B_3 + A_3(B_2 + B_3 - 5B_2B_3^2) + B_3(B_3 -$

$B_2(1+2B_3^2)))f^2 - B_2^2B_3^2(A_3+B_3)^3f^3)/(B_3+B_2B_3f)^2$, $s = (A_3(A_3+B_3(A_3-B_2+B_3)f)(B_3+B_2((A_3+B_3)^2+B_3(1+(A_3+B_3)^2)f)))/(B_3^3(1+B_2f)^2)$, $A_2 = (B_2(A_3+B_3(A_3-B_2+B_3)f))/(B_3+B_2B_3f)$, $(1+B_2f)af(A_3^2B_2+B_3+A_3B_2B_3+B_2B_3f+A_3^2B_2B_3f+A_3B_2B_3^2f) \neq 0\}$;

2) $\{b = (1-aB_{2,3})f/(a+f)$, $c = -(a^2+2aB_{2,3}+2af+2B_{2,3}f+aB_{2,3}^2f+f^2)/(B_{2,3}(a+f))$, $d = a(a(a+B_{2,3})+a(2+B_{2,3}^2)f+f^2)/(a+f)^2-(1+B_{2,3}^2+B_{2,3}^3f)/B_{2,3}$, $g = f(-1+2aB_{2,3}+B_{2,3}f)/(a+f)$, $s = -B_{2,3}f$, $A_2 = A_3 = -B_{2,3}f/(a+f)$, $B_{3,2} = -1/(a+f)\}$;

3) $\{b = 1$, $c = (1-af-A_2^2f^2)/(A_2f-1)$, $d = 2(a+f)$, $g = (1-a^2+aA_2-af-aA_2^2f-A_2^2f^2)/(A_2f-1)$, $s = A_2(a+f)(1-af-A_2f))/(1-A_2f)$, $A_3 = (1-af-A_2f)/(f(1-A_2f))$, $B_2 = B_3 = -1/f\}$;

4) $\{b = (1-aB_3)f/(a+f)$, $c = -(a^2+2aB_3+2af+2B_3f+aB_3^2f+f^2)/(B_3(a+f))$, $d = -(a^2-2a^3B_3+2af-5a^2B_3f+aB_3^2f+f^2-4aB_3f^2+aB_3^3f^2-B_3f^3)/(B_3(a+f)^2)$, $g = -(a^3+3a^2f+aB_3f+3af^2+2B_3f^2-aB_3^2f^2+f^3)/(B_3f(a+f))$, $s = (a^2+2af+B_3f-aB_3^2f+f^2)/(B_3(a+f))$, $A_3 = -B_3f/(a+f)$, $A_2 = (a+f)(a^2+2af+B_3f-aB_3^2f+f^2)/(B_3f(a^2-aB_3+2af-aB_3^2f+f^2))$, $B_2 = -(a+f)^2/(f(a^2-aB_3+2af-aB_3^2f+f^2))\}$.

From these five conditions the following five systems result, respectively:

$$\begin{aligned}
\dot{x} &= (1-x)[B_2(A_3+B_3)^2f(1+B_3f)x^2+B_3(1+B_2f)y-(A_3B_2f\\
&\quad +B_3(1+2B_2f)(1+(A_3+B_3)f))xy+B_3f(1+B_2f)y^2]/(B_3(1+B_2f)),\\
&\quad (1+B_2f)af(A_3^2B_2+B_3+A_3B_2B_3+B_2B_3f+A_3^2B_2B_3f+A_3B_2B_3^2f) \neq 0,\\
\dot{y} &= [-B_2B_3^3(1+B_2f)^2x+B_2B_3(-A_3(2B_3+B_2(A_3+B_3)^2)\\
&\quad -B_3(A_3B_2(3+A_3^2-A_3B_2)+(A_3-B_2)(1+2A_3B_2)B_3+(1+(A_3\\
&\quad -B_2)B_2)B_3^2)f+B_2B_3(A_3(-1+A_3^2)B_2+(B_2+A_3(-1+4A_3B_2))B_3\\
&\quad +(-1+5A_3B_2)B_3^2+2B_2B_3^3)f^2+B_2^2B_3^2(A_3+B_3)^3f^3)x^2-A_3B_2(A_3\\
&\quad +B_3(A_3-B_2+B_3)f)(B_3+B_2((A_3+B_3)^2+B_3(1+(A_3+B_3)^2)f))x^3\\
&\quad -B_3^2(1+B_2f)(-B_3+B_2(-1-B_3f+B_2f(-1+2(A_3+B_3)^2(1\\
&\quad +B_3f))))xy+(A_3^3B_2(1+B_3f)(B_2+B_3+2B_2B_3f)-B_2B_3^2f(B_2-B_3\\
&\quad -B_2B_3f+B_2^2(f+B_3(1+B_3f)^2))-A_3^2B_2B_3(1+B_3f)(-2B_3+B_2(-2\\
&\quad +f(B_2-4B_3+B_2B_3f)))+A_3B_3(B_3+B_2(1+B_3^2+B_3(3+B_3^2)f\\
&\quad -2B_3f(B_2+B_2B_3f)^2+B_2(1+2B_3f)(B_3+f+B_3^2f))))x^2y\\
&\quad +B_2B_3^2(1+B_2f)^2(A_3+B_3(A_3+B_3)f)y^2-B_3(1+B_2f)(B_3+B_2(A_3(A_3\\
&\quad +2B_3)+B_3(1+A_3^2-2B_2B_3+A_3(-B_2+B_3))f-2B_2B_3^2(A_3\\
&\quad +B_3)f^2))xy^2-B_2B_3^3f(1+B_2f)^2y^3]/(B_2B_3^3(1+B_2f)^2);
\end{aligned} \tag{25}$$

$$\begin{aligned}
\dot{x} &= (1-x)(aB_{2,3}(a+f)x^2+B_{2,3}(a+f)y\\
&\quad -(a^2+aB_{2,3}+2af+B_{2,3}f+aB_{2,3}^2f+f^2)xy\\
&\quad +B_{2,3}f(a+f)y^2)/(B_{2,3}(a+f)),\\
\dot{y} &= (-B_{2,3}(a+f)^2x-B_{2,3}f(a+f)(-1+2aB_{2,3}+B_{2,3}f)x^2\\
&\quad +B_{2,3}^2f(a+f)^2x^3-(-a^2+a^3B_{2,3}-2af+2a^2B_{2,3}f-2aB_{2,3}^2f\\
&\quad -f^2+aB_{2,3}f^2-B_{2,3}^2f^2-2aB_{2,3}^3f^2-B_{2,3}^3f^3)xy\\
&\quad +B_{2,3}f(a+f)(a-B_{2,3}+f-B_{2,3}^2f)x^2y
\end{aligned} \tag{26}$$

