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On the number of topologies on countable skew fields

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Abstract. If a countable skew field R admits a non-discrete metrizable topology τ_0, then the lattice of all topologies of this skew fields admits:
– Continuum of non-discrete metrizable topologies of the skew fields stronger than the topology τ_0 and such that $\sup\{\tau_1, \tau_2\}$ is the discrete topology for any different topologies τ_1 and τ_2;
– Continuum of non-discrete metrizable topologies of the skew fields stronger than τ_0 and such that any two of these topologies are comparable;
– Two to the power of continuum of topologies of the skew fields stronger than τ_0, each of them is a coatom in the lattice of all topologies of the skew fields.

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1 Introduction

The study of possibility to set a non-discrete Hausdorff topology on infinite algebraic systems in which existing operations are continuous was begun in [1]. In this article, for any countable group, a method of constructing such group topologies was given.

For countable rings the problem of the possibility to set non-discrete Hausdorff ring topologies was studied in [2, 3].

For infinite fields the problem of the possibility to set non-discrete field topologies was studied in [2].

For countable skew field the problem of the possibility to set non-discrete Hausdorff topologies has not been solved.

The present article is a continuation of research in this direction. The main result of this paper is Theorem 3.1, in which for any countable skew field R which admits a nondiscrete, Hausdorff topology we got the numbers of some topologies.

For countable groups, countable rings and countable fields similar results were obtained in [4–7].

2 Notations and preliminaries

To present the main results we remind the following well-known result:

Theorem 2.1. *A set Ω of subsets of a skew field R is a basis of filter of neighborhoods of zero for some Hausdorff skew field topology τ on the skew field R if and only if the following conditions are satisfied:*

1) $\bigcap\limits_{V \in \Omega} V = \{0\}$;

2) *For any subsets V_1 and $V_2 \in \Omega$ there exists a subset $V_3 \in \Omega$ such that $V_3 \subseteq V_1 \cap V_2$;*

3) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $V_2 + V_2 \subseteq V_1$;*

4) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $-V_2 \subseteq V_1$;*

5) *For any subset $V_1 \in \Omega$ and any element $r \in R$ there exists a subset $V_2 \in \Omega$ such that $r \cdot V_2 \subseteq V_1$;*

6) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $V_2 \cdot V_2 \subseteq V_1$.*

7) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $(e+V_2)^{-1} - e \subseteq V_1$.*

Proof. According to ([2], Proposition 1.2.2, Theorems 1.2.5 and 1.2.12) for the proof of the theorem it suffices to verify that for any subset $V_1 \in \Omega$ and any element $r \neq 0$ there exists a subset $V \in \Omega$ such that $(r+V)^{-1} \subseteq r^{-1} + V_1$.

In fact, as any skew field topology is a ring topology on R then any basis of the filter of neighborhoods of zero of the skew field topology τ satisfies the condition 7.

Conversely, let Ω satisfy the condition 7. Then for any subset $V_0 \in \Omega$ and any element $0 \neq r \in R$ there exist sets $V_1, V_2, V \in \Omega$ such that $r^{-1} \cdot V_1 \subseteq V_0$, $(e+V_2)^{-1} - e \subseteq V_1$ and $V \cdot r^{-1} \cdot \subseteq V_2$. Then

$$(r+V)^{-1} = (r^{-1} \cdot r) \cdot (r+V)^{-1} = r^{-1} \cdot \big((r^{-1})^{-1} \cdot (r+V)^{-1}\big) =$$

$$r^{-1} \cdot \big((r+V) \cdot r^{-1}\big)^{-1} = r^{-1} \cdot (r \cdot r^{-1} + V \cdot r^{-1})^{-1} = r^{-1} \cdot (e + V \cdot r^{-1})^{-1} \subseteq$$

$$r^{-1} \cdot \big((e+V_2)^{-1} - e + e\big) \subseteq r^{-1} \cdot (V_1 + e) = r^{-1} + r^{-1} \cdot V_1 \subseteq r^{-1} + V_0.$$

From the arbitrariness of the element r and the set V_0, it follows that the operation of taking the inverse element in (R, τ) is continuous, and hence the theorem is completely proved. □

Definition 2.2. *A subset V of an Abelian group $R(+)$ is called the symmetric subset* if $V = -V$.

