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Closure operators in modules and their characterizations

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Abstract. This work is dedicated to the investigation of closure operators of a module category R-Mod. The principal types of closure operators of R-Mod are studied and their characterizations are indicated, using dense or (and) closed submodules. The method of description of the closure operators consists in the elucidation of properties of functions which separate in every module, the set of dense submodules and the set of closed submodules. The main properties of the closure operators of R-Mod are studied: weakly heredity – idempotency, maximality – minimality, heredity – coheredity, as well as diverse combinations of them. Altogether, 16 types of the closure operators are described, among which 7 types possess double characterizations (by dense submodules and by closed ones).

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Introduction

The main notion of this work is the *closure operator* of a module category R-Mod. It is defined as a function C which associates to every pair $N \subseteq M$, where N is a submodule of the module $M \in R$-Mod, a new submodule $C_M(N) \subseteq M$ such that the properties of *extension, monotony* and *continuity* are satisfied (see Definition 1.1). This notion possesses numerous relations with different problems of algebra, topology, theory of categories etc. Various aspects of closure operators were studied by many authors, especially in the book [4], where a broad spectrum of questions on closure operators and their relations with other notions are exposed in categorial language.

The relations between the closure operators of R-Mod and the *preradicals* of this category are of special interest for us. We will use this connection, as well as the method of description of preradicals by the associated classes of modules, adopting it to the case of closure operators.

The aim of this work is the characterization of principal types of closure operators of R-Mod by the *dense* submodules or (and) by the *closed* submodules. More exactly, every closure operator C of R-Mod defines two functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$, which distinguish in every module $M \in R$-Mod the set $\mathcal{F}_1^C(M)$ of C-*dense* submodules and the set of C-*closed* submodules $\mathcal{F}_2^C(M)$. The studied types of closure operators are characterized by the function $\mathcal{F}_1^C$ or (and) by the function $\mathcal{F}_2^C$, indicating, for every studied type of operators, properties of the associated functions, which reestablish the given operators.

The following pairs of properties of closure operators have a particular importance: 1) weakly heredity, idempotence; 2) maximality, minimality; 3) heredity, coheredity. On the base of these conditions, different combinations of them are considered, which define the most important classes of closure operators of R-Mod. The method of characterization of diverse types of closure operators consists in the following. For every type of operators the properties of associated functions $\mathcal{F}_1^C$ or (and) $\mathcal{F}_2^C$ are indicated, which uniquely determine the studied type of operators. In such a way, the description of closure operators is expressed in the form of a bijection between the operators (C) of the studied type and the abstract functions $(\mathcal{F})$ with the necessary properties. In this procedure both the well known conditions of the functions are used ([4, 5]), as well as some new properties are introduced (see Table 1).

We will consider *all* possible cases (determined by the indicated three pairs of properties) and will show the characterizations by associated functions for 16 types of closure operators, among which 7 types are described both by dense submodules and by closed submodules. These facts supplement and complete the known results on this question exposed in [4] and [5]. Figure 1 shows the relations between the studied classes of closure operators, as well as the properties of associated functions, which characterize the given type of closure operators.

1 Preliminary notions and results

This section is of preparatory character and contains the notions and main results which are necessary for the exposition of the basic material. We begin with the central notion: closure operator of R-Mod [2–7].

Let R be a ring with unity and R-Mod be the category of unitary left R-modules. For every module $M \in R$-Mod we denote by $\mathbb{L}(M)$ the lattice of submodules of M.

Definition 1.1. A *closure operator* of R-Mod is a function C which associates to every pair $N \subseteq M$, where $M \in R$-Mod and $N \in \mathbb{L}(M)$, a submodule of M, denoted by $C_M(N)$, such that the following conditions are satisfied:

(c_1) $N \subseteq C_M(N)$ (*extension*);

(c_2) If $N, L \in \mathbb{L}(M)$ and $N \subseteq L$, then $C_M(N) \subseteq C_M(L)$ (*monotony*);

(c_3) If $f : M \to M'$ is an R-morphism and $N \subseteq M$, then $f\big(C_M(N)\big) \subseteq C_{M'}\big(f(N)\big)$ (*continuity*).

By $\mathbb{CO}(R)$ we denote the class of all closure operators of R-Mod.

Definition 1.2. Let $C \in \mathbb{CO}(R)$ and $M \in R$-Mod. A submodule $N \in \mathbb{L}(M)$ is called:

a) *C-dense* in M if $C_M(N) = M$;

b) *C-closed* in M if $C_M(N) = N$.

For every closure operator $C \in \mathbb{CO}(R)$ and every module $M \in R$-Mod we denote by:

$\mathcal{F}_1^C(M) = \{N \in \mathbb{L}(M) \mid C_M(N) = M\}$ the set of C-dense submodules of M;

$\mathcal{F}_2^C(M) = \{N \in \mathbb{L}(M) \mid C_M(N) = N\}$ the set of C-closed submodules of M.

It is obvious that $\mathcal{F}_1^C(M) \bigcap \mathcal{F}_2^C(M) = \{M\}$.

In such a way every closure operator $C \in \mathbb{CO}(R)$ defines two functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$, which separate in every module $M \in R$-Mod the set of C-dense submodules $\mathcal{F}_1^C(M)$ and the set of C-closed submodules $\mathcal{F}_2^C(M)$. In Section 2 we will show properties of these functions.

In the class $\mathbb{CO}(R)$ of the closure operators of R-Mod the order relation $(\leq)$ can be defined as follows:

$$C \leq D \Leftrightarrow C_M(N) \subseteq D_M(N) \ \text{ for every pair } \ N \subseteq M.$$

Moreover, the class $\mathbb{CO}(R)$ can be transformed in a ("big") complete lattice by the following operations:

1) the *meet* (intersection): $\bigwedge\limits_{\alpha \in \mathfrak{A}} C_\alpha$, $\{C_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}(R)$, where

$$\big(\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha\big)_M(N) \overset{\text{def}}{=\!=} \bigcap_{\alpha \in \mathfrak{A}} \big[(C_\alpha)_M(N)\big] \ \text{ for every } \ N \subseteq M;$$

2) the *join* (sum): $\bigvee\limits_{\alpha \in \mathfrak{A}} C_\alpha$, $\{C_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}(R)$, where

$$\big(\bigvee_{\alpha \in \mathfrak{A}} C_\alpha\big)_M(N) \overset{\text{def}}{=\!=} \sum_{\alpha \in \mathfrak{A}} \big[(C_\alpha)_M(N)\big] \ \text{ for every } \ N \subseteq M.$$

Besides, in the class $\mathbb{CO}(R)$ two more operations are defined, namely:

3) the *composition* (multiplication): $C \cdot D$, where $C, D \in \mathbb{CO}(R)$ and

$$\big(C \cdot D\big)_M(N) \overset{\text{def}}{=\!=} C_M\big(D_M(N)\big) \text{ for every } \ N \subseteq M;$$

4) the *cocomposition* (comultiplication): $C * D$, where $C, D \in \mathbb{CO}(R)$ and

$$\big(C * D\big)_M(N) \overset{\text{def}}{=\!=} C_{D_M(N)}(N) \text{ for every } \ N \subseteq M.$$

The properties of these operations are shown in [2–7]. As an example, we indicate the distributivity properties:

$$\big(\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha\big) \cdot D = \bigwedge_{\alpha \in \mathfrak{A}} (C_\alpha \cdot D), \quad \big(\bigvee_{\alpha \in \mathfrak{A}} C_\alpha\big) \cdot D = \bigvee_{\alpha \in \mathfrak{A}} (C_\alpha \cdot D);$$

$$\big(\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha\big) * D = \bigwedge_{\alpha \in \mathfrak{A}} (C_\alpha * D), \quad \big(\bigvee_{\alpha \in \mathfrak{A}} C_\alpha\big) * D = \bigvee_{\alpha \in \mathfrak{A}} (C_\alpha * D).$$

In continuation we remind an important notion related to the closure operators.

Definition 1.3. A *preradical* r of R-Mod is a subfunctor of the identity functor of R-Mod, i.e. r is a function which separates, in every module $M \in R$-Mod, a submodule $r(M) \subseteq M$ such that for every R-morphism $f : M \to M'$ the relation $f\big(r(M)\big) \subseteq r(M')$ holds [1].

We denote by $\mathbb{PR}(R)$ the class of all preradicals of R-Mod. Similarly to the class of closure operators $\mathbb{CO}(R)$, it possesses four operations [1].

The classes $\mathbb{CO}(R)$ and $\mathbb{PR}(R)$ are closely related by the following three mappings:

$$\mathbb{CO}(R) \quad \underset{\Psi_2}{\overset{\Psi_1}{\leftleftarrows}} \!\!\!\xrightarrow{\ \Phi\ } \quad \mathbb{PR}(R)\,,$$

which are defined by such rules.

1) $\Phi : \mathbb{CO}(R) \longrightarrow \mathbb{PR}(R)$. For every $C \in \mathbb{CO}(R)$ we denote $\Phi(C) = r_C$ and define $r_C(M) \overset{\text{def}}{=\!=} C_M(0)$ for every $M \in R$-Mod.