$$+B_{2,3}(aB_{2,3}-1)f(a+f)y^2+(a+f)((a+f)^2-B_{2,3}^2f^2)xy^2$$
$$-B_{2,3}f(a+f)^2y^3)/(B_{2,3}(a+f)^2);$$

$$\begin{aligned}\dot{x}=&-(x-1)(a(1-A_2f)x^2+(1-A_2f)y+f(a-A_2\\&+A_2^2f)xy+f(1-A_2f)y^2))/(1-A_2f),\ af\neq 0,\\ \dot{y}=&((A_2f-1)x+(1-a^2-A_2^2f^2-a(f+A_2(A_2f-1)))x^2\\&+A_2(a+f)(-1+(a+A_2)f)x^3+2(a+f)(A_2f-1)xy\\&+f(1-a^2-A_2^2f^2-a(f+A_2(A_2f-1)))x^2y\\&+(A_2f-1)y^2+f(A_2-f+A_2f(f-A_2)\\&+a(A_2f-2))xy^2+f(A_2f-1)y^3)/(1-A_2f);\end{aligned}\tag{27}$$

$$\begin{aligned}\dot{x}=&(x-1)(aB_3(a+f)x^2+B_3(a+f)y-(a^2+aB_3+2af+B_3f\\&+aB_3^2f+f^2)xy+B_3f(a+f)y^2)/(B_3(a+f)),\ B_3\neq -1/f,\\ \dot{y}=&-(B_3^2f(a+f)^2x-B_3(a+f)(a^3+3a^2f+aB3f+3af^2+2B_3f^2\\&-aB_3^2f^2+f^3)x^2+B_3f(a+f)(a^2+2af+B_3f-aB_3^2f+f^2)x^3\\&+B_3f(-a^2+2a^3B_3-2af+5a^2B_3f-aB_3^2f-f^2+4aB_3f^2\\&-aB_3^3f^2+B_3f^3)xy+(a^4+4a^3f+a^2B_3f+6a^2f^2\\&+2aB_3f^2-a^2B_3^2f^2+aB_3^3f^2+4af^3+B_3f^3-2aB_3^2f^3\\&+aB_3^4f^3+f^4-B_3^2f^4)x^2y-B_3^2(-1+aB_3)f^2(a+f)y^2\\&-B_3f(a+f)(2a^2+4af+B_3f+2f^2)xy^2\\&+B_3^2f^2(a+f)^2y^3/(B_3^2f(a+f)^2).\end{aligned}\tag{28}$$

3 Integrability of the systems (9)–(17), (19)–(28)

System (9) has first integral of the form

$$F(x,y)=f_1^{\alpha_1}f_2^{\alpha_2}f_3^{\alpha_3}f_4^{\alpha_4},\tag{29}$$

where

$$f_1=x-1,\ f_2=1+x+cx,\ f_3=1-x+dy,\ f_4=\exp\left[\tfrac{d(g+2)+(c+2)y}{x-1}\right],$$
$$\alpha_1=-(c+1)((c+2)^2+d^2(1+(3+c)(1+g))),\ \alpha_2=d^2(c-g),$$
$$\alpha_3=(c+1)(c+2)^2,\ \alpha_4=d(c+1)(c+2).$$

For *system* (10) (*respectively,* (11), (12)) the first Lyapunov quantity $L_1=-(c+2)/4$ (respectively, $L_1=(c+2)/4$, $L_1=-(d+q)/4$) is not equal to zero and therefore (10) (respectively, (11), (12)) has a focus at $(0,0)$.

System (13) has invariant straight lines $f_1=x-1$, $f_{2,3}=y-lB_2B_3x-B_{2,3}$ and exponential factor $f_4=\exp[1/(x-1)]$. Their cofactors are: $K_{f_1}(x,y)=y(x-1)$, $K_{f_{2,3}}(x,y)=(B_{3,2}+lB_2B_3x-y)(x+lB_2B_3y)/(B_2B_3)$, $K_{f_4}(x,y)=-y$. The system (13) has an integrating factor of the Darboux form

$$\mu(x,y)=f_1^{\alpha_1}f_2^{\alpha_2}f_3^{\alpha_3}f_4^{\alpha_4}\tag{30}$$

if and only if in x and y the identity $M(x,y)\equiv 0$ holds, where

$$M(x,y)=\alpha_1K_{f_1}+\alpha_2K_{f_2}+\alpha_3K_{f_3}+\alpha_4K_{f_4}+\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial x}.\tag{31}$$

For (13) the first Lyapunov quantity is $L_1 = -l(1 - B_2B_3 + B_2^2B_3^2l^2)/4$. If we take $\alpha_1 = -(1 + 2B_2B_3 + B_2^2B_3^2l^2)/(B_2B_3)$, $\alpha_2 = -(2B_2 - B_3 + lB_2^2B_3^2(2 + B_2l + B_3l))/(B_2 - B_3)$, $\alpha_3 = -3 - \alpha_2$, $\alpha_4 = (1 + l^2B_2^2B_3^2(1 + 2B_2B_3) + lB_2B_3(B_2 + B_3)(1 + l^2B_2^2B_3^2))/(B_2B_3)$ then $M(x, y) = 4L_1x^2$. Therefore, the system $\{(13),\ L_1 = 0\}$ has an integrating factor of the form (30) and for it the singular point $(0, 0)$ is of center type.