Notation 2.3. Let $V_1, V_2, \dots$ and $S_1, S_2, \dots$ be sequences of non-empty symmetric subsets of a skew field R, and e is the unit of the field R. If $S_1 \subseteq S_2 \subseteq \dots$ and $e \in S_1$ then for any natural number k we define by induction the subset $F_k\big(S_1, \dots, S_k; V_1, \dots, V_k\big)$ of the skew field R as follows:

$F_1(S_1; V_1) = (e + V_1 \setminus \{0\})^{-1} \cdot V_1 \cdot S_1 + V_1 \cdot V_1 + S_1 \cdot V_1 \cdot (e+V_1)^{-1}$, and

$$F_{k+1}(S_1, S_2, \dots, S_{k+1}; V_1, V_2, \dots, V_{k+1}) = F_1\big(S_1; V_1 + F_k(S_2, \dots, S_{k+1}; V_2, \dots, V_{k+1})\big).$$

Proposition 2.4. *Let $V_1, V_2, \dots$ and $S_1, S_2, \dots$ be some sequences of non-empty finite and symmetric subsets of a skew field R. If $e \in S_1 \subseteq S_2 \subseteq \dots$, and $0 \in V_i$ for any natural numbers i, then the following statements are true:*

Statement 1. *The following inclusions are true:*

1. $F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) + F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_{n+k}) \subseteq F_k(S_1, \dots, S_k; V_n, \dots, V_k)$ *for any natural number* $k > 1$;

2. $F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \cdot F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_{n+k}) \subseteq F_k(S_1, \dots, S_k; V_n, \dots, V_k)$ *for any natural number* $k > 1$;

3. $F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \cdot (e + F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k))^{-1} \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ and
$(e + F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k))^{-1} \cdot F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ *for any natural number* $k > 1$;

4. $S_1 \cdot F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \subseteq F_{k1}(S_1, \dots, S_k; V_1, \dots, V_k)$ and $F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \cdot S_1 \subseteq F_{k1}(S_1, \dots, S_k; V_1, \dots, V_k)$ *for any natural number* $k > 1$.

Statement 2. *For any natural number* k*, the set* $F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ *is a finite and symmetric set;*

Statement 3. $F_k(S_1, \dots, S_k; \{0\}, \dots, \{0\}) = \{0\}$ *for any natural number* k;

Statement 4. *If* $0 \in U_i \subseteq V_i \subseteq R$ *and* $e \in T_i \subseteq S_i \subseteq R$ *for any natural number* i*, then*

$$F_k(T_1, \dots, T_k; U_1, \dots, U_k) \subseteq F_k(S_1, \dots, S_k; V_1, \dots, V_k);$$

Statement 5. *If* k *and* p *are natural numbers and* $V_{k+j} = \{0\}$ *for any natural number* $1 \leq j \leq p$*, then*

$$F_k(S_1, \dots, S_k; V_1, \dots, V_k) = F_{k+p}(S_1, \dots, S_{k+p}; V_1, \dots, V_{k+p});$$

Statement 6.

$$F_{k+1}(S_1, \dots, S_{k+1}; V_1, \dots, V_{k+1}) = F_k\big(S_1, \dots, S_k; V_1, \dots, V_{n-1}, V_k + F_1(S_{k+1}, V_{k+1})\big).$$

for any natural number n;

Statement 7. *If* k *and* p *are natural numbers then*

$$F_k(S_{p+1}, \dots, S_{k+p}; V_{p+1}, \dots, V_{k+p}) \subseteq F_{k+p}(S_1, \dots, S_{k+p}; V_1, \dots, V_{k+p}).$$

Proof. As $0 \in V_i$ and $e \in S_i$ for any natural number i then Statement 1 follows from definition of the set $F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ for any $k > 1$.

Statements 2–5 are easily proved by induction on the number k similar to the proof of Proposition 5.3.2 in [2].

We prove Statement 6 by induction on the number k.

If $k = 2$, then $F_2(S_1, S_2; V_1, V_2) = F_1\big(S_1; V_1 + F_1(S_2; V_2)\big)$.

Assume that the required inclusion is proved for the number $k = n \geq 2$ and let $k = n + 1$. Then, from the induction assumption it follows

$$F_k(S_1, \dots, S_k; V_1, \dots, V_k) = F_{n+1}(S_1, \dots, S_{n+1}; V_1, \dots, V_{n+1}) =$$

$$F_1\big(S_1; V_1 + F_n(S_2, \dots, S_{n+1}; V_2, \dots, V_{n+1})\big) =$$

$$F_1\Big(S_1; V_1 + F_{n-1}\big(S_2, \dots, S_{n+1}; V_2, \dots, V_{n-1}, V_n + F_1(S_{n+1}, V_{n+1})\big)\Big) =$$

$$F_n\big(S_1, \dots, S_n; V_1, \dots, V_{n-1}, V_n + F_1(S_{n+1}, V_{n+1})\big).$$

Thus Statement 6 is proved.

We prove Statement 7 by induction on the number p. If $p = 1$ then from Statements 3 and 4 and inclusion 1 of Statement 1 it follows

$$F_k(S_2, \dots, S_{k+1}; V_2, \dots, V_{k+1}) \subseteq F_1\big(S_1; V_1 + F_k(S_2, \dots, S_{k+1}; V_2, \dots, V_{k+1})\big) =$$

$$F_{1+k}(S_1, \dots, S_{k+1}; V_1, \dots, V_{k+1}).$$

Assume that the required inclusion is proved for the number $p = n$ and any natural number k and let $p = n + 1$. Then