2) $\Psi_1 : \mathbb{PR}(R) \longrightarrow \mathbb{CO}(R)$. For every $r \in \mathbb{PR}(R)$ we denote $\Psi_1(r) = C^r$ and define $(C^r)_M(N)$ by such relation:

$$[(C^r)_M(N)]\,/N \overset{\text{def}}{=\!=} r(M/N) \text{ for every } N \subseteq M.$$

3) $\Psi_2 : \mathbb{PR}(R) \longrightarrow \mathbb{CO}(R)$. For every $r \in \mathbb{PR}(R)$ we denote $\Psi_2(r) = C_r$ and define:

$$(C_r)_M(N) \overset{\text{def}}{=\!=} r(M) + N \text{ for every } N \subseteq M.$$

For any preradical $r \in \mathbb{PR}(R)$ the operator C^r is the greatest between the operators $C \in \mathbb{CO}(R)$ with the property $\Phi(C) = r$, while C_r is the least operator of such type. Therefore every preradical $r \in \mathbb{PR}(R)$ defines the equivalence class (interval) $[C^r, C_r]$ and there exists a bijection between the preradicals of R-Mod and such classes. The operators of the form C^r for $r \in \mathbb{PR}(R)$ are called *maximal* and the operators of the form C_r are called *minimal* (the direct definitions are formulated in continuation).

In the literature on this subject many types of closure operators are studied [2–7]. We distinguish the most important types of closure operators in R-Mod, which play a fundamental role in the theory of closure operators in module categories. On the base of our investigations the following types of closure operators are.

Definition 1.4. A closure operator $C \in \mathbb{CO}(R)$ is called:

– *weakly hereditary* if $C_{C_M(N)}(N) = C_M(N)$ for every $N \subseteq M$;

– *idempotent* if $C_M\big(C_M(N)\big) = C_M(N)$ for every $N \subseteq M$;

– *maximal* if $C_M(N)/N = C_{M/N}(\bar{0})$ for every $N \subseteq M$
$\big($or: $C_M(N)/K = C_{M/K}(N/K)$ for every $K \subseteq N \subseteq M\big)$;

– *minimal* if $C_M(N) = C_M(0) + N$ for every $N \subseteq M$
$\big($or: $C_M(N) = C_M(K) + N$ for every $K \subseteq N \subseteq M\big)$;

– *hereditary* if $C_N(K) = C_M(K) \bigcap N$ for every $K \subseteq N \subseteq M$;

– *cohereditary* if $\big(C_M(N)+K\big)/K = C_{M/K}\big((N+K)/K\big)$ for every $K, N \in \mathbb{L}(M)$

$\big($or: $f\big(C_M(N)\big) = C_{f(M)}\big(f(N)\big)$ for every R-morphism $f : M \to M'$ and $N \subseteq M\big)$.

From the definitions of the operations $(\cdot)$ and $(*)$ in $\mathbb{CO}(R)$ it is clear that C is weakly hereditary iff $C * C = C$ and C is idempotent iff $C \cdot C = C$.

Besides the indicated three pairs of types of closure operators, in this work different combinations of these conditions are studied (see Figure 1). Now we remark some simple relations between the indicated classes of closure operators of R-Mod, which will be used in the next proofs.

Lemma 1.1. *Every hereditary closure operator is weakly hereditary.*

Proof. If $C \in \mathbb{CO}(R)$ is hereditary and $N \subseteq M$, then in the situation $N \subseteq C_M(N) \subseteq M$ the heredity of C implies:

$$C_{C_M(N)}(N) = C_M(N) \bigcap C_M(N) = C_M(N),$$

so C is weakly hereditary. □

Lemma 1.2. *Every minimal closure operator is idempotent.*

Proof. If $C \in \mathbb{CO}(R)$ is minimal and $N \subseteq M$, then:

$$C_M\big(C_M(N)\big) = C_M\big(C_M(0)+N\big) = C_M(0)+\big(C_M(0)+N\big) = C_M(0)+N = C_M(N).$$ □

Lemma 1.3. *A closure operator $C \in \mathbb{CO}(R)$ is cohereditary if and only if C is maximal and minimal.*

Proof. $(\Rightarrow)$ Let $C \in \mathbb{CO}(R)$ be a cohereditary closure operator and $K \subseteq N \subseteq M$. Then $C_M(N)\big/K = C_{M/K}(N/K)$, i.e. C is maximal.

To verify the minimality of C let $N \subseteq M$ and consider the natural morphism $\pi : M \to M/N$. In M we have the situation $N \subseteq C_M(N) \subseteq M$, which is transferred by π in $\bar{0} \subseteq C_M(N)\big/N \subseteq M/N$ and by coheredity of C we have $C_M(N)\big/N = C_{M/N}(\bar{0})$.

On the other hand, we have $0 \subseteq C_M(0) \subseteq M$, which by π passes to $\bar{0} \subseteq \big(C_M(0)+N\big)\big/N \subseteq M/N$ and the coheredity of C now implies $\big(C_M(0)+N\big)\big/N = C_{M/N}(\bar{0})$. Comparing with the above relation, we have $C_M(N)\big/N = \big(C_M(0)+N\big)\big/N$, i.e. $C_M(N) = C_M(0)+N$ and C is minimal.

$(\Leftarrow)$ Let $C \in \mathbb{CO}(R)$ be maximal and minimal. For submodules $K, N \in \mathbb{L}(M)$ we apply the minimality of C in the situation $N \subseteq N+K \subseteq M$ and obtain:

$$C_M(N+K) = C_M(N) + (N+K) = C_M(N) + K.$$

Therefore $\big(C_M(N)+K\big)\big/K = \big(C_M(N+K)\big)\big/K$.

Now we will use the maximality of C in the situation $K \subseteq N+K \subseteq M$ and obtain: $\big(C_M(N+K)\big)\big/K = C_{M/K}\big((N+K)\big/K\big)$. Comparing with the previous relation, we have $\big(C_M(N)+K\big)\big/K = C_{M/K}\big((N+K)\big/K\big)$, i.e. C is cohereditary. □

Now it is clear that every cohereditary closure operator, since being minimal (Lemma 1.3), is idempotent (Lemma 1.2). The rest of relations indicated in Figure 1 are obvious.

In continuation we remind some more facts on the principal types of closure operators. On the pair of properties "weakly hereditary – idempotent" we can say that every operator $C \in \mathbb{CO}(R)$ possesses an approximation in the respective classes of operators. More exactly, for every $C \in \mathbb{CO}(R)$ there exists *the greatest weakly hereditary* operator C_* which is contained in C (*weakly hereditary core* of C). Dually, for $C \in \mathbb{CO}(R)$ there exists *the least idempotent operator* C^* which contains C (*idempotent hull* of C). These approximations C_* and C^* can be obtained by the operations $(*)$ and $(\cdot)$, indicated above [4].

Namely, for $C \in \mathbb{CO}(R)$ we consider the descending ordinal chain:

$$C_1 \geq C_2 \geq \cdots \geq C_\alpha \geq C_{\alpha+1} \geq \cdots \geq C_\beta \geq \cdots ,$$

where $C_1 = C,\ C_{\alpha+1} = C * C_\alpha$ and $C_\beta = \bigcap\limits_{\gamma<\beta} C_\gamma$ for every ordinal number α and limit ordinal β. Then the *intersection* C_* of all members of this chain is the greatest weakly hereditary operator which is contained in C.

Dually, we can define the ascending ordinal chain for $C \in \mathbb{CO}(R)$:

$$C^1 \leq C^2 \leq \cdots \leq C^\alpha \leq C^{\alpha+1} \leq \cdots \leq C^\beta \leq \cdots ,$$

where $C^1 = C,\ C^{\alpha+1} = C \cdot C^\alpha$ and $C^\beta = \bigvee\limits_{\gamma<\beta} C^\gamma$ for all α and β as above. Then the *join* (sum) C^* of all members of this chain is the least idempotent closure operator containing C.

Similar constructions for preradicals are shown in [1].

We remark that in Section 2 the other method will be shown for the construction of operators C_* and C^* by the associated functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$.

Some similar results on approximations are true also for the maximal and minimal operators, using the mappings Φ, Ψ_1 and Ψ_2, mentioned above. Namely, for $C \in \mathbb{CO}(R)$ and $r = r_C$, C^r is *the least maximal* operator containing C, while C_r is *the greatest minimal* operator which is contained in C.

Now we will remind the method of characterization of some types of *preradicals* by means of the associated classes of modules [1]. Every preradical $r \in \mathbb{P}R(R)$ defines two classes of modules:

$\mathcal{T}_r = \{M \in R\text{-Mod} \mid r(M) = M\}$ is the class of *r-torsion* modules;

$\mathcal{F}_r = \{M \in R\text{-Mod} \mid r(M) = 0\}$ is the class of *r-torsionfree* modules.

In some cases the classes $\mathcal{T}_r$ or (and) $\mathcal{F}_r$ reestablish the preradical r and thus uniquely determine r. The inverse transition from the classes of modules to preradicals is defined as follows.