System (14). The invariant straight lines $f_1 = x - 1$, $f_2 = f(c + 2)y - f^2x + c + 2$, $f_3 = (c + 2)y - fx + f$ and exponential factor $f_4 = \exp[((c + 2)(1 + fy) - f^2)/(x - 1)]$ have the cofactors, respectively, $K_{f_1} = -y(1 + x + cx + fy)$, $K_{f_2} = -(f - fx + 2y + cy)(2x + cx + fy)/(c+2)$, $K_{f_3} = (2x + cx + fy)(-2 - c + f^2x - 2fy - cfy)/(f(c+2))$, $K_{f_4} = (2 + c - f^2x + 2fy + cfy)(2fx + cfx - 2y - cy + f^2y)/(c+2)$. If $\alpha_1 = -3 - f^2(f^2 - (c + 2)(c + 3))/(c + 2)^3$, $\alpha_2 = (f^2 - c - 2)/(c + 2)^2$, $\alpha_3 = -1 + f^2 - f^4/(c+2)^2$, $\alpha_4 = 1 - f^2/(c+2)^2$, then (see, (31)) $M(x, y) = 4f(c+2)^2y^2L_1$, where $L_1 = -f(6 + 5c + c^2 - f^2)(2 + 3c + c^2 + f^2)/(4(c + 2)^3)$ is the first Lyapunov quantity for singular point $(0, 0)$ of (14).

System (16) has first integral of the form (29), where

$$
\begin{aligned}
&f_1 = x - 1,\ f_{2,3} = y - B_2B_3fx - B_{2,3},\ f_4 = \exp\left[\tfrac{\omega(B_2 - B_3)(1 + B_2B_3fy)}{x - 1}\right],\\
&\alpha_1 = \omega(B_3 - B_2)(1 + B_2^2B_3^2f^2),\ \alpha_2 = -\omega B_2^2B_3(1 + B_2f),\\
&\alpha_3 = \omega B_2B_3^2(1 + B_3f),\ \alpha_4 = 1,
\end{aligned}
$$

where $\omega = (1 + B_2f)(1 + B_3f)$.

Systems (17). Let the function $\mu(x, y)$ have the form (30), where

$$
\begin{aligned}
&f_1 = x - 1,\ f_2 = y - B_2B_3fx - B_{2,3},\ f_3 = y - B_2B_3fx - B_{3,2},\\
&f_4 = \exp[(1 + B_{2,3}f)(B_{2,3} + B_2B_3f - y)/(B_{2,3}(x - 1))],\\
&\alpha_1 = -2 + 2B_{3,2}f + B_2B_3f^2 + 1/(B_2B_3),\\
&\alpha_2 = -(1 + B_{3,2}^2 + 2B_{2,3}B_{3,2}^2f + B_2^2B_3^2f^2)/(B_{3,2}(B_{2,3} - B_{3,2})),\\
&\alpha_3 = (-1 + B_{2,3}^2 - 2B_2B_3 - 2B_{2,3}^2B_{3,2}(1 + B_2B_3 - B_{3,2}^2)f\\
&\qquad -B_{2,3}^2(3B_{2,3} - 4B_{3,2})B_{3,2}^3f^2 - B_{2,3}(2B_{2,3} + B_{2,3}^3 - B_{3,2})B_{3,2}^2f^2\\
&\qquad -B_{2,3}^2B_{3,2}^3f^3(B_{2,3} - B_{3,2})(2B_2 + 2B_3 + B_2B_3f))/(B_{2,3}(B_{3,2} - B_{2,3})),\\
&\alpha_4 = B_{2,3}f(1 + B_{3,2}f)(1 + 2B_{2,3}B_{3,2}^2f + B_2^2B_3^2f^2)/(1 + B_{2,3}f).
\end{aligned}
$$

Then $div(\mu(x, y)\mathbb{X}) = 4L_1\mu(x, y)x^2$, where $L_1 = (1 + B_{2,3}^2B_{3,2}f + B_2^2B_3^2f^2)(1 + B_2B_3 + 2B_{2,3}B_{3,2}^2f + B_2^2B_3^2f^2)/(4B_{2,3}^2B_{3,2})$ is a first Lyapunov quantity and $\mathbb{X}$ is the vector field associated to system (17). Therefore, if $L_1 = 0$, then $\mu(x, y)$ is an integrating factor for (17).

System (19). If the first Lyapunov quantity $L_1 = -l(1 + B_2^2B_3^2l^2)/4$ vanishes, i.e. $l = 0$, then $\{(19),\ l = 0\}$ has the first integral of the form (29), where $f_1 = x - 1$, $f_{2,3} = y + B_{2,3}x - B_{2,3}$, $f_4 = \exp[1/x - 1)]$, $\alpha_1 = (B_2 - B_3)(1 - B_2B_3)$, $\alpha_2 = B_2^2B_3$, $\alpha_3 = -B_2B_3^2$, $\alpha_4 = B_3 - B_2$.

System (20) has the first integral of the form (29), where $f_1 = x - 1$, $f_{2,3} = y + B_{2,3}x - B_{2,3}$, $f_4 = \exp[(1 + B_2B_3fy)/x - 1)]$, $\alpha_1 = (B_2 - B_3)(B_2B_3(1 + B_2f + B_3f) - 1)$, $\alpha_2 = -B_2^2B_3(1 + B_2f)$, $\alpha_3 = B_2B_3^2(1 + B_3f)$, $\alpha_4 = B_2 - B_3$.

Systems (21), (22), (24), (25), (27), (28) are integrable, because they have the following integrating factor, respectively:

$\mu(x,y) = 1/((x-1)^2(-B_2 + bB_2x + y)(-B_3 + bB_3x + y))$;

$\mu(x,y) = 1/((x-1)^2(-B_2 + B_2x + y)(-B_3 + B_3x + y))$;

$\mu(x,y) = 1/((x-1)^2(-A_3 + A_3x - y)(-1 + x - A_3y - fy + A_3^2 fy))$;

$\mu(x,y) = 1/((x-1)^2(B_3 + A_3x - y)(-B_2B_3 - B_2^2B_3f - A_3B_2x - A_3B_2B_3fx + B_2^2B_3fx - B_2B_3^2fx + B_3y + B_2B_3fy))$;

$\mu(x,y) = 1/((x-1)^2(1 - A_2fx + fy)(1 - A_2f - x + afx + A_2fx + fy - A_2f^2y))$;

$\mu(x,y) = 1/((x-1)^2(B_3(a+f-fx) - (a+f)y)(-B_3(a+f)^2 + (a+f)(a^2 + 2af + B_3f - aB_3^2f + f^2)x - B_3f(a^2 - aB_3 + 2af - aB_3^2f + f^2)y)))$.