$$F_k(S_{n+1+1}, \dots, S_{k+n+1}; V_{n+1+1}, \dots, V_{k+n+1}) \subseteq$$

$$F_1\big(S_{n+1}; V_{n+1} + F_k(S_{n+1+1}, \dots, S_{k+n+1}; V_{n+1+1}, \dots, V_{k+n+1})\big) \subseteq$$

$$F_1\big(S_1; V_1 + F_{k+n}(S_2, \dots, S_{k+n+1}; V_2, \dots, V_{k+n+1})\big) \subseteq$$

$$F_{k+n+1}(S_1, \dots, S_{k+n+1}; V_1, \dots, V_{k+n+1}).$$

Thus Statement 7 is proved, and hence, Proposition 2.4 is proved. □

Proposition 2.5. *Let (R, τ) be a Hausdorff topological skew field. If $S_1, S_2, \dots$ is a sequence of finite sets, then the following statements are true:*

Statement 1. *For any neighborhood W of zero there exists a neighborhood W_1 of zero such that $F_1(S_1; W_1) \subseteq W$;*

Statement 2. *If A is a finite symmetric set such that $e \notin A$ and $0 \in A$ then for any neighborhood U of zero there exists a neighborhood W_1 of zero such that $F_1(S_1; A + W_1) \subseteq F_1(S_1, A) + U$.*

Statement 3. *If $2 \le n$ is a natural number and $A_1, A_2, \dots, A_n$ are finite symmetric sets such that $-e \notin A_i$ and $0 \in A_i$ for any $1 \le i \le n$, then for any neighborhood U of zero there exists a neighborhood W_n of zero such that*

$$F_n(S_1, S_2, \dots, S_n; A_1, A_2, \dots, A_{n-1}, W_n) \subseteq F_{n-1}(S_1, S_2, \dots S_{n-1}; A_1, A_2, \dots A_{n-1}) + U.$$

Proof. We prove Statement 1. There exist neighborhoods U_1, U_2 of zero such that $U_1 + U_1 + U_1 \subseteq W$ and $U_2 \cdot U_2 \subseteq U_1$. As the set S is a finite set then there exits a neighborhoods U_3 of zero such that $S \cdot U_3 \subseteq U_1$ and $U_3 \cdot S \subseteq U_1$ and since (R, τ) is a topological skew field then there exits a neighborhood U_4 of zero such that $(e + U_4 \setminus \{0\})^{-1} \cdot U_4 \subseteq U_3$ and $U_4 \cdot (e + U_4 \setminus \{0\})^{-1} \subseteq U_3$.

If $W_1 = U_2 \bigcap U_3 \bigcap U_4$ then

$$F_1(S; W_1) = (e + W_1 \setminus \{0\})^{-1} \cdot W_1 \cdot S + W_1 \cdot W_1 + S \cdot W_1 \cdot (e + W_1 \setminus \{0\})^{-1} \subseteq$$

$$(e + U_4 \setminus \{0\})^{-1} \cdot U_4 \cdot S + U_2 \cdot U_2 + S \cdot U_4 \cdot (e + U_4)^{-1} \subseteq U_3 \cdot S + U_2 \cdot U_2 + S \cdot U_3 \subseteq U_1 + U_1 + U_1 \subseteq W.$$

Thus Statement 1 is proved.

We prove Statement 2.

There exists a neighborhood U_1 of zero such that $U_1 + U_1 \subseteq U$ and since sets A and S are finite sets, then there exists a neighborhood U_2 of zero such that

$$U_2 \cdot A \cdot S + S \cdot A \cdot U_2 + U_2 \cdot U_2 \cdot S + U_2 \cdot U_2 \subseteq U_1.$$

As $0 \neq e + a$ for any element $a \in A$ and (R, τ) is a topological skew field and the topology τ is a Hausdorff topology then there exists a neighborhood U_a of zero such that $-e - a \notin U_a$ and $(e + a + U_a)^{-1} \subseteq (e + a)^{-1} + U_2$. Then

$$\big(e + A + (\bigcap_{a\in A} U_a)\big)^{-1} = \big(e + (\bigcup_{a\in A} \{a\}) + (\bigcap_{a\in A} U_a)\big)^{-1} \subseteq (e + A)^{-1} + U_2.$$

As sets A and S are finite sets, then there exists a neighborhood W_1 of zero such that $W_1 \subseteq (\bigcap\limits_{a\in A} U_a) \cap U_2$ and

$$A \cdot W_1 + W_1 \cdot A + (e + A)^{-1} \cdot W_1 \cdot S + S \cdot W_1 \cdot (e + A)^{-1} + W_1 \cdot W_1 \subseteq U_1$$

and $W_1 \cdot W_1 \cdot S + W_1 \cdot W_1 + S \cdot W_1 \cdot W_1 \subseteq U_1$. Then $F_1(S; A + W_1) =$

$$(e+A+W_1)^{-1} \cdot (A+W_1) \cdot S + (A+W_1) \cdot (A+W_1) + S \cdot (A+W_1) \cdot (e+A+W_1)^{-1} \subseteq$$