Let $\mathcal{K}$ be an *abstract class of modules*, i.e. $\mathcal{K}$ is closed under isomorphisms $(M \in \mathcal{K}, M \cong M' \Rightarrow M' \in \mathcal{K})$ and $0 \in \mathcal{K}$. Then we can obtain two preradicals $r^{\mathcal{K}}$ and $r_{\mathcal{K}}$ by the rules:

$$r^{\mathcal{K}}(M) \stackrel{\text{def}}{=\!=} \sum_{\alpha\in\mathfrak{A}} \{N_\alpha \subseteq M \mid N_\alpha \in \mathcal{K}\},$$
$$r_{\mathcal{K}}(M) \stackrel{\text{def}}{=\!=} \bigcap_{\alpha\in\mathfrak{A}} \{N_\alpha \subseteq M \mid M/N_\alpha \in \mathcal{K}\},$$

for every $M \in R$-Mod. Using the mappings $r \rightsquigarrow \mathcal{T}_r$, $\mathcal{K} \rightsquigarrow r^{\mathcal{K}}$ and $r \rightsquigarrow \mathcal{F}_r$, $\mathcal{K} \rightsquigarrow r_{\mathcal{K}}$, the characterizations of all principal types of preradicals r by associated classes $\mathcal{T}_r$ or (and) $\mathcal{F}_r$ can be obtained [1]. For example, the mappings $r \rightsquigarrow \mathcal{T}_r$, $\mathcal{K} \rightsquigarrow r^{\mathcal{K}}$ establish a monotone bijection between the *idempotent radicals* of R-Mod and the classes of modules closed under homomorphic images and direct sums. Dually, the mappings $r \rightsquigarrow \mathcal{F}_r$, $\mathcal{K} \rightsquigarrow r_{\mathcal{K}}$ define an antimonotone bijection between the *radicals* of R-Mod and the classes of modules closed under submodules and direct products. In more complicated cases some new properties of classes of modules are added.

The general method of characterization of principal types of closure operators, accepted in this work, is similar to the indicated above method for preradicals. Namely, for every $C \in \mathbb{CO}(R)$ the associated functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$, defined by the dense and closed submodules, are considered. The studied types of operators are described by the properties of corresponding functions $\mathcal{F}_1^C$ or (and) $\mathcal{F}_2^C$. With this aim the inverse transitions are defined: from an abstract function $\mathcal{F}$ of the necessary form to the closure operators $C^{\mathcal{F}}$ and $C_{\mathcal{F}}$.

We will use the term *abstract function* $\mathcal{F}$ of R-Mod in the sense that $\mathcal{F}$ is a function which separates, in every module $M \in R$-Mod, a set of submodules $\mathcal{F}(M) \subseteq \mathbb{L}(M)$ such that it is concordant with isomorphisms $\big(M \overset{f}{\cong} M',\ N \in \mathcal{F}(M) \Rightarrow f(N) \in \mathcal{F}(M')\big)$ and $M \in \mathcal{F}(M)$. Every abstract function $\mathcal{F}$ of R-Mod defines two operators $C^{\mathcal{F}}$ and $C_{\mathcal{F}}$ by the rules:

$$(C^{\mathcal{F}})_M\,(N) = \sum_{\alpha\in\mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\ N \in \mathcal{F}(N_\alpha)\},$$
$$(C_{\mathcal{F}})_M\,(N) = \bigcap_{\alpha\in\mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\ N_\alpha \in \mathcal{F}(M)\},$$

for every $N \subseteq M$. The mappings $C \rightsquigarrow \mathcal{F}_1^C$, $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ and $C \rightsquigarrow \mathcal{F}_2^C$, $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ give us the possibility to characterize the principal types of closure operators, using the abstract functions of R-Mod with the suitable conditions (properties). The list of all used conditions is attached in the end of this work (see Table 1).

2 Dense and closed submodules. Properties of the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$

Let $C \in \mathbb{CO}(R)$ and we consider the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$, defined as above by C-dense and C-closed submodules:

$$\mathcal{F}_1^C(M) \stackrel{\text{def}}{=\!=} \{N \in \mathbb{L}(M) \,\big|\, C_M(N) = M\}, \quad \mathcal{F}_2^C(M) \stackrel{\text{def}}{=\!=} \{N \in \mathbb{L}(M) \,\big|\, C_M(N) = N\}.$$

In this section we will show the properties of the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$ for an arbitrary $C \in \mathbb{CO}(R)$.

For convenience in the following exposition we formulate some conditions (properties) in a general form, considering an abstract function $\mathcal{F}$ (which preserves the isomorphisms and $M \in \mathcal{F}(M)$ for every $M \in R$-Mod):

C 1. If $M_\alpha \subseteq M$ and $N \in \mathcal{F}(M_\alpha)$ for every $\alpha \in \mathfrak{A}$, then $N \in \mathcal{F}(\sum_{\alpha \in \mathfrak{A}} M_\alpha)$;

C 2. If $N \subseteq L \subseteq M$ and $N \in \mathcal{F}(L)$, then $N + K \in \mathcal{F}(L + K)$ for every $K \subseteq M$;

C 3. If $K \subseteq N \subseteq M$ and $N \in \mathcal{F}(M)$, then $N/K \in \mathcal{F}(M/K)$.

The reason of the choice of these conditions consists in the following.

Proposition 2.1. *For every closure operator $C \in \mathbb{CO}(R)$ the associated function $\mathcal{F}_1^C$, defined by C-dense submodules, satisfies the conditions* C 1, C 2 and C 3.

Proof. C 1. Suppose that $M_\alpha \subseteq M$ and $N \in \mathcal{F}_1^C(M_\alpha)$ for every $\alpha \in \mathfrak{A}$. Then $C_{M_\alpha}(N) = M_\alpha$ ($\alpha \in \mathfrak{A}$) and from the monotony of C in the first term (which follows from (c_3)) we have $C_{M_\alpha}(N) \subseteq C_{\sum_{\alpha \in \mathfrak{A}} M_\alpha}(N)$, i.e. $M_\alpha \subseteq C_{\sum_{\alpha \in \mathfrak{A}} M_\alpha}(N)$ for every $\alpha \in \mathfrak{A}$. Therefore $\sum_{\alpha \in \mathfrak{A}} M_\alpha \subseteq C_{\sum_{\alpha \in \mathfrak{A}} M_\alpha}(N)$, so $\sum_{\alpha \in \mathfrak{A}} M_\alpha = C_{\sum_{\alpha \in \mathfrak{A}} M_\alpha}(N)$ and $N \in \mathcal{F}_1^C(\sum_{\alpha \in \mathfrak{A}} M_\alpha)$.

C 2. Let $N \subseteq L \subseteq M$ and $N \in \mathcal{F}_1^C(L)$. Then $C_L(N) = L$ and for any submodule $K \subseteq M$ we have $C_L(N) + K = L + K$. From the monotony of C (in both terms) it follows that $C_L(N) \subseteq C_{L+K}(N+K)$, therefore $C_L(N)+K \subseteq C_{L+K}(N+K)$. Then $L + K \subseteq C_{L+K}(N + K)$, which means that $N + K \in \mathcal{F}_1^C(L + K)$.

C 3. Let $K \subseteq N \subseteq M$ and $N \in \mathcal{F}_1^C(M)$, i.e. $C_M(N) = M$. By (c_3) we have $\pi\big(C_M(N)\big) \subseteq C_{M/K}(N/K)$, where $\pi : M \to M/K$ is the natural morphism. Then $C_M(N)/K \subseteq C_{M/K}(N/K)$ and by assumption $M/K \subseteq C_{M/K}(N/K)$, i.e. $N/K \in \mathcal{F}_1^C(M/K)$. $\square$

The abstract functions of R-Mod with the conditions C1$-$C 3 will play a special role in the following investigations and for simplification of exposition we need such notion.

Definition 2.1. An abstract function $\mathcal{F}$ of R-Mod which satisfies the conditions C 1$-$C 3 will be called *a function of the type* $\mathcal{F}_1$.

For the next characterizations some modifications of the conditions C 2 and C 3 are convenient. By this reason we formulate the following two conditions, indicating their relations with the previous ones:

C 4. If $N \subseteq L \subseteq M$ and $N \in \mathcal{F}(M)$, then $L \in \mathcal{F}(M)$;

C 5. If $f : M \to M'$ is an R-morphism and $N \in \mathcal{F}(M)$, then $f(N) \in \mathcal{F}\big(f(M)\big)$.

Lemma 2.2. *The following implications are true:*

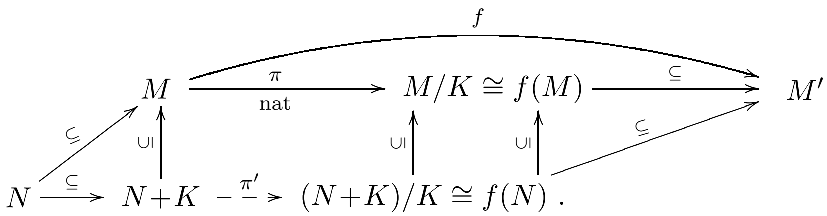
$$\text{C}\,2 \Rightarrow \text{C}\,4, \quad \text{C}\,5 \Rightarrow \text{C}\,3, \quad \text{C}\,3 + \text{C}\,4 \Rightarrow \text{C}\,5 \quad (\textit{so}\ \ \text{C}\,2 + \text{C}\,3 \Rightarrow \text{C}\,5).$$

Therefore, every abstract function of the type $\mathcal{F}_1$ satisfies the conditions C 4 *and* C 5.