Systems (23). If the first Lyapunov quantity $L_1 = -2(B_{2,3} - f + B_{2,3}^2f)(B_{2,3} + f + B_{2,3}^2f)/B_{2,3}$ vanishes, then the function $\mu(x,y) = (B_{2,3}(x-1) + y)^{-1+B_{2,3}^2+B_{2,3}^3f}(1 + B_{2,3}fx + fy)^{B_{2,3}(B_{2,3}+1/f)}Exp[B_{2,3}(1 + B_{2,3}f)(1 + B_{2,3}f + fy)/(f(1-x))]/(x-1)^3$ is an integrating factor for (23).

Systems (26). For these systems we have $L_1 = 2(a^2 - aB_{2,3} + 2af - B_{2,3}f - 2aB_{2,3}^2f + f^2 - B_{2,3}^2f^2)(a^2 + 2af + B_{2,3}f + f^2 + B_{2,3}^2f^2)/(B_{2,3}(a+f)^2)$. If $L_1 = 0$ the expression (30) is an integrating factor for (26), where

$f_1 = aB_{2,3} + B_{2,3}f - B_{2,3}fx - ay - fy$, $f_2 = 1 + B_{2,3}fx + ay + fy$, $f_3 = x - 1$, $f_4 = Exp[((1 + B_{2,3}f)x + (a+f)y)/(x-1)]$;

$\alpha_1 = (-a^3 + a^2(B_{2,3} - 3f) + f(B_{2,3}^2 + B_{2,3}(-1 + 3B_{2,3}^2)f + (-1 - B_{2,3}^2 + 3B_{2,3}^4)f^2 + B_{2,3}^5f^3) + af(-3f + B_{2,3}^2(-f + B_{2,3}(1 + B_{2,3}f)^2)))/(B_{2,3}f(a+f)(1 + B_{2,3}(a+f)))$, $\alpha_2 = (a^2 + f^2 - a(B_{2,3} + (-2 + B_{2,3}^2)f))/(B_{2,3}f(1 + B_{2,3}(a+f)))$, $\alpha_3 = -3$, $\alpha_4 = -B_{2,3}f(1 + B_{2,3}f)/(a+f)^2$.

4 Cubic systems with a monodromic critical point and three invariant straight lines l_1, l_2, l_3, $m(l_3) \geq 2$, $m(l_1) + m(l_2) + m(l_3) \geq 4$, where l_1, l_2 are complex lines and l_3 is a real one

4.1 The cases when l_1 and l_2 are parallel complex lines

Without loss of generality we consider $l_3 = x - 1$, $m(l_3) \geq 2$ and $l_{1,2} = y - Ax - \alpha \mp \beta i$, $A, \alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, $i^2 = -1$. To get the class of cubic systems for which these straight lines are invariant it is sufficient to put $B_{2,3} = \alpha \pm \beta i$ in (13), (16) and (17). Denote $A = lB_2B_3$ in the case of system (13) and $A = fB_2B_3$ in the cases of systems (16), (17).

Via a transformation (2) we can consider that invariant straight line l_1 is described by the equation $l_1 \equiv x - \alpha - i - 0$ and system (1) has the form

$$\begin{aligned} \dot{x} &= y[(x-\alpha)^2 + 1]/(\alpha^2 + 1), \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3), \end{aligned} \tag{32}$$

i.e. $a = f = k = p = r = 0$, $c = -2\alpha/(\alpha^2 + 1)$, $m = 1/(\alpha^2 + 1)$.

We will determinate conditions under which the system (32) has the real invariant straight line $l_3 \equiv y - Ax - B = 0$, $A, B \in \mathbb{R}$, $B \neq 0$ and then conditions under which this straight line has the multiplicity two.

Note $\varphi(x) = \Big(A \cdot P(x,y) + B \cdot Q(x,y)\Big)\Big|_{y=Ax+B}$, $\sigma(x,y) = E_1(\mathbb{X})/(y - Ax - B)$ and $H_2(x) = \sigma(x,y)|_{y=Ax+B}$.

The system (32) has the invariant straight line $l_3 \equiv y - Ax - B = 0$, $A, B \in \mathbb{R}$, $B \neq 0$ if the polynomial $\varphi(x) = -(B(A+B(b+Bl))(1+\alpha^2)+[A^2(1+\alpha^2)+(1+B(d+Bn))(1+\alpha^2)+AB(-2\alpha+2b(1+\alpha^2)+3Bl(1+\alpha^2)]x+[(g+Bq)(1+\alpha^2)+A^2(b+3Bl-2\alpha+(b+3Bl)\alpha^2)+A(B+d+2Bn+(d+2Bn)\alpha^2)]x^2+[A(A(1+Al+n)+q)+s+(A(A(Al+n)+q)+s)\alpha^2]x^3)/(1+\alpha^2)$ is equivalent to zero, i. e. $\{A = -B(b+Bl)$, $q = (-g+(b+Bl)(-2+B(-d+B(1+(b+Bl)^2)))-2B^2(b+Bl)^2\alpha+(-g+(b+Bl)(-2+B(-d+B(b+Bl)^2)))\alpha^2)/(B(1+\alpha^2))$, $s = -(b+Bl)(b+g+Bl)$, $n = (-1+B(-d+B(b+Bl)(b+2Bl)))/B^2 - 2(b+Bl)\alpha/(1+\alpha^2)\}$.