$$(e+A+W_1)^{-1} \cdot A \cdot S + (e+A+W_1)^{-1} \cdot W_1 \cdot S + A \cdot A + W_1 \cdot A + A \cdot W_1 + W_1 \cdot W_1 +$$
$$S \cdot A \cdot (e + A + W_1)^{-1} + S \cdot W_1 \cdot (e + A + W_1)^{-1} + S \cdot (A + W_1) \cdot (e + A + W_1)^{-1} \subseteq$$
$$\big((e + A)^{-1} + U_2\big) \cdot A \cdot S + \big((e + A)^{-1} + U_2\big) \cdot W_1 \cdot S +$$
$$A \cdot A + A \cdot W_1 + W_1 \cdot A + W_1 \cdot W_1 + S \cdot A \cdot \big((e+A)^{-1} + U_2\big) + S \cdot W_1 \cdot \big((e+A)^{-1} + U_2\big) \subseteq$$
$$(e+A)^{-1} \cdot A \cdot S + U_2 \cdot A \cdot S + (e+A)^{-1} \cdot W_1 \cdot S + U_2 \cdot W_1 \cdot S + A \cdot A + A \cdot W_1 + W_1 \cdot A + W_1 \cdot W_1 +$$
$$S \cdot A \cdot (e + A)^{-1} + S \cdot A \cdot U_2 + S \cdot W_1 \cdot (e + A)^{-1} + S \cdot W_1 \cdot U_2 \subseteq$$
$$\big((e + A)^{-1} \cdot A \cdot S + S \cdot A \cdot (e + A)^{-1} + A \cdot A\big) + \big(A \cdot W_1 + W_1 \cdot A + (e + A)^{-1} \cdot W_1 \cdot S +$$
$$S \cdot W_1 \cdot (e + A)^{-1} + W_1 \cdot W_1\big) + \big(U_2 \cdot A \cdot S + S \cdot A \cdot U_2 + U_2 \cdot U_2 \cdot S + S \cdot U_2 \cdot U_2\big) \subseteq$$
$$F_1(S_1; A) + U_1 + U_1 \subseteq F_1(S_1; A) + U.$$

Thus Statement 2 is proved.

We prove Statement 3 by induction on the number n.

If $n = 2$ then (see Statement 2) there exists a neighborhood V of zero such that $F_1(S_1; A_1 + V) \subseteq F_1(S_1; A_1) + U$ and (see Statement 1) there exists a neighborhood W_2 of zero such that $F_1(S_2; W_2) \subseteq U$. Then $F_2(S_1, S_2; A_1, W_2) =$

$$F_1\big(S_1; A_1 + F_1(S_2, W_2)\big) \subseteq F_1\big(S_1; A_1 + F_1(S_2, W_2)\big) \subseteq F_1(S_1; A_1 + V) \subseteq F_1(S_1; A_1) + U.$$

Assume that the required inclusion is proved for the number $n = k$ and let $A_1, A_2, \ldots, A_{k+1}$ be finite symmetric sets such that $-e \notin A_i$ and $0 \in A_i$ for any $1 \leq i \leq k + 1$.

Then from the induction assumption it follows that there exists a neighborhood V of zero such that

$$F_k(S_1,\ldots,S_k;A_2,\ldots,A_k+V)\subseteq F_k(S_1,\ldots S_k;A_1,A_2,\ldots A_k)+U,$$

and (see Statement 1) there exists a neighborhood W_{k+1} of zero such that $F_1(S_{k+1};W_{k+1})\subseteq V$.

Then (see Statement 6 of Proposition 2.4)

$$F_{k+1}(S_1,S_2,\ldots,S_{k+1};A_1,A_2,\ldots,A_k,W_{k+1})=$$

$$F_k(S_1,S_2,\ldots,S_k;A_1,A_2,\ldots,A_k+F_1(S_{k+1};W_{k+1})=$$

$$F_k(S_1,S_2,\ldots,S_k;A_1,\ldots,A_{k-1},A_k+V)\subseteq F_k(S_1,\ldots S_k;A_1,A_2,\ldots A_k)+U.$$

Thus Statement 3 is proved, and hence Proposition 2.5 is proved.

□

Notation 2.6. If $R=\{0,\pm1,\pm r_1,\pm r_2,\ldots\}$ is a countable skew field, then for any natural number k we put $S_k=\{\pm e,\pm r_1,\pm r_2,\ldots,\pm r_k\}$.