Proof. C 2 $\Rightarrow$ C 4. If $N \subseteq L \subseteq M$ and $N \in \mathcal{F}(M)$, then from C 2 we have $N + L \in \mathcal{F}(M + L)$, i.e. $L \in \mathcal{F}(M)$.

C 5 $\Rightarrow$ C 3. If $K \subseteq N \subseteq M$ and $N \in \mathcal{F}(M)$, then from C 5 (applied to the natural morphism $\pi : M \to M/K$) we obtain $N/K \in \mathcal{F}(M/K)$.

C 3 + C 4 $\Rightarrow$ C 5. Let $f : M \to M'$ be an R-morphism and $N \in \mathcal{F}(M)$. Denote $K = \operatorname{Ker} f$ and consider the situation:



Since $N \in \mathcal{F}(M)$, by C 4 we have $N + K \in \mathcal{F}(M)$. Now from C 3 we obtain $(N+K)/K \in \mathcal{F}(M/K)$. By the definition $\mathcal{F}$ preserves the isomorphisms, so $f(N) \in \mathcal{F}\big(f(M)\big)$ and C 5 is true.

From the implication C 2 $\Rightarrow$ C 4 now it is clear that C 2 + C 3 $\Rightarrow$ C 5. Therefore every function of the type $\mathcal{F}_1$ satisfies the conditions C 4 and C 5. $\square$

In the rest of this section we will study in a similar manner the function $\mathcal{F}_2^C$, defined by C-closed submodules for $C \in \mathbb{CO}(R)$. This part has a dual character relative to the previous. Now we consider the following conditions for an abstract function $\mathcal{F}$ of R-mod:

C 1*. If $N_\alpha \subseteq M$ and $N_\alpha \in \mathcal{F}(M)$ for every $\alpha \in \mathfrak{A}$, then $\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha \in \mathcal{F}(M)$;

C 2*. If $N \subseteq L \subseteq M$ and $N \in \mathcal{F}(L)$, then $N \cap K \in \mathcal{F}(L \cap K)$ for every $K \subseteq M$;

C 3*. If $K \subseteq N \subseteq M$ and $N/K \in \mathcal{F}(M/K)$, then $N \in \mathcal{F}(M)$.

Proposition 2.3. *For every closure operator $C \in \mathbb{CO}(R)$ the associated function $\mathcal{F}_2^C$, defined by C-closed submodules, satisfies the conditions* C1*, C2* *and* C3*.

Proof. C 1*. Suppose that $N_\alpha \subseteq M$ and $N_\alpha \in \mathcal{F}_2^C(M)$ for every $\alpha \in \mathfrak{A}$. Then $C_M(N_\alpha) = N_\alpha$ $(\alpha \in \mathfrak{A})$ and by the monotony of C we have $C_M(\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha) \subseteq C_M(N_\alpha) = N_\alpha$ for every $\alpha \in \mathfrak{A}$. Therefore $C_M(\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha) \subseteq \bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha$, so $C_M(\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha) = \bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha$, i.e. $\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha \in \mathcal{F}_2^C(M)$.

C 2*. Let $N \subseteq L \subseteq M$ and $N \in \mathcal{F}_2^C(L)$, i.e. $C_L(N) = N$. For any submodule $K \subseteq M$ from the monotony of C it follows that $C_{L \cap K}(N \cap K) \subseteq C_L(N) = N$.

From the other hand, the monotony of C implies $C_{L \cap K}(N \cap K) \subseteq C_K(N \cap K) \subseteq K$. By the previous relation now it is clear that $C_{L \cap K}(N \cap K) \subseteq N \cap K$, i.e. $C_{L \cap K}(N \cap K) = N \cap K$. Therefore $N \cap K \in \mathcal{F}_2^C(L \cap K)$ and C 2* for $\mathcal{F}_2^C$ is true.

C 3*. Let $K \subseteq N \subseteq M$ and $N/K \in \mathcal{F}_2^C(M/K)$, i.e. $C_{M/K}(N/K) = N/K$. By (c_3) we have $\pi\big(C_M(N)\big) = C_M(N)/K \subseteq C_{M/K}(N/K) = N/K$, i.e. $C_M(N)/K \subseteq N/K$, where $\pi : M \to M/K$ is the natural morphism. Therefore $C_M(N) \subseteq N$, i.e. $C_M(N) = N$ and $N \in \mathcal{F}_2^C(M)$. $\square$

Definition 2.2. If an abstract function $\mathcal{F}$ of R-Mod satisfies the conditions $\mathrm{C}\,1^*, \mathrm{C}\,2^*$ and $\mathrm{C}\,3^*$, then it will be called a *function of the type* $\mathcal{F}_2$.

As in the previous case, now we formulate two auxiliary conditions, which are closely related to $\mathrm{C}\,2^*$ and $\mathrm{C}\,3^*$:

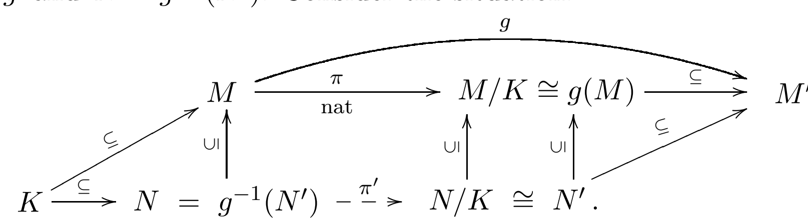
$\mathrm{C}\,4^*$. If $N \subseteq L \subseteq M$ and $N \in \mathcal{F}(M)$, then $N \in \mathcal{F}(L)$;

$\mathrm{C}\,5^*$. If $g : M \to M'$ is an R-morphism, $N' \subseteq g(M)$ and $N' \in \mathcal{F}\big(g(M)\big)$, then $g^{-1}(N') \in \mathcal{F}(M)$.

Lemma 2.4. *The implications* $\mathrm{C}\,2^* \Rightarrow \mathrm{C}\,4^*$, $\mathrm{C}\,3^* \Leftrightarrow \mathrm{C}\,5^*$ *are true. Therefore every function of the type* $\mathcal{F}_2$ *satisfies the conditions* $\mathrm{C}\,4^*$ *and* $\mathrm{C}\,5^*$.

Proof. $\mathrm{C2}^* \Rightarrow \mathrm{C4}^*$. If $N \subseteq L \subseteq M$ and $N \in \mathcal{F}(M)$, then from $\mathrm{C}\,2^*$ it follows that $N \cap L \in \mathcal{F}(M \cap L)$, i.e. $N \in \mathcal{F}(L)$ and $\mathrm{C}\,4^*$ is true.

$\mathrm{C}\,3^* \Rightarrow \mathrm{C}\,5^*$. Let $g : M \to M'$ be an R-morphism and $N' \in \mathcal{F}\big(g(M)\big)$. Denote $K = \operatorname{Ker} g$ and $N = g^{-1}(N')$. Consider the situation:



Since $N' \in \mathcal{F}\big(g(M)\big)$ and $\mathcal{F}$ preserves the isomorphisms, we have $N/K \in \mathcal{F}(M/K)$. From $\mathrm{C}\,3^*$ now we obtain $N = g^{-1}(N') \in \mathcal{F}(M)$ and $\mathrm{C}\,5^*$ holds.

$\mathrm{C5}^* \Rightarrow \mathrm{C3}^*$. Let $K \subseteq N \subseteq M$ and $N/K \in \mathcal{F}(M/K)$. Applying $\mathrm{C5}^*$ to the natural morphism $\pi : M \to M/K$ we obtain $\pi^{-1}(N/K) \in \mathcal{F}(M)$, i.e. $N \in \mathcal{F}(M)$ and $\mathrm{C}\,3^*$ holds. □

3 Weakly hereditary and idempotent closure operators

In Section 2 it is shown that every closure operator $C \in \mathbb{CO}(R)$ defines two functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$, specifying their properties. In general case the operator C can not be reestablished uniquely by $\mathcal{F}_1^C$ or by $\mathcal{F}_2^C$. The aim of this section is to prove the following results:

a) operator C can be uniquely described by $\mathcal{F}_1^C$ if and only if it is *weakly hereditary*;

b) operator C can be characterized by $\mathcal{F}_2^C$ if and only if it is *idempotent.*

Moreover, we will prove that the properties $\mathrm{C}\,1$–$\mathrm{C}\,3$ and $\mathrm{C}\,1^*$–$\mathrm{C}\,3^*$ are necessary and sufficient for realization of the respective characterizations. The results of this section constitute the base of all other facts established in this work, since all further characterizations contain the conditions $\mathrm{C}\,1$–$\mathrm{C}\,3$ or $\mathrm{C}\,1^*$–$\mathrm{C}\,3^*$, adding in every case some necessary auxiliary conditions.