In these conditions $H_2(x) = (1-(b+Bl)x)H_{21}(x)/(B(1+\alpha^2)^3)$ where $H_{21}(x) = B^2(1+\alpha^2)(2+2\alpha^2+Bd(1+\alpha^2)+4B^4l^2(1+\alpha^2)+B^3l(-2\alpha+3b(1+\alpha^2)))+2B^2(1+\alpha^2)(-2b^2B^2\alpha+b^3B^2(1+\alpha^2)+g(1+\alpha^2)-Bl(4-2B^2+2B^3l\alpha+4\alpha^2+3Bd(1+\alpha^2)+4B^4l^2(1+\alpha^2))-b(2-B^2+4B^3l\alpha+2\alpha^2+2Bd(1+\alpha^2)+5B^4l^2(1+\alpha^2)))x+(4(1+\alpha^2)^2+4Bd(1+\alpha^2)^2+4B^8l^4(1+\alpha^2)^2+B^7l^3(1+\alpha^2)(14\alpha+3b(1+\alpha^2))-B^6l^2(8+4\alpha^2+9b^2(1+\alpha^2)^2-32b(\alpha+\alpha^3))+B^5l(-2\alpha-11b^3(1+\alpha^2)^2+9dl(1+\alpha^2)^2-b(11+3\alpha^2)+22b^2(\alpha+\alpha^3))+B^3(1+\alpha^2)(d(-1+4b\alpha+3b^2(1+\alpha^2))+l(4\alpha+8b(1+\alpha^2)-5g(1+\alpha^2)))+B^2(1+\alpha^2)(-2-2g\alpha+d^2(1+\alpha^2)+b(4\alpha-3g(1+\alpha^2)))+B^4(b^2(-3+\alpha^2)-3b^4(1+\alpha^2)^2+4b^3(\alpha+\alpha^3)+4l(1+\alpha^2)(d\alpha+2l(1+\alpha^2))+2b(-\alpha+6dl(1+\alpha^2)^2)))x^2+2(2+2bB^2\alpha+2\alpha^2+Bd(1+\alpha^2)+2B^4l^2(1+\alpha^2)+2B^3l(b+\alpha+b\alpha^2))(g+B(b+Bl)(-d+B(1+b^2+bBl))-2B^2(b+Bl)^2\alpha+(g+B(b+Bl)(-d+bB(b+Bl)))\alpha^2)x^3+(g+B(b+Bl)(-d+bB(b+Bl))-2B^2(b+Bl)^2\alpha+(g+B(b+Bl)(-d+bB(b+Bl)))\alpha^2)(g+B(b+Bl)(-d+B(1+b^2+bBl))-2B^2(b+Bl)^2\alpha+(g+B(b+Bl)(-d+bB(b+Bl)))\alpha^2)x^4$. The identity $H_2(x) \equiv 0$, which is equivalent with $H_{21}(x) \equiv 0$, gives as the following sets of conditions:

$$\{d = (-4+b^2B^2-2bB^2\alpha-4\alpha^2+b^2B^2\alpha^2)/(2B(1+\alpha^2)),\ g = -b,\ l = -b/(2B)\}$$

or

$$\{d = -2/B,\ g = -b(2+B^2+b^2B^2-2bB^2\alpha+2\alpha^2+b^2B^2\alpha^2)/(1+\alpha^2),\ l = 0\}.$$

Therefore, the systems (1) with three invariant straight lines, two of them are parallel complex lines, and the third one is an invariant straight line of multiplicity two, look like that:

$$\begin{aligned}\dot{x} &= y(1+(x-\alpha)^2)/(1+\alpha^2),\\ \dot{y} &= (-8xy^2(1+\alpha^2)-2B^2(bx-2)(x(bx-2)-2by^2)(1+\alpha^2)\\ &\quad +4By(4x-2bx^2+by^2)(1+\alpha^2)+bB^3xy(-4x+8\alpha\\ &\quad +b(bx-4)(1+\alpha^2)))/(8B^2(1+\alpha^2));\end{aligned} \tag{33}$$

$$\begin{aligned}\dot{x} &= y(1+(x-\alpha)^2)/(1+\alpha^2),\\ \dot{y} &= (2Bx(bx-1)y(1+\alpha)^2)+xy^2(1+\alpha)^2)+bB^4x^2(bx\\ &\quad -1)(1+b(b-2\alpha)+b\alpha)^2))+2bB^3x^2y(1+b(b-2\alpha)+b\alpha)^2))\\ &\quad +B^2(-2bx^2(1+\alpha)^2)+b^2x^3(1+\alpha)^2)+by^2(1+\alpha)^2)\\ &\quad +x(1-2by^2\alpha)+\alpha)^2+b^2y^2(1+\alpha)^2)))))/(B^2(1+\alpha^2)).\end{aligned} \tag{34}$$

4.2 The cases when $l_{1,2}$ are homogeneous nonparallel complex lines

In these cases the straight lines $l_{1,2}$ are invariant for (1) if they are described by the equations $y \mp ix = 0$ and if the conditions

$$f = a + d,\ g = b + c,\ q = l - k + p,\ s = m + n - r \tag{35}$$

hold.

We can consider the real straight line $l_3 = x - 1$. The line l_3 is an invariant one for the system {(1),(35)} if $P(1, y) = a + k + (1 + c + m)y + (a + d + p)y^2 + ry^3 \equiv 0$, i. e. $\{r = 0,\ k = -a,\ m = -1 - c,\ p = -a - d\}$.

In these conditions we have $\sigma(x, y)|_{x=1} = (b + n + ly)\psi(y)$, where $\psi(y) = a^2 - al + (2+c)(b+n) + 2(a(2+c) + d(b+n))y + (2a^2 + 2a(d-l) + dl + (2+c)(2-b+c-n))y^2 + 2(2+c)(a+d-l)y^3 + (a+d)(a+d-l)y^4$. Taking into account $\gcd(P, Q) = 1$, the identity $\sigma(x, y)|_{x=1} \equiv 0$ holds if $\psi(y) \equiv 0$, i.e. $\{c = -2,\ d = 0,\ l = a\}$ or $\{a = 0,\ c = -2,\ d = 0\}$. Then the cubic system (1) looks like one of these two forms:

$$\begin{aligned} \dot{x} &= (1 - x)(ax^2 + y - xy + ay^2), \\ \dot{y} &= -x + (2 - b)x^2 - (1 + n)x^3 - ax^2y - by^2 - nxy^2 - ay^3; \end{aligned} \tag{36}$$

$$\begin{aligned} \dot{x} &= (x - 1)^2 y, \\ \dot{y} &= -x + (2 - b)x^2 - (1 + n)x^3 - lx^2y - by^2 - nxy^2 - ly^3. \end{aligned} \tag{37}$$

4.3 The cases when $l_{1,2}$ are nonhomogeneous nonparallel complex lines

The case when $l_3 = x - A,\ A \neq 0$.