3 Basic results

Theorem 3.1. *If $R=\{0,\pm r_{1,},\pm r_2,\ldots\}$ is a countable skew field and τ_0 is a non-discrete, Hausdorff, skew field topology such that the topological skew field (R,τ_0) has a countable basis of the filter of neighborhoods of zero, then the following statements are true:*

Statement 1. *For any infinite set A of natural numbers there exists a skew field topology $\tau(A)$ such that $\tau_0\le\tau(A)$ and the topological skew field $(R,\tau(A))$ has a countable basis of the filter of neighborhoods of zero*;

Statement 2. $\sup\{\tau(A),\tau(B)\}$ *is the discrete topology for any infinite sets A and B of natural numbers such that $A\cap B$ is a finite set;*

Statement 3. *There are continuum of skew field topologies stronger than τ_0 and such that any two of them are comparable to each other;*

Statement 4. *There are two to the power of continuum of field topologies such that* $\sup\{\tau_1,\tau_2\}$ *is the discrete topology for any two different topologies τ_1 and τ_2;*

Statement 5. *There are two to the power of continuum of coatoms in the lattice of all skew field topologies of the skew field R.*

Proof. **Proof of Statement 1.** Since (R,τ_0) is a topological skew field and it is a Hausdorff space, then there exists a countable basis $\{V_1,V_2,\ldots\}$ of the filter of neighborhoods of zero such that $-V_k=V_k$, $V_k\cap S_k=\emptyset$ and $F_1(S_{k+1};V_{k+1}+V_{k+1})\subseteq V_k$ for any natural number k (see Statement 1 of Proposition 2.5).

As (see Statement 6 of Proposition 2.4)

$$F_{k+1}(S_1,\ldots,S_{k+1};V_1,\ldots,V_{k+1})=F_k\big(S_1,\ldots,S_k;V_1,\ldots,V_{n-1},V_k+F_1(S_{k+1},V_{k+1})\big).$$

then by induction on n it is easy to prove that $F_n(S_{i+1}, \ldots, S_{i+n}; V_{i+1} \ldots, V_{i+n}) \subseteq V_i$ for any natural numbers i and n.

Further the proof of Statement 1 will be realized in several steps.

Step I. By induction we construct a sequence $k_1, k_2, \ldots$ of natural numbers such that $k_i \geq i$, for any positive integer number i and we construct a sequence $h_1, h_2, \ldots$ of nonzero elements of the skew field R such that $\{-h_i, h_i\} \subseteq V_{k_i}$ and

$$F_n\big(S_1, \ldots, S_k; U_{A,1}, \ldots, U_{A,n}\big) \bigcap F_n\big(S_1, \ldots, S_k; U_{B,1}, \ldots, U_{B,n}\big) = \{0\}$$

for all subsets A and B of the set of natural numbers such that $A \cap B = \emptyset$, where $UC, i = \{h_i, 0, -h_i\}$ if $i \in C$ and $UC, i = \{0\}$ if $i \notin C$, for any set C of natural numbers.

We take $k_1 = 2$, and as h_1 we take an arbitrary element of the set $V_2 \backslash \{0\}$.

If A and B are some sets of natural numbers such that $A \bigcap B = \emptyset$, then $k_1 \notin A$ and hence $U_{A,1} = \{0\}$ or $k_1 = 2 \notin B$, and hence $U_{B,1} = \{0\}$. Then $F_1(S_1; U_{A,1}) \cap F_1(S_1; U_{B,1}) = \{0\}$.

Suppose that we defined natural numbers $k_1 < k_2 < \ldots < k_n$ such that $k_i \geq i$ and we defined nonzero elements $h_1, h_2, \ldots, h_n$ of the skew field R such that $\{-h_i, h_i\} \subseteq V_{k_i}$ and

$$F_n\big(S_1, \ldots, S_n; U_{A,1}, \ldots, U_{A,n}\big) \cap F_n\big(S_1, \ldots, S_n; U_{B,1}, \ldots, U_{B,n}\big) = \{0\}$$

for any sets A and B of natural numbers such that $A \cap B = \emptyset$.

If $\Omega = \{A_1, \ldots, A_{2^n}\}$ is the set all subsets of the set $\{1, \ldots, 2^n\}$ and $B_i = \{1, \ldots, 2^n\} \setminus A_i$ for $1 \leq i \leq 2^n$ then from the induction assumption

$$F_n\big(S_1, \ldots, S_n; U_{A_i,1}, \ldots, U_{A_i,n}\big) \cap F_n\big(S_1, \ldots, S_n; U_{B_i,1}, \ldots, U_{B_i,1}\big) = \{0\}$$

for any $1 \leq i \leq 2^n$.

As (R, τ_0) is a Hausdorff skew field and sets $F_n\big(S_1, \ldots, S_n; U_{A_i,1}, \ldots, U_{A_i,n}\big)$ and $F_n\big(S_1, \ldots, S_n; U_{B_i,1}, \ldots, U_{B_i,1}\big)$ are finite sets then for any $1 \leq i \leq 2^n$ there exists a neighborhood W_i of zero such that

$$\Big(\big(F_n(S_1, \ldots, S_n; U_{A_i,1}, \ldots, U_{A_i,n}) \setminus \{0\}\big) + W_i\Big) \cap$$

$$\Big(\big(F_n(S_1, \ldots, S_n; U_{B_i,1}, \ldots, U_{B_i,1}) \setminus \{0\}\big) + W_i\Big) = \emptyset,$$

and hence

$$\Big(\big(F_n(S_1, \ldots, S_n; U_{A_i,1}, \ldots, U_{A_i,n})\big) + W_i\Big) \cap$$

$$\Big(\big(F_n(S_1, \ldots, S_n; U_{B_i,1}, \ldots, U_{B_i,n})\big) + W_i\Big) = \{0\}.$$