Remind that the operator $C \in \mathbb{CO}(R)$ is called *weakly hereditary* if $C_{C_M(N)}(N) = C_M(N)$ for every pair $N \subseteq M$, i.e. $C * C = C$. Dually, the operator C is called *idempotent* if $C_M\big(C_M(N)\big) = C_M(N)$ for every $N \subseteq M$, i.e. $C * C = C$ (see Definition 1.4).

In the previous part of this work the mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $C \rightsquigarrow \mathcal{F}_2^C$ are used. Now we will show the inverse transition: from an abstract function $\mathcal{F}$ of R-Mod to the operators $C^{\mathcal{F}}$ and $C_{\mathcal{F}}$, which are defined by the following rules:

$$(C^{\mathcal{F}})_M(N) \stackrel{\text{def}}{=\!=} \sum_{\alpha \in \mathfrak{A}} \{M_\alpha \subseteq M \mid N \subseteq M_\alpha\,,\; N \in \mathcal{F}(M_\alpha)\}, \tag{3.1}$$

$$(C_{\mathcal{F}})_M(N) \stackrel{\text{def}}{=\!=} \bigcap_{\alpha \in \mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\; N_\alpha \in \mathcal{F}(M)\}, \tag{3.2}$$

for every $N \subseteq M$.

From the definitions of these operators it is clear that the mapping $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ is *monotone*, while the mapping $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ is *antimonotone*.

In this section we will study three cases, considering separately three types of closure operators of R-Mod: a) *weakly hereditary*; b) *idempotent*; c) *weakly hereditary and idempotent*.

a) *Weakly hereditary closure operators*

We begin with the characterization of weakly hereditary closure operators C by means of the associated functions $\mathcal{F}_1^C$, i.e. with the help of C-*dense* submodules.

Proposition 3.1. *Let $\mathcal{F}$ be an abstract function of R-Mod of the type $\mathcal{F}$ (i.e. with the conditions* C 1–C 3*). Then $C^{\mathcal{F}}$ (defined by* (3.1)*) is a closure operator of R-Mod.*

Proof. We will verify the conditions $(c_1)-(c_3)$ of Definition 1.1.

(c_1). Since in the rule (3.1) we have $N \subseteq M_\alpha$ for every $\alpha \in \mathfrak{A}$, it is clear that $N \subseteq \sum_{\alpha \in \mathfrak{A}} M_\alpha \stackrel{\text{def}}{=\!=} (C^{\mathcal{F}})_M(N)$.

(c_2). Suppose that $N \subseteq L \subseteq M$. Then by (3.1) we have:

$$(C^{\mathcal{F}})_M(N) = \sum_{\alpha \in \mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\; N \in \mathcal{F}(N_\alpha)\},$$

$$(C^{\mathcal{F}})_M(L) = \sum_{\beta \in \mathfrak{B}} \{L_\beta \subseteq M \mid L \subseteq L_\beta\,,\; L \in \mathcal{F}(L_\beta)\}.$$

Since $N \in \mathcal{F}(N_\alpha)$, from C 2 we have $N + L \in \mathcal{F}(N_\alpha + L)$, i.e. $L \in \mathcal{F}(N_\alpha + L)$. Denote $L_\beta = N_\alpha + L$. Then $N_\alpha \subseteq L_\beta$ and $L \in \mathcal{F}(L_\beta)$, so every submodule N_α is contained in some L_β from the definition of $(C^{\mathcal{F}})_M(L)$. Therefore $\sum_{\alpha \in \mathfrak{A}} N_\alpha \subseteq \sum_{\beta \in \mathfrak{B}} L_\beta$, i.e. $(C^{\mathcal{F}})_M(N) \subseteq (C^{\mathcal{F}})_M(L)$ and (c_2) is true for $C^{\mathcal{F}}$.

(c_3). Let $f : M \to M'$ be an arbitrary R-morphism and $N \subseteq M$. Then as above $(C^{\mathcal{F}})_M(N) = \sum_{\alpha \in \mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\; N \in \mathcal{F}(N_\alpha)\}$ and

$$(C^{\mathcal{F}})_{M'}\big(f(N)\big) = \sum_{\beta \in \mathfrak{B}} \{N'_\beta \subseteq M' \mid f(N) \subseteq N'_\beta\,,\; f(N) \in \mathcal{F}(N'_\beta)\}.$$

Since $\mathcal{F}$ is a function of the type $\mathcal{F}_1$, it satisfies the condition C 5 (Lemma 2.2), from which it follows that the relation $N \in \mathcal{F}(N_\alpha)$ implies $f(N) \in \mathcal{F}\big(f(N_\alpha)\big)$. Therefore $f(N_\alpha)$ is one of the submodules N'_β and so $f(N_\alpha) \subseteq \sum\limits_{\beta\in\mathfrak{B}} N'_\beta$ for every $\alpha \in \mathfrak{A}$. In such a way $\sum\limits_{\alpha\in\mathfrak{A}} f(N_\alpha) \subseteq \sum\limits_{\beta\in\mathfrak{B}} N'_\beta$, which implies the relations:

$$f\big((C^{\mathcal{F}})_M(N)\big) = f\big(\sum_{\alpha\in\mathfrak{A}} N_\alpha\big) = \sum_{\alpha\in\mathfrak{A}} f(N_\alpha) \subseteq \sum_{\beta\in\mathfrak{B}} N'_\beta \overset{\text{def}}{=\!=} (C^{\mathcal{F}})_{M'}\big(f(N)\big).$$

This means that $C^{\mathcal{F}}$ satisfies also the condition (c_3), therefore it is a closure operator of R-Mod. $\square$

Now we will clarify the situation related with the transitions $\mathcal{F} \rightsquigarrow C^{\mathcal{F}} \rightsquigarrow \mathcal{F}_1^{C^{\mathcal{F}}}$, beginning with an abstract function $\mathcal{F}$ of the type $\mathcal{F}_1$ (conditions C 1$-$C 3).

Proposition 3.2. *Let $\mathcal{F}$ be an abstract function of the type $\mathcal{F}_1$ of R-Mod. Then the closure operator $C^{\mathcal{F}}$ is **weakly hereditary** and the associated function $\mathcal{F}_1^{C^{\mathcal{F}}}$ coincides with the initial function $\mathcal{F}$ (i.e. $\mathcal{F} = \mathcal{F}_1^{C^{\mathcal{F}}}$).*

Proof. By Proposition 3.1 $C^{\mathcal{F}}$ is a closure operator of R-Mod. Now we verify that $C^{\mathcal{F}}$ is weakly hereditary. If $N \subseteq M$ then by the definition of $C^{\mathcal{F}}$ we have:

$$(C^{\mathcal{F}})_M(N) = \sum_{\alpha\in\mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha,\ N \in \mathcal{F}(N_\alpha)\},$$

$$(C^{\mathcal{F}})_{\sum\limits_{\alpha\in\mathfrak{A}} N_\alpha}(N) = \sum_{\beta\in\mathfrak{B}} \{L_\beta \subseteq \sum_{\alpha\in\mathfrak{A}} N_\alpha \mid N \subseteq L_\beta,\ N \in \mathcal{F}(L_\beta)\}.$$

Since $\mathcal{F}$ satisfies C 1, from the relations $N \in \mathcal{F}(N_\alpha)$ $(\alpha \in \mathfrak{A})$ it follows that $N \in \mathcal{F}\big(\sum\limits_{\alpha\in\mathfrak{A}} N_\alpha\big)$. Therefore $\sum\limits_{\alpha\in\mathfrak{A}} N_\alpha$ is one of submodules L_β and so $\sum\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq \sum\limits_{\beta\in\mathfrak{B}} L_\beta$, where the inverse inclusion is trivial, because $\sum\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq M$. So we have $C^{\mathcal{F}}_{(C^{\mathcal{F}})_M(N)}(N) = (C^{\mathcal{F}})_M(N)$ for every $N \subseteq M$, i.e. $C^{\mathcal{F}}$ is weakly hereditary.

Now it remains to prove the relation $\mathcal{F} = \mathcal{F}_1^{C^{\mathcal{F}}}$. The inclusion $\mathcal{F} \leq \mathcal{F}_1^{C^{\mathcal{F}}}$ follows from the definition: if $N \in \mathcal{F}(M)$, then by (3.1) it is clear that $(C^{\mathcal{F}})_M(N) = M$, i.e. $N \in \mathcal{F}_1^{C^{\mathcal{F}}}(M)$. The inverse inclusion $\mathcal{F}_1^{C^{\mathcal{F}}} \leq \mathcal{F}$ follows from the condition C 1 of $\mathcal{F}$: if $N \in \mathcal{F}_1^{C^{\mathcal{F}}}(M)$, then $(C^{\mathcal{F}})_M(N) = M$, i.e. $\sum\limits_{\alpha\in\mathfrak{A}} N_\alpha = M$ and now from the relations $N \in \mathcal{F}(N_\alpha)$ $(\alpha \in \mathfrak{A})$ by C 1 we conclude that $N \in \mathcal{F}(\sum\limits_{\alpha\in\mathfrak{A}} N_\alpha) = \mathcal{F}(M)$. $\square$

In continuation we analyze the transitions $C \rightsquigarrow \mathcal{F}_1^C \rightsquigarrow C^{\mathcal{F}_1^C}$ for an arbitrary closure operator $C \in \mathbb{CO}(R)$. By Proposition 2.1 $\mathcal{F}_1^C$ is a function of the type $\mathcal{F}_1$, therefore by Proposition 3.1 $\mathcal{F}_1^C$ defines the closure operator $C^{\mathcal{F}_1^C}$. Denote $C_* = C^{\mathcal{F}_1^C}$ and examine its relation with the operator C.