Let $l_{1,2} \in \mathbb{C}[x, y]\backslash\mathbb{R}[x, y]$, $l_2 = \overline{l_1}$ and $l_1 \nparallel l_2$. Via a transformation of the form (2) we can do that the line l_1 (l_2) be described by equation $y - (\alpha + \beta i)x - 1 = 0$, $\beta \neq 0$ ($y - (\alpha - \beta i)x - 1 = 0$, $\beta \neq 0$), i.e. we can do that the straight lines l_1 and l_2 pass through the point $(0, 1)$. The lines $l_{1,2}$ are invariant for (1) if and only if the following conditions

$$\begin{aligned} &k = g - 2\alpha + 2a\alpha,\ l = -b,\ m = 2 - a + d + 2c\alpha - 3\alpha^2 + \beta^2, \\ &n = -1 - d - 2\alpha^2 - f\alpha^2 - 2\beta^2 - f\beta^2,\ p = b - c + 4\alpha + 2f\alpha, \\ &q = -g - c\alpha^2 + 2\alpha^3 - c\beta^2 + 2\alpha\beta^2,\ r = -1 - f, \\ &s = -(-1 + a)(\alpha^2 + \beta^2), \end{aligned} \tag{38}$$

hold, i.e. if (1) has the form

$$\begin{aligned} \dot{x} &= y + ax^2 + cxy + fy^2 + (2 - a + d + 2c\alpha - 3\alpha^2 + \beta^2)x^2y \\ &\quad + (g - 2\alpha + 2a\alpha)x^3 + (b - c + 4\alpha + 2f\alpha)xy^2 - (f + 1)y^3, \\ \dot{y} &= -x - gx^2 - dxy - by^2 + (g + (c - 2\alpha)(\alpha^2 + \beta^2))x^2y \\ &\quad + (a - 1)(\alpha^2 + \beta^2)x^3 + (1 + d + (f + 2)(\alpha^2 + \beta^2))xy^2 \\ &\quad + by^3,\ \beta \neq 0. \end{aligned} \tag{39}$$

Let $l_3 = x - A,\ A \neq 0$. The straight line $l_3 = x - A,\ A \neq 0$ is an invariant one for the system (39) if $P(A, y) = A^2(a + Ag - 2A\alpha + 2aA\alpha) + (1 + 2A^2 - aA^2 + Ac +$

$A^2d+2A^2c\alpha-3A^2\alpha^2+A^2\beta^2)y+(Ab-Ac+f+4A\alpha+2Af\alpha)y^2-(1+f)y^3\equiv 0$, i. e. $\{f=-1,\ g=-(a-2A\alpha+2aA\alpha)/A,\ c=-(1-Ab-2A\alpha)/A,\ d=-(2A-aA+b+2Ab\alpha+A\alpha^2+A\beta^2)/A\}$.

In these conditions we have $\sigma(x,y)|_{x=A}=(-A+aA+by)\psi(y)/A^2$, where $\psi(y)=A^2(a^2A^2(1+2A\alpha)-(1+A(b+2\alpha))((1+A\alpha)^2+A^2\beta^2)+a(1+A(4\alpha+A(-2+5\alpha^2+\beta^2+2A\alpha(-1+\alpha^2+\beta^2)))))-2(A^2)(a^2A^2-(1+A\alpha)^2-A^2\beta^2+a(2-A(b-4\alpha)+A^2(-1-2b\alpha+\alpha^2+\beta^2)))y+(-1-A(b+4\alpha-A(-1+3a+b^2-4\alpha^2)+A^2b(-1+4a-2b\alpha-3\alpha^2+\beta^2)))y^2-2(-1+Ab)(1+Ab+2A\alpha)y^3+(-1+Ab)y^4$. If $(-A+aA+by)\equiv 0$, then the polynomials $P(x,y)$ and $Q(x,y)$ have a common factor different from one. If $\psi(y)\equiv 0$, then $\{A=1/b,\ a=-b^2-2b\alpha-\alpha^2-\beta^2\}$. So, the cubic system (1) looks like that :

$$\begin{aligned}
\dot{x} &= (bx-1)(((b+\alpha)^2x^2+\beta^2)-y-(b+2\alpha)xy+y^2),\\
\dot{y} &= -x-x^2(b^3+5b\alpha^2+2\alpha^3+(b+2\alpha)\beta^2)+2xy(1\\
&\quad +(b+\alpha)^2+\beta^2)+x^2y(2\alpha+(b+2\alpha)((b+\alpha)^2+\beta^2))\\
&\quad -by^2-(1+2b^2+4b\alpha+\alpha^2+\beta^2)xy^2+by^3\\
&\quad -x^3(\alpha^2+\beta^2)(1+(b+\alpha)^2+\beta^2).
\end{aligned}\tag{40}$$

The case when $l_3=y-Ax-B,\ A,B\in\mathbb{R},\ B\neq 0$.

Let $l_3\equiv y-Ax-B=0,\ A,B\in\mathbb{R},\ B\neq 0$. This straight line is an invariant one for the system (39) if $\varphi(x)=(B-1)B(A+AB+bB+ABf)(-1-Bd+A^2(-1-2Bf+3B^2(1+f))+AB((-1+B)(2b+c)-2B(2+f)\alpha)+B^2(1+d+(2+f)\alpha^2+(2+f)\beta^2))x+(aA(-1+B)+A^3(-f+3B(1+f))-g+A^2(b(-1+B)+(-1+2B)c-4B(2+f)\alpha)+A((-1+B)d+B\alpha(-2c+7\alpha+2f\alpha)+B(3+2f)\beta^2)+B(g+(c-2\alpha)(\alpha^2+\beta^2)))x^2+(-1+a+A^2+Ac+A^2f-2A\alpha)(A^2-2A\alpha+\alpha^2+\beta^2)x^3\equiv 0$ or:

1) $\{a=1,\ f=-2,\ A=c-2\alpha,\ B=1\}$;

2) $\{a=1-A(A+c+Af-2\alpha),\ b=-A(1+B+Bf)/B,\ d=(1-A^2(1+B^2(1+f))+AB(c-Bc+2B(2+f)\alpha)-B^2(1+(2+f)\alpha^2+(2+f)\beta^2))/((B-1)B),\ g=(-A^3B(1+f)+A^2B(-c+2(1+B+Bf)\alpha)-B^2(c-2\alpha)(\alpha^2+\beta^2)+A(-1+B+B^2(\alpha(2c-(5+f)\alpha)-(1+f)\beta^2)))/((B-1)B)\}$.