Then (see Statement 3 of Proposition 2.4) for any $1 \leq i \leq 2^n$ there exists a neighborhood W_i' of zero such that

$$F_{n+1}\big(S_1, \ldots, S_n, S_{n+1}; U_{A_i,1}, \ldots, U_{A_i,n}, W_i'\big) \subseteq$$

$$F_n(S_1, \dots, S_n; U_{A_i,1}, \dots, U_{A_i,n}) + W_i$$

and

$$F_{n+1}\big(S_1, \dots, S_n, S_{n+1}; U_{B_i,1}, \dots, U_{B_i,n}, W_i')\big) \subseteq$$
$$F_n(S_1, \dots, S_n; U_{B_i,1}, \dots, U_{B_i,n}) + W_i.$$

If k_{n+1} is a natural number such that $k_{n+1} > k_n$ and $V_{k_{n+1}} \subseteq \bigcap\limits_{j=1}^{2^n} W_i'$ and h_{n+1} is some element of $V_{k_{n+1}} \setminus \{0\}$ then we take as h_{n+1} an arbitrary element of the set $V_{k_{n+1}} \setminus \{0\}$.

We prove that

$$F_{n+1}(S_1, \dots, S_{n+1}; U_{A,1}, \dots, U_{A,n+1}) \cap F_{n+1}(S_1, \dots, S_{n+1}; U_{B,1}, \dots, U_{B,n+1}) = \{0\}$$

for any subsets A and B of natural numbers such that $A \cap B = \emptyset$ (definition of sets $U_{C,k}$ see above).

Let A and B be some subsets of natural numbers such that $A \cap B = \emptyset$. Then $A \cap \{1, \dots n\} = A_s \in \Omega$ and $B \cap \{1, \dots n\} \subseteq B_s$ (definition of sets Ω and B_s see above) for $1 \le s \le 2^n$.

It is easy to see that $U_{A,i} = U_{A_s,i}$ and $U_{B,i} \subseteq U_{B_s,i}$ for any $1 \le i \le n$, and hence

$$F_n(S_1, \dots, S_n; U_{A,1}, \dots, U_{A,n}) \cap F_n(S_1, \dots, S_n; U_{B,1}, \dots, U_{B,n}) \subseteq$$

$$F_n(S_1, \dots, S_n; U_{A_s,1}, \dots, U_{A_s,n}) \cap F_n(S_1, \dots, S_n; U_{B_s,1}, \dots, U_{B_s,n}).$$

Since $A \cap B = \emptyset$ then $U_{B,n+1} = \{0\}$ or $U_{A,n+1} = \{0\}$.

Assume, for definiteness, that $U_{A,n+1} = \{0\}$. Then from Statement 5 of Proposition 2.4 and definition of sets W_i (see above) it follows that

$$\{0\} \subseteq F_{n+1}(S_1, \dots, S_n; U_{A,1}, \dots, U_{A,n+1}) \bigcap F_{n+1}(S_1, \dots, S_{n+1}; U_{B,1}, \dots, U_{B,n+1}) \subseteq$$

$$F_{n+1}(S_1, \dots, S_{n+1}; U_{A_s,1}, \dots, U_{A_s,n}, \{0\}) \bigcap$$
$$F_{n+1}(S_1, \dots, S_{n+1}; U_{B_s,1}, \dots, U_{B_s,n}, \{-h_{n+1}, 0, h_{n+1}\}) \subseteq$$

$$F_n(S_1, \dots, S_n; U_{A_s,1}, \dots, U_{A_s,n}) \bigcap F_{n+1}(S_1, \dots, S_{n+1}; U_{B_s,1}, \dots, U_{B_s,n}, V_{k_{n+1}}) \subseteq$$

$$F_n(S_1, \dots, S_n; U_{A_s,1}, \dots, U_{A_s,n}) \bigcap F_n(S_1, \dots, S_n; U_{B,1}, \dots, U_{B,n}) + W_s = \{0\},$$

and hence

$$F_{n+1}(S_1, \dots, S_{n+1}; U_{A,1}, \dots, U_{A,n+1}) \cap F_{n+1}(S_1, \dots, S_{n+1}; U_{B,1}, \dots, U_{B,n+1}) = \{0\}.$$

So, we defined the sequence $k_1, k_2, \dots$ of natural numbers such that $k_i \ge i$ for any number i and we defined the sequence $h_1, h_2, \dots$ of nonzero elements of the skew field R such that $\{-h_i, h_i\} \subseteq V_{k_i}$ for any natural number i and

$$F_n(S_1, \dots, S_k; U_{A,1}, \dots, U_{A,n}) \cap \tilde{F}_n\ (S_1, \dots, S_k; U_{B,1}, \dots, U_{B,n}) = \{0\}$$

for any natural number n and any sets A and B of natural numbers such that $A \cap B = \emptyset$.