Proposition 3.3. *Let $C \in \mathbb{CO}(R)$. Then:*

a) *$C_* \leq C$;*

b) *C_* is weakly hereditary;*

c) *C_* is the greatest weakly hereditary closure operator of $\mathbb{CO}(R)$ which is contained in C.*

Proof. a) By the rule (3.1) for every $N \subseteq M$ we have $(C_*)_M(N) = \sum\limits_{\alpha \in \mathfrak{A}} \{M_\alpha \subseteq M \mid N \subseteq M_\alpha\,,\ N \in \mathcal{F}_1^C(M_\alpha)\}$. Since $\mathcal{F}_1^C$ satisfies C 1 (Proposition 2.1), from the relations $N \in \mathcal{F}_1^C(M_\alpha)$ $(\alpha \in \mathfrak{A})$ it follows that $N \in \mathcal{F}_1^C(\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha)$. Therefore $C_{\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha}(N) = \sum\limits_{\alpha \in \mathfrak{A}} M_\alpha$ and from the monotony of C we have $C_{\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha}(N) \subseteq C_M(N)$, i.e. $\sum\limits_{\alpha \in \mathfrak{A}} M_\alpha \subseteq C_M(N)$. This means that $(C_*)_M(N) \subseteq C_M(N)$ for every $N \subseteq M$, i.e. $C_* \leq C$.

b) Since $\mathcal{F}_1^C$ satisfies C 1 – C 3 (Proposition 2.1), the operator $C_* = C^{\,\mathcal{F}_1^C}$ is weakly hereditary by Proposition 3.2.

c) Let $D \in \mathbb{CO}(R)$ be a weakly hereditary closure operator such that $D \leq C$. We must verify that $D \leq C_* = C^{\,\mathcal{F}_1^C}$. The assumptions on D imply the relations:

$$D_M(N) \,=\, D_{D_M(N)}(N) \,\subseteq\, C_{D_M(N)}(N) \,\subseteq\, D_M(N).$$

Therefore $C_{D_M(N)}(N) = D_M(N)$, i.e. $N \in \mathcal{F}_1^C\big(D_M(N)\big)$. Thus $D_M(N)$ is one of submodules M_α, so $D_M(N) \subseteq \sum\limits_{\alpha \in \mathfrak{A}} M_\alpha \overset{\text{def}}{=\!=} (C_*)_M(N)$ for every $N \subseteq M$. This means that $D \leq C_*$ and c) is true. □

Corollary 3.4. *A closure operator $C \in \mathbb{CO}(R)$ is weakly hereditary if and only if $C = C_*$, where $C_* = C^{\,\mathcal{F}_1^C}$.* □

From Proposition 3.3 it follows that the operator C_* coincides with the *weakly hereditary core* of C, which was constructed in Section 1 by the operation of cocomposition $(*)$ in $\mathbb{CO}(R)$.

For the transitions $C \rightsquigarrow \mathcal{F}_1^C \rightsquigarrow C^{\,\mathcal{F}_1^C}$ the equality $C = C^{\,\mathcal{F}_1^C}$ means that C is weakly hereditary (Corollary 3.4), therefore the characterization of $C \in \mathbb{CO}(R)$ by the function $\mathcal{F}_1^C$ is possible if and only if C is weakly hereditary. This is the reason why in all the following characterizations the function $\mathcal{F}_1^C$ is used only for the class of weakly hereditary operators and for its subclasses.

Now we can formulate the main result of this part, which shows the characterization of weakly hereditary closure operators $C \in \mathbb{CO}(R)$ by the associated functions $\mathcal{F}_1^C$.

Theorem 3.5. *The mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $\mathcal{F} \rightsquigarrow C^{\,\mathcal{F}}$ define a monotone bijection between the **weakly hereditary** closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ of this category.*

Proof. If $C \in \mathbb{CO}(R)$ is weakly hereditary then $C = C^{\,\mathcal{F}_1^C}$ (Corollary 3.4). From the other hand, for every abstract function $\mathcal{F}$ of the type $\mathcal{F}_1$ we have $\mathcal{F} = \mathcal{F}_1^{C^{\mathcal{F}}}$ (Proposition 3.2). Therefore the indicated mappings define a monotone bijection. □

b) *Idempotent closure operators*

In a similar manner as in the previous case, now we will describe the idempotent closure operators $C \in \mathbb{CO}(R)$ by the associated functions $\mathcal{F}_2^C$ defined by C-closed submodules. In Section 2 it was proved that for every $C \in \mathbb{CO}(R)$ the function $\mathcal{F}_2^C$ is of the type $\mathcal{F}_2$, i.e. it satisfies the conditions $\mathrm{C}\,1^* - \mathrm{C}\,3^*$ (Proposition 2.3). Moreover, $\mathcal{F}_2^C$ satisfies also the conditions $\mathrm{C}\,4^*$ and $\mathrm{C}\,5^*$ (Lemma 2.4).

Now we will study the inverse transition $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$, where $C_{\mathcal{F}}$ is defined by the rule (3.2): $(C_{\mathcal{F}})_M(N) \stackrel{\text{def}}{=\!=} \bigcap\limits_{\alpha \in \mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\, N_\alpha \in \mathcal{F}(M)\}$ for every $N \subseteq M$.

Proposition 3.6. *Let $\mathcal{F}$ be an abstract function of the type $\mathcal{F}_2$. Then $C_{\mathcal{F}}$ is a closure operator of R-Mod.*

Proof. (c_1) Since $N \subseteq N_\alpha$ for every $\alpha \in \mathfrak{A}$, it is obvious that $N \subseteq \bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha \stackrel{\text{def}}{=\!=} (C_{\mathcal{F}})_M(N)$.

(c_2) Let $N \subseteq L \subseteq M$. By the definition of $C_{\mathcal{F}}$ we have as above $(C_{\mathcal{F}})_M(N) = \bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha$ with $N_\alpha \in \mathcal{F}(M)$ for every $\alpha \in \mathfrak{A}$, and

$$(C_{\mathcal{F}})_M(L) = \bigcap_{\beta \in \mathfrak{B}} \{L_\beta \subseteq M \mid L \subseteq L_\beta\,,\; L_\beta \in \mathcal{F}(M)\}.$$

Then $N \subseteq L \subseteq L_\beta$ and $L_\beta \in \mathcal{F}(M)$, therefore L_β is one of submodules N_α and so $\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha \subseteq L_\beta$ for every $\beta \in \mathfrak{B}$. In such a way we obtain $\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha \subseteq \bigcap\limits_{\beta \in \mathfrak{B}} L_\beta$, i.e. $(C_{\mathcal{F}})_M(N) \subseteq (C_{\mathcal{F}})_M(L)$, which shows the monotony of $C_{\mathcal{F}}$.

(c_3) Let $f : M \to M'$ be an R-morphism and $N \subseteq M$. Then we have the submodule $(C_{\mathcal{F}})_M(N)$, defined by (3.2) and

$$(C_{\mathcal{F}})_{M'}\big(f(M)\big) = \bigcap_{\beta \in \mathfrak{B}} \{N'_\beta \subseteq M' \mid f(N) \subseteq N'_\beta\,,\; N'_\beta \in \mathcal{F}(M')\}.$$

Since $\mathcal{F}$ is a function of the type $\mathcal{F}_2$, it satisfies the condition $\mathrm{C}\,5^*$ (Lemma 2.4), from which we conclude that the relation $N'_\beta \in \mathcal{F}(M')$ implies $f^{-1}(N'_\beta) \in \mathcal{F}(M)$, where $N'_\beta \supseteq f(N)$, so $f^{-1}(N'_\beta) \supseteq f^{-1}\big(f(N)\big) \supseteq N$. This means that $f^{-1}(N'_\beta)$ is one of the submodules N_α from the definition of $(C_{\mathcal{F}})_M(N)$. Therefore $\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha \subseteq f^{-1}(N'_\beta)$ for every $\beta \in \mathfrak{B}$ and

$$\bigcap_{\alpha \in \mathfrak{A}} N_\alpha \subseteq \bigcap_{\beta \in \mathfrak{B}} \{f^{-1}(N'_\beta) \mid f(N) \subseteq N'_\beta\,,\; N'_\beta \in \mathcal{F}(M')\}.$$

Using this relation we obtain:

$$f[(C_{\mathcal{F}})_M(N)] = f(\bigcap_{\alpha \in \mathfrak{A}} N_\alpha) \subseteq f\,\Big[\bigcap_{\beta \in \mathfrak{B}} f^{-1}(N'_\beta)\Big] \subseteq \bigcap_{\beta \in \mathfrak{B}} \big[f\big(f^{-1}(N'_\beta)\big)\big] =$$
$$= \bigcap_{\beta \in \mathfrak{B}} \big[N'_\beta \cap f(M)\big] \subseteq \bigcap_{\beta \in \mathfrak{B}} N'_\beta \stackrel{\text{def}}{=\!=} (C_{\mathcal{F}})_{M'}\big(f(N)\big),$$

so $C_{\mathcal{F}}$ satisfies (c_3). □

Now we will study the transitions $\mathcal{F} \rightsquigarrow C_{\mathcal{F}} \rightsquigarrow \mathcal{F}_2^{C_{\mathcal{F}}}$, where $\mathcal{F}$ is an abstract function of the type $\mathcal{F}_2$.