In the conditions 1) the cubic system (39) has $H_2(x)=-H_{21}(x)H_{22}(x)$, where $H_{21}(x)=b+(2+2bc+d-4b\alpha+\alpha^2+\beta^2)x+(c+bc^2+cd+g-2\alpha-4bc\alpha-2d\alpha+4b\alpha^2+c\alpha^2-2\alpha^3+c\beta^2-2\alpha\beta)x^2$, and $H_{22}(x)=2b+(2+5bc-c^2+d-10b\alpha+6c\alpha-8\alpha^2)x-2(c-2\alpha)(-2-2bc+c^2-d+4b\alpha-6c\alpha+8\alpha^2)x^2-(c-2\alpha)(-c-bc^2+c^3-cd-g+2\alpha+4bc\alpha-8c^2\alpha+2d\alpha-4b\alpha^2+20c\alpha^2-16\alpha^3)x^3$. If $H_{21}(x)\equiv 0$ then the cubic system is degenerate. Let $H_{21}(x)\not\equiv 0$, and $H_{22}(x)\equiv 0\Rightarrow\{b=0,\ d=-2,\ c=2\alpha\}$ or $\{b=0,\ d=-2+g^2-2g\alpha,\ c=g+2\alpha\}$.

In this way we got the systems:

$$\begin{aligned}
\dot{x} &= x^2+gx^3+y-2y^2+y^3+2xy\alpha-2xy^2\alpha+(\alpha^2+\beta^2-1)x^2y,\\
\dot{y} &= x(1+gx-y)(y-1);
\end{aligned}\tag{41}$$

$$\begin{aligned}\dot{x} &= x^2 + gx^3 + y + (g + 2\alpha)xy + (g^2 - 1 + \alpha^2 \\ &\quad +\beta^2)x^2y - 2y^2 - (g + 2\alpha)xy^2 + y^3, \\ \dot{y} &= x(-1 - gx + (2 - g^2 + 2g\alpha)y + gxy + (g^2 - 1 - 2g\alpha)y^2 \\ &\quad +g(\alpha^2 + \beta^2)xy).\end{aligned} \tag{42}$$

In the case of conditions 2) the cubic system (39) has $H_2(x) = -H_{23}(x)H_{24}(x)/((B-1)^2B^2)$, where $H_{23}(x) = B(1+B+Bf)+(A+2AB+Bc+2ABf-2B)x$, and $H_{24}(x) = (-1+B)(B^2)(A^2(1+B(-1+B(5+B(-1+f)))(1+f))+AB(-(-1+B)^2c-2B(2+f)(2+Bf)\alpha)+(1+B+Bf)(1+B(-2+B(1+(2+f)\alpha^2+(2+f)\beta^2))))+2(-1+B)B(A^3(1+B^2(7+2B(-1+f))(1+f))+A^2B(Bc(3+B(-1+f))-2(1+B(8+5f+B(-1+f(3+2f))))\alpha)+B^2(1+B+Bf)(c-2\alpha)(\alpha^2+\beta^2)+A(1+B(-2+B(1-2c(2+Bf)\alpha+(3(4+f)+2B(1+f(4+f)))\alpha^2+(4+3f+2B(1+f)^2)\beta^2))))x+(-1+3B+A^4(-1+B(1+B(-12+B(19+3B(-3+f)-4f))(1+f)))+A^3B((-1+B)Bc(5+B(-2+4f))+2\alpha+2B(2(7+5f)+B(-23+4(-2+f)f-2B(-6+(-2+f)f)))\alpha)+B^2(-3+2c\alpha-(7+2f)\alpha^2-(3+2f)\beta^2)+B^3(1+3(4+f)\alpha^2+(4+3f)\beta^2+2c\alpha(-2+\alpha^2+\beta^2)-c^2(\alpha^2+\beta^2))-B^4(\alpha^2(5+f+(1+f)(2+f)\alpha^2)+(1+f)(1+2(2+f)\alpha^2)\beta^2+(1+f)(2+f)\beta^4+2c\alpha(-1+\alpha^2+\beta^2)-c^2(\alpha^2+\beta^2))+A^2(-2+B(4+B^3(-5+c^2-5f-8cf\alpha-(23+f(5+2f))\alpha^2-7\beta^2+f(-5+2f)\beta^2)-B(9+6f-8c\alpha+25\alpha^2+8f\alpha^2+(9+8f)\beta^2)+B^2(-c^2+8c(-1+f)\alpha+12(1+3\alpha^2+\beta^2)+f(11-(5+4f)\alpha^2+(7-4f)\beta^2))))+AB(2\alpha+B(-2(-1+B)Bc^2\alpha+(-1+B)c(2+3\alpha^2+3\beta^2+B(-2+4(1+f)\alpha^2+4f\beta^2))+2\alpha(5+4f+3\alpha^2+3\beta^2+B(-11-7f-4\alpha^2+3f\alpha^2+(-4+3f)\beta^2+B(5(1+\alpha^2+\beta^2)+f(3+(3+2f)\alpha^2+(3+2f)\beta^2)))))))x^2-2B(2A+c+2Af-2\alpha)(A^2-2A\alpha+\alpha^2+\beta^2)(1+A^2-2B-2A^2B+B^2+3A^2B^2+A^2B^2f-4AB^2\alpha-2AB^2f\alpha+2B^2\alpha^2+B^2f\alpha^2+2B^2\beta^2+B^2f\beta^2)x^3-B^2(2A+c+2Af-2\alpha)(3A+c+2Af-2\alpha)(A^2-2A\alpha+\alpha^2+\beta^2)^2x^4$.

If $H_{23}(x) \equiv 0$ then the polynomials $P(x,y)$ and $Q(x,y)$ have a common factor different from one. Let $H_{23}(x) \not\equiv 0$, and $H_{24}(x) \equiv 0 \Rightarrow \{c = 2(A+A^3-2AB+AB^2-2A^2B\alpha+A^2B^2\alpha-AB^2\alpha^2+B^2\alpha^3+AB^2\beta^2+B^2\alpha\beta^2)/(B^2(A^2-2A\alpha+\alpha^2+\beta^2)),\ f = (-1-A^2+2B-B^2-A^2B^2+2AB\alpha+2AB^2\alpha-2B^2\alpha^2-2B^2\beta^2)/(B^2(A^2-2A\alpha+\alpha^2+\beta^2))\}$ and the cubic system (1) looks like that:

$$\begin{aligned}\dot{x} = {} & [A^4x^2((1-B)(1+y)+2(1+B)x\alpha) - A^3x((B-1)y(y+By-2) \\ & -2(B-1)x(B+(2+B)y)\alpha + x^2(1+B((5+B)\alpha^2+(1+B)\beta^2-1))) \\ & +A^2((1-B)(B^2-y)(y-1)y + 2(B-1)xy(-y+B(B+2y-2))\alpha \\ & -(B-1)x^2(1+y+B((B+8y)\alpha^2+B\beta^2-2) + 2x^3\alpha(1+B(2B(\alpha^2 \\ & +\beta^2)-1))) - A(2(1-B)B(B-y)(y-1)y\alpha + Bx^3(\alpha^2+\beta^2) \\ & \times(2+B(B(1+\alpha^2+\beta^2)-3)) - 2(B-1)x^2\alpha(2y-2By \\ & +B^2(y(1+\alpha^2+\beta^2)-1)) + (B-1)xy(y+B(4+y(-2-3\alpha^2+\beta^2) \\ & +B(y+2\alpha^2+y\alpha^2+(y-2)\beta^2-2)-2))) + (B-1)(B^2x^2(\alpha^2+\beta^2) \\ & +y^3(1+B(B(1+\alpha^2+\beta^2)-2)) + By(\alpha^2+\beta^2)(B(1 \\ & +x(2\alpha+x(1+\alpha^2+\beta^2))) - 2x^2) - y^2(1+2x\alpha \\ & +B(B(1+2\beta^2+2\alpha(\alpha+x(1+\alpha^2+\beta^2))) - 2 - 4x\alpha))) \\ & -A^5x^3]/((B-1)B^2(A^2-2A\alpha+\alpha^2+\beta^2)),\end{aligned}$$

(43)

$$\dot{y} = [(1-B)x(B-y)^2(\alpha^2+\beta^2) + A^4x(2(1-B)y^2 + 2y(B+2Bx\alpha-1) \\ +x(\alpha(x\alpha - B(4+x\alpha)) - (B-1)x\beta^2)) + A(Bx(\alpha(2x\alpha + B(B(2+x(\alpha+\alpha^3)) \\ -2-3x\alpha)) + x(2+B(B+2B\alpha^2-3))\beta^2 + B^2x\beta^4) + xy(\alpha(B(4+4x\alpha \\ +B(x\alpha(B+(B-2)\alpha^2-3)-4)) - 2x\alpha) + x(B(4+B(B+2(B-2)\alpha^2-3)) \\ -2)\beta^2 + (B-2)B^2x\beta^4) + (B-1)^2y^3(B(1+\alpha^2+\beta^2)-1) \\ +(B-1)y^2(B(1+2x\alpha)(2+\alpha^2+\beta^2 - B(1+\alpha^2+\beta^2)) - 1)) \\ +A^3((1-B)^2(y-1)y^2 + 4(B-1)Bx(y-1)y\alpha + 2(B-1)Bx^3\alpha(\alpha^2+\beta^2) \\ +x^2(1-y-2y(\alpha^2+\beta^2) - B^2(y-1)(5\alpha^2+\beta^2) + B(\alpha^2+\beta^2-1+y(1 \\ +\alpha^2+\beta^2)))) - A^2x(-x^2(\alpha^2+\beta^2) - y(-2+y(2+\alpha^2+\beta^2)) + B^3(1 \\ +y^2(1+\alpha^2+\beta^2) - 2y(1+x\alpha)(1+\alpha^2+\beta^2) + x(1+\alpha^2+\beta^2) \\ \times(\alpha(2+x\alpha)+x\beta^2)) + B(4x\alpha + 3x^2(\alpha^2+\beta^2) + y^2(4+\alpha^2+\beta^2) - 4y(1 \\ +x\alpha(1+\alpha^2+\beta^2))) + B^2(-1-6x\alpha - y^2(3+\alpha^2+\beta^2) - x(\alpha^2+\beta^2)(-2\alpha \\ +x(3+\alpha^2+\beta^2)) + 2y(2+\alpha^2+\beta^2+x\alpha(3+\alpha^2+\beta^2)))) \\ -A^5x^2(y-1)]/((B-1)B^2(A^2-2A\alpha+\alpha^2+\beta^2)).$$

5 Integrability of the systems (33), (34), (36), (37), (40)–(43).

System (33). For this system the first Lyapunov quantity L_1 looks like $L_1 = b(4(1+\alpha^2) + B^2(-4+b^2(1+\alpha^2)))/(4B(1+\alpha^2))$. If $L_1 = 0$ then the system is integrable and has an integrating factor of the form (30) where

$f_1 = (x-\alpha-i)$, $f_2 = (x-\alpha+i)$,

$f_3 = -2B + bBx + 2y$, $f_4 = Exp[1/(-2B+bBx+2y)]$;

$\alpha_1 = -i(-4(\alpha-i)(i+\alpha)^2 + b^2B^4(b-2\alpha+b\alpha^2) - B^2(8i+b(1+\alpha^2)(-4 + b(i+\alpha)))))/(8B^2)$,

$\alpha_2 = i(-4(\alpha-i)^2(i+\alpha) + b^2B^4(b-2\alpha+b\alpha^2) + B^2(8i - b(1+\alpha^2)(-4 + b(\alpha-i)))))/(8B^2)$,

$\alpha_3 = -3$, $\alpha_4 = 2B + bB^3(b-(2\alpha)/(1+\alpha^2))$.

The Systems (34), (36), (40), (41), (43) are integrable and they have the following integrating factors, respectively:

$\mu(x,y) = 1/((-B+bBx+y)^2(1+x^2-2x\alpha+\alpha^2))$;

$\mu(x,y) = 1/((-1+x)^2(x^2+y^2))$;

$\mu(x,y) = 1/((-1+bx)^2(1-2y+y^2+2x\alpha-2xy\alpha+x^2\alpha^2+x^2\beta^2))$;

$\mu(x,y) = 1/((y-1)^2(1-2y+y^2+2x\alpha-2xy\alpha+x^2\alpha^2+x^2\beta^2))$;

$\mu(x,y) = 1/((-B-Ax+y)^2(1-2y+y^2+2x\alpha-2xy\alpha+x^2\alpha^2+x^2\beta^2))$.

Therefore, all of these systems have a center at origin of coordinates $(0,0)$.

System (37) (*System* (42)). For this system the first Lyapunov quantity $L_1 = -8l$ ($L_1 = 2g(g^2-2g\alpha+\alpha^2+\beta^2)$) vanishes for $l = 0$ ($g = 0$). Then the system (37) ((42)) is integrable and has the integrating factor $\mu(x,y) = 1/((-1+x)^2(x^2+y^2))$ ($\mu(x,y) = 1/((y-1)^2(1-2y+y^2+2x\alpha-2xy\alpha+x^2\alpha^2+x^2\beta^2)))$.

The above results prove the Main Theorem.

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