Step II. For any pair (i, j) of natural numbers we consider the set

$$U_{A,(i,j)} = F_j(S_{i+1}, \ldots, S_{i+j}; U_{A,i+1}, \ldots, U_{A,i+j}),$$

where (as before) $U_{A,k} = \{0\}$ if $k \notin A$ and $U_{A,i} = \{0, h_k, -h_k\}$ if $k \in A$.

For the sets $U_{(i,j),A}$ we prove the following inclusions:

1. From Statement 3 of Proposition 2.4 it follows that $0 \in U_{A,(i,j)}$ for any natural numbers i, j and

$$U_{A,(i,j)} = F_n(S_{i+1}, \ldots, S_{i+n}; U_{A,i+1}, \ldots, U_{A,i+n}) \subseteq$$

$$F_n(S_{i+1}, \ldots, S_{i+n}; V_{i+1}, \ldots, V_{i+n}) \subseteq V_i$$

for any natural numbers i, n and any set A of natural numbers.

2. From Statements 4 and 5 of Proposition 2.4 it follows that $U_{A,(k,j)} \subseteq U_{A,(k,n)}$ for any natural numbers n and $j \leq n$.

3. From Statement 7 of Proposition 2.4 it follows that $U_{(i,j),A} \subseteq U_{A,(k,j+i-k)}$ for any natural numbers $k \leq i$ and any $j > i$.

4. From Statement 2 of Proposition 2.4 it follows that $U_{A,(i,j)}$ is a symmetric set, i.e. $-U_{A,(i,j)} = U_{A,(i,j)}$ for any natural numbers i, j.

5. From inclusion 2 of Statement 1 of Proposition 2.4 it follows that

$$U_{A,(i+1,n+1)} \cdot U_{A,(i+1,n+1)} = F_{n+1}(S_{i+2}, \ldots, S_{i+n+2};$$

$$U_{A,i+2}, \ldots, U_{A,i+n+2}) \cdot F_{n+1}(S_{i+2}, \ldots, S_{i+n+2}; U_{A,i+2}, \ldots, U_{A,i+n+2}) \subseteq$$

$$F_{n+1}(S_{i+1} \ldots, S_{i+n+2}; U_{A,i+1}, \ldots, U_{A,i+n+2}) = U_{A,(i,n+1)};$$

6. From inclusion 1 of Statement 1 of Proposition 2.4 it follows that

$$U_{A,(i+1,j+1)} + U_{A,(i+1,j+1)} =$$

$$F_{j+1}(S_{i+2}, \ldots, S_{i+j+2}; U_{A,i+2}, \ldots, U_{A,i+j+2}) +$$

$$F_{j+1}(S_{i+2}, \ldots, S_{i+j+2}; U_{A,i+2}, \ldots, U_{A,i+j+2}) \subseteq$$

$$F_{j+1}(S_{i+1}, \ldots, S_{i+j+2}; U_{A,i+1}, \ldots, U_{A,i+j+2}) = U_{A,(i,j+1)};$$

7. From inclusion 3 of Statement 1 of Proposition 2.4 it follows that

$$U_{A,(i+1,j+1)} \cdot (e + U_{A,(i+1,j+1)})^{-1} =$$

$$\big(F_{j+1}(S_{i+2}, \ldots, S_{i+j+2}; U_{A,i+2}, \ldots, U_{A,i+j+2})\big) \cdot \big(e+$$

$$F_{j+1}(S_{i+2},\dots,S_{i+j+2};U_{A,i+2},\dots,U_{A,i+j+2})\big)^{-1}\subseteq$$
$$F_{j+2}(S_{i+1}\dots,S_{i+j+2};U_{A,i+1},\dots,U_{A,i+j+2})=U_{A,(i,j+2)}$$

8. From inclusion 4 of Statement 1 of Proposition 2.4 it follows that

$$r_n\cdot U_{A,(i+n,j)}\subseteq S_{i+n}\cdot F_{n+i+j}(S_{n+i+1},\dots,S_{n+i+j};U_{A,n+i+1},\dots,U_{A,n+i+j})\subseteq$$
$$F_{n+i+j+1}(S_{n+i},\dots,S_{n+i+j};U_{A,n+i},\dots,U_{A,n+i+j})=$$

$U_{A,(i+n-1,j)}\subseteq U_{A,(i,j+1)}$.

Step III. For every infinite set A of natural numbers and any natural number i we take $\hat{U}_i(A)=\bigcup\limits_{j=1}^{\infty}U_{A,(i,j)}$ and we show that the set $\{\hat{U}_i(A)|i\in\mathbb{N}\}$ satisfies the conditions of Theorem 2.1, and hence this set is a basis of the filter of neighborhoods of zero for a field topology $\tau(A)$ on the skew field R.