Proposition 3.7. *Let $\mathcal{F}$ be an abstract function of the type $\mathcal{F}_2$ of R-Mod. Then the corresponding closure operator $C_{\mathcal{F}}$ is* ***idempotent*** *and the associated function $\mathcal{F}_2^{C_{\mathcal{F}}}$ coincides with the initial function $\mathcal{F}$ (i.e. $\mathcal{F} = \mathcal{F}_2^{C_{\mathcal{F}}}$).*

Proof. By Proposition 3.6 $C_{\mathcal{F}}$ is a closure operator. Let $N \subseteq M$. By the definition of $C_{\mathcal{F}}$ we have:

$$(C_{\mathcal{F}})_M(N) = \bigcap_{\alpha\in\mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha ,\ N_\alpha \in \mathcal{F}(M)\},$$

$$(C_{\mathcal{F}})_M\big[(C_{\mathcal{F}})_M(N)\big] = \bigcap_{\beta\in\mathfrak{B}} \{L_\beta \subseteq M \mid (C_{\mathcal{F}})_M(N) \subseteq L_\beta ,\ L_\beta \in \mathcal{F}(M)\}.$$

Since $\mathcal{F}$ satisfies C 1^*, from the relations $N_\alpha \in \mathcal{F}(M)$ $(\alpha \in \mathfrak{A})$ we obtain $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \in \mathcal{F}(M)$. Therefore $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha$ is some L_β, so $\bigcap\limits_{\beta\in\mathfrak{B}} L_\beta \subseteq \bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha$, which means that $(C_{\mathcal{F}})_M\big[(C_{\mathcal{F}})_M(N)\big] \subseteq (C_{\mathcal{F}})_M(N)$ and the inverse inclusion is trivial. This shows that $C_{\mathcal{F}}$ is idempotent.

Now we will verify that $\mathcal{F} = \mathcal{F}_2^{C_{\mathcal{F}}}$. The relation $\mathcal{F} \le \mathcal{F}_2^{C_{\mathcal{F}}}$ follows from the construction. Indeed, if $N \in \mathcal{F}(M)$, then N is one of N_α from the definition of $(C_{\mathcal{F}})_M(N)$, therefore $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha = N$, i.e. $(C_{\mathcal{F}})_M(N) = N$ and $N \in \mathcal{F}_2^{C_{\mathcal{F}}}(M)$.

The inverse relation $\mathcal{F}_2^{C_{\mathcal{F}}} \le \mathcal{F}$ follows from the condition C 1^* of the function $\mathcal{F}$. If $N \in \mathcal{F}_2^{C_{\mathcal{F}}}(M)$, then $(C_{\mathcal{F}})_M(N) = N$, i.e. $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha = N$. Since $N_\alpha \in \mathcal{F}(M)$ for every $\alpha \in \mathfrak{A}$, by C 1^* we have $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \in \mathcal{F}(M)$, i.e. $N \in \mathcal{F}(M)$. □

Further we consider the transitions $C \rightsquigarrow \mathcal{F}_2^C \rightsquigarrow C_{\mathcal{F}_2^C}$ for an arbitrary closure operator $C \in \mathbb{CO}(R)$. Denote: $C^* = C_{\mathcal{F}_2^C}$.

Proposition 3.8. *Let $C \in \mathbb{CO}(R)$. Then:*

a) *$C^* \ge C$;*

b) *C^* is idempotent;*

c) *C^* is the least idempotent closure operator of R-Mod, containing C.*

Proof. a) Let $N \subseteq M$. By the definition of C^* we have:

$$(C^*)_M(N) = \bigcap_{\alpha\in\mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha ,\ N_\alpha \in \mathcal{F}_2^C(M)\}.$$

Since $\mathcal{F}_2^C$ satisfies C 1^* and $N_\alpha \in \mathcal{F}_2^C(M)$ for every $\alpha \in \mathfrak{A}$, we have $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \in \mathcal{F}_2^C(M)$, i.e. $C_M(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha) = \bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha$. The monotony of C and the relation $N \subseteq \bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha$ imply that $C_M(N) \subseteq C_M(\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha) = \bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha$. This means that $C_M(N) \subseteq (C^*)_M(N)$ for every $N \subseteq M$, i.e. $C \le C^*$.

b) For every $C \in \mathbb{CO}(R)$ $\mathcal{F}_2^C$ is a function of the type $\mathcal{F}_2$ (Proposition 2.3), therefore by Proposition 3.7 the operator $C^* = C_{\mathcal{F}_2^C}$ is idempotent.

c) Let $D \in \mathbb{CO}(R)$ be an idempotent closure operator such that $D \ge C$. We will verify that $C^* \le D$. By the definition of C^* we have:

$$(C^*)_M(N) = (C_{\mathcal{F}_2^C})_M(N) = \bigcap_{\alpha\in\mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\ N_\alpha \in \mathcal{F}_2^C(M)\}.$$

Since D is idempotent and $D \geq C$, we obtain:

$$D_M(N) = D_M\big(D_M(N)\big) \geq C_M\big(D_M(N)\big) \geq D_M(N),$$

therefore $D_M(N) = C_M\big(D_M(N)\big)$, i.e. $D_M(N) \in \mathcal{F}_2^C(M)$. Then $D_M(N)$ is one of submodules N_α from the definition of $(C^*)_M(N)$ and so $\bigcap\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq D_M(N)$. This means that $(C^*)_M(N) \subseteq D_M(N)$ for every $N \subseteq M$, i.e. $C^* \leq D$. □

Corollary 3.9. *A closure operator $C \in \mathbb{CO}(R)$ is idempotent if and only if $C = C^*$, where $C^* = C_{\mathcal{F}_2^C}$.* □

Therefore the characterization of an operator $C \in \mathbb{CO}(R)$ by the associated function $\mathcal{F}_2^C$ is possible if and only if the operator C is *idempotent*. By this reason in all subsequent descriptions the function $\mathcal{F}_2^C$ is used if and only if the studied type of closure operators is contained in the class of idempotent operators.

From Proposition 3.8 it is clear that the operator $C^* = C_{\mathcal{F}_2^C}$ coincides with the *idempotent hull* of C, which was constructed in Section 1 by the operation of composition $(*)$ in $\in \mathbb{CO}(R)$.

The previous results show the characterization of an idempotent closure operator $C \in \mathbb{CO}(R)$ by the associated functions $\mathcal{F}_2^C$.

Theorem 3.10. *The mappings $C \rightsquigarrow \mathcal{F}_2^C$ and $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ define an antimonotone bijection between the* ***idempotent*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_2$ of this category.* □

c) *Weakly hereditary and idempotent closure operators*

In this subsection we will study closure operators of R-Mod which are *weakly hereditary and idempotent*. From the foregoing it follows that every operator C of such type can be uniquely reestablished both by the function $\mathcal{F}_1^C$ (Theorem 3.5) and by the function $\mathcal{F}_2^C$ (Theorem 3.10). Now we will supplement the bijections of these theorems, showing the necessary and sufficient conditions for an abstract function $\mathcal{F}$ of R-Mod such that the corresponding operators $C^{\mathcal{F}}$ and $C_{\mathcal{F}}$ are weakly hereditary and idempotent. With this purpose, besides the previous conditions (C 1 − C 3, C 1*−C 3*), we will use the following condition:

C 6 = C 6*. If $N \subseteq L \subseteq M$, $N \in \mathcal{F}(L)$ and $L \in \mathcal{F}(M)$, then $N \in \mathcal{F}(M)$.

If is obvious that this condition is autodual. It will be called in continuation the property of *transitivity* and will be used both for the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$.

Proposition 3.11. *If the closure operator $C \in \mathbb{CO}(R)$ is* ***idempotent****, then the associated function $\mathcal{F}_1^C$ possesses the property of transitivity* C 6 = C 6*.

Proof. Let $C \in \mathbb{CO}(R)$ be an idempotent closure operator and consider the situation: $N \subseteq L \subseteq M$, $N \in \mathcal{F}_1^C(L)$, $L \in \mathcal{F}_1^C(M)$. Then $C_L(N) = L$ and $C_M(L) = M$. By the monotony of C we have $C_L(N) \subseteq C_M(N)$ and so $L \subseteq C_M(N)$. The idempotency and monotony of C imply $C_M(L) \subseteq C_M\big(C_M(N)\big) = C_M(N)$, i.e. $M \subseteq C_M(N)$ and $M = C_M(N)$. So we have $N \in \mathcal{F}_1^C(M)$, i.e. for $\mathcal{F}_1^C$ C 6 is true. □

The following statement in some sense is inverse to the previous: transitivity implies idempotency.