In fact, since

$$\{0\}\subseteq U_{A,(i,n+1)}=F_{n+1}(S_{i+1},\dots,S_{i+n+1};U_{A,i+1},\dots,U_{A,i+n+1})\subseteq$$
$$F_{n+1}(S_{i+1}\dots,S_{i+n+1};V_{i+1},\dots,V_{i+n+1})\subseteq V_i$$

for any natural numbers i and n, then $\hat{U}_i(A)=\bigcup\limits_{j=1}^{\infty}U_{A,(i,j)}\subseteq V_i$. Then $\{0\}\subseteq\bigcap\limits_{i=1}^{\infty}\hat{U}_i(A)\subseteq\bigcap\limits_{i=1}^{\infty}V_i=\{0\}$, and hence the condition 1 of Theorem 2.1 is satisfied.

From inclusions 2 and 3 (see Step II), it follows

$$\hat{U}_i(A)\bigcap\hat{U}_k(A)=(\bigcup_{j=1}^{\infty}(U_{A,(i,j)})\bigcap(\bigcup_{l=1}^{\infty}U_{A,(k,l)})=$$

$$\bigcup_{j=1}^{\infty}\bigcup_{l=1}^{\infty}(U_{A,(i,j)}\bigcap U_{A,(k,l)})=\bigcup_{j=1}^{\infty}U_{A,(t,j)}=\hat{U}_t(A),$$

where $t=\max\{i,k\}$, and hence the condition 2 of Theorem 2.1 is satisfied.

From inclusions 2 and 5 (see Step II) it follows

$$\hat{U}_i(A)+\hat{U}_k(A)=(\bigcup_{j=1}^{\infty}U_{A,(i,j)})+(\bigcup_{l=1}^{\infty}U_{A,(i,l)})=$$

$$\bigcup_{j=1}\bigcup_{l=1}(U_{A,(i,j)}+U_{A,(i,l)})\subseteq\bigcup_{t=1}^{\infty}U_{A,(i-1,t)}=\hat{U}_{i-1}(A)$$

and

$$\hat{U}_i(A)\cdot\hat{U}_k(A)=(\bigcup_{j=1}^{\infty}U_{A,(i,j)})\cdot(\bigcup_{l=1}^{\infty}U_{A,(i,l)})=$$

$$\bigcup_{j=1}^{\infty}\bigcup_{l=1}^{\infty}(U_{A,(i,j)} \cdot U_{A,(i,l)}) \subseteq \bigcup_{t=1}^{\infty} U_{A,(i-1,t)} = \hat{U}_{i-1}(A)$$

for any natural number $i > 1$, and hence conditions 3 and 6 of Theorem 2.1 are satisfied.

From inclusion 4 (see Step II) it follows

$$-\hat{U}_i(A) = -(\bigcup_{j=1}^{\infty} U_{A,(i,j)}) = \bigcup_{j=1}^{\infty}\bigcup_{l=1}^{\infty}(-U_{A,(i,j)}) = \bigcup_{j=1}^{\infty} U_{A,j} = \hat{U}_i(A)$$

for any natural number i, and hence, the condition 4 of Theorem 2.1 is satisfied.

Now, let $r \in R$.

If $r = 0$, then $r \cdot \hat{U}_i(A) = \{0\} \subseteq \hat{U}_i(A)$ and $\hat{U}_i(A) \cdot r = \{0\} \subseteq \hat{U}_i(A)$ for any natural number i and any set A of natural numbers.

If $r \neq 0$, then $r = r_n$ or $r = -r_n$ for some natural number n. Then, from the inclusion 8 (see Step II) it follows $r_n \cdot \hat{U}_{i+n}(A) \subseteq \hat{U}_i(A)$ for any natural number i, and hence the condition 5 of Theorem 2.1 is satisfied.

If now $a \in \hat{U}_{i+1}(A) = \bigcup\limits_{j=1}^{\infty} U_{A,(i+1,j)}$ then there exists a natural number n such that $a \in U_{A,(i+1,n)}$. Then from inclusion 7 (see Step II) it follows

$$(e+a)^{-1}-e = (e-e-a)\cdot(e+a)^{-1} = (-a)\cdot(e+a)^{-1} \in U_{A,(i,n+1)} \subseteq \bigcup_{j=1}^{\infty} U_{A,(i,j)} = \hat{U}_i(A).$$

From the arbitrariness of the element $a\hat{U}_{i+1}(A)$ it follows that $(\hat{U}_{i+1}(A)){-}1{-}e \subseteq \hat{U}_i(A)$ for any natural number i, and hence the condition 7 of Theorem 2.1 is satisfied.

Thus, we have shown that the set $\{\hat{U}_i(A) | i \in \mathbb{N}\}$ satisfies conditions 1 – 7 of Theorem 2.1, and hence this set is a basis of the filter of neighborhoods of zero for a skew field topology $\tau(A)$ on the skew field R.

Since $\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{A,(i,j)} \subseteq V_i$ for any natural number i, then $\tau_0 \leq \tau(A)$.

Thus Statement 1 of this theorem is proved.

Proof of Statements 2 – 5 can be obtained if we repeat word for word the proof of the corresponding statements 3.1.2 – 3.1.5 in [6].

The theorem is proved. □

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