Proposition 3.12. *Let $\mathcal{F}$ be an abstract function of the type $\mathcal{F}_1$ which satisfies the condition* C 6*. Then the closure operator $C^{\mathcal{F}}$ is idempotent.*

Proof. If $\mathcal{F}$ is a function of the type $\mathcal{F}_1$, then it defines the closure operator $C^{\mathcal{F}}$ (Proposition 3.1), from the definition of which we have:

$$(C^{\mathcal{F}})_M(N) = \sum_{\alpha\in\mathfrak{A}} \{M_\alpha \subseteq M \mid N \subseteq M_\alpha\,,\ N \in \mathcal{F}(M_\alpha)\},$$

$$(C^{\mathcal{F}})_M\big[(C^{\mathcal{F}})_M(N)\big] = \sum_{\beta\in\mathfrak{B}} \{L_\beta \subseteq M \mid (C^{\mathcal{F}})_M(N) \subseteq L_\beta\,,\ (C^{\mathcal{F}})_M(N) \in \mathcal{F}(L_\beta)\},$$

for every $N \subseteq M$. From the relations $N \in \mathcal{F}(M_\alpha)$ $(\alpha \in \mathfrak{A})$ by C 1 we conclude that $N \in \mathcal{F}\big(\sum_{\alpha\in\mathfrak{A}} M_\alpha\big)$. Having also the relation $\sum_{\alpha\in\mathfrak{A}} M_\alpha \in \mathcal{F}(L_\beta)$ for every $\beta \in \mathfrak{B}$, we can apply the condition C 6 in the situation $N \subseteq \sum_{\alpha\in\mathfrak{A}} M_\alpha \subseteq L_\beta$, obtaining $N \in \mathcal{F}(L_\beta)$ $(\beta \in \mathfrak{B})$. Now from C 1 we have $N \in \mathcal{F}\big(\sum_{\beta\in\mathfrak{B}} L_\beta\big)$, therefore $\sum_{\beta\in\mathfrak{B}} L_\beta$ is one of submodules M_α. Hence $\sum_{\beta\in\mathfrak{B}} L_\beta \subseteq \sum_{\alpha\in\mathfrak{A}} M_\alpha$, which means that $(C^{\mathcal{F}})_M\big[(C^{\mathcal{F}})_M(N)\big] \subseteq (C^{\mathcal{F}})_M(N)$, i.e. $(C^{\mathcal{F}})_M\big[(C^{\mathcal{F}})_M(N)\big] = (C^{\mathcal{F}})_M(N)$ and the operator $C^{\mathcal{F}}$ is idempotent. □

The indicated above results permit us to show the characterization of weakly hereditary and idempotent operators $C \in \mathbb{CO}(R)$ by the associated functions $\mathcal{F}_1^C$ (i.e. by C-*dense* submodules).

Theorem 3.13. *The mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ define a monotone bijection between the* ***weakly hereditary and idempotent*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ which satisfies the condition* C 6.

Proof. By Theorem 3.5 the indicated mappings define a monotone bijection between the weakly hereditary closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ of R-Mod. In this bijection if C is idempotent then by Proposition 3.1 the function $\mathcal{F}_1^C$ is transitive.

From the other hand, in the same situation if a function $\mathcal{F}$ of the type $\mathcal{F}_1$ is transitive, then the corresponding weakly hereditary closure operator $C^{\mathcal{F}}$ is idempotent (Proposition 3.12). In such a way, by the restriction of the bijection of Theorem 3.5 we obtain the formulated result. □

In a dual manner the weakly hereditary and idempotent closure operators of R-Mod can be described by the associated functions $\mathcal{F}_2^C$, i.e. by C-*closed* submodules. We remark that in this case the same autodual condition of transitivity $\mathrm{C}\,6 = \mathrm{C}\,6^*$ is used.

Proposition 3.14. *If the closure operator $C \in \mathbb{CO}(R)$ is weakly hereditary, then the associated function $\mathcal{F}_2^C$ satisfies the condition* $\mathrm{C}\,6 = \mathrm{C}\,6^*$.

Proof. Let $C \in \mathbb{CO}(R)$ be a weakly hereditary closure operator. Consider the situation: $N \subseteq L \subseteq M$, $N \in \mathcal{F}_2^C(L)$, $L \in \mathcal{F}_2^C(M)$. Then $C_L(N) = N$ and $C_M(L) = L$. The monotony of C implies $C_M(N) \subseteq C_M(L) = L$ and so $C_M(N) \subseteq L$. This relation gives us $C_{C_M(N)}(N) \subseteq C_L(N)$, i.e. $C_{C_M(N)}(N) = N$. Since C is weakly hereditary, we have $C_{C_M(N)}(N) = C_M(N)$ and from the previous equality we obtain $C_M(N) = N$ and $N \in \mathcal{F}_2^C(M)$. This means that the function $\mathcal{F}_2^C$ is transitive. □

Proposition 3.15. *If an abstract function $\mathcal{F}$ is of the type $\mathcal{F}_2$ and satisfies the condition* C 6, *then the operator $C_{\mathcal{F}}$ is weakly hereditary.*

Proof. By the definition of $C_{\mathcal{F}}$, for every $N \subseteq M$ we have:

$$(C_{\mathcal{F}})_M(N) = \bigcap_{\alpha \in \mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\ N_\alpha \in \mathcal{F}(M)\},$$

$$(C_{\mathcal{F}})_{(C_{\mathcal{F}})_M(N)}(N) = \bigcap_{\beta \in \mathfrak{B}} \{L_\beta \subseteq M \mid N \subseteq L_\beta \subseteq (C_{\mathcal{F}})_M(N)\,,\ L_\beta \in \mathcal{F}\big((C_{\mathcal{F}})_M(N)\big)\}.$$

Since $\mathcal{F}$ satisfies $\mathrm{C}\,1^*$, the relations $N_\alpha \in \mathcal{F}(M)$ $(\alpha \in \mathfrak{A})$ imply $\bigcap_{\alpha \in \mathfrak{A}} N_\alpha \in \mathcal{F}(M)$, i.e. $(C_{\mathcal{F}})_M(N) \in \mathcal{F}(M)$.

From the other hand, the relations $L_\beta \in \mathcal{F}\big((C_{\mathcal{F}})_M(N)\big)$ $(\beta \in \mathfrak{B})$ and $\mathrm{C}\,1^*$ imply $\bigcap_{\beta \in \mathfrak{B}} L_\beta \in \mathcal{F}\big((C_{\mathcal{F}})_M(N)\big)$. Now we can use the transitivity of $\mathcal{F}$ in the situation: $\bigcap_{\beta \in \mathfrak{B}} L_\beta \subseteq (C_{\mathcal{F}})_M(N) \subseteq M$ and conclude from the foregoing that $\bigcap_{\beta \in \mathfrak{B}} L_\beta \in \mathcal{F}(M)$. Therefore $\bigcap_{\beta \in \mathfrak{B}} L_\beta$ is one of the submodules N_α and so $\bigcap_{\alpha \in \mathfrak{A}} N_\alpha \subseteq \bigcap_{\beta \in \mathfrak{B}} L_\beta$. This means that $(C_{\mathcal{F}})_M(N) \subseteq (C_{\mathcal{F}})_{(C_{\mathcal{F}})_M(N)}(N)$ and the inverse inclusion follows from the relation $M \supseteq (C_{\mathcal{F}})_M(N)$. Thus $(C_{\mathcal{F}})_M(N) = (C_{\mathcal{F}})_{(C_{\mathcal{F}})_M(N)}(N)$ for every $N \subseteq M$, i.e. the operator $C_{\mathcal{F}}$ is weakly hereditary. □

The last two statements permit us to restrict the bijection of Theorem 3.10 and to obtain the characterization of the weakly hereditary and idempotent closure operators C by the functions $\mathcal{F}_2^C$ (i.e. by *C-closed* submodules).

Theorem 3.16. *The mappings $C \rightsquigarrow \mathcal{F}_2^C$ and $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ define an antimonotone bijection between the* ***weakly hereditary and idempotent*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_2$ which satisfy the condition* $\mathrm{C}\,6 = \mathrm{C}\,6^*$. □

As a conclusion of this section we can say that every *weakly hereditary* closure operator C can be uniquely reestablished by the associated function $\mathcal{F}_1^C$ which satisfies the conditions C 1 − C 3. Dually, every *idempotent* closure operator C can be described by the function $\mathcal{F}_2^C$ which possesses the properties $\mathrm{C}\,1^*$–$\mathrm{C}\,3^*$. In the case of *weakly hereditary and idempotent* operators the previous conditions are completed by the transitivity ($\mathrm{C}\,6 = \mathrm{C}\,6^*$).

4 Maximal and minimal closure operators

We continue the study of principal types of closure operators of R-Mod. To the conditions of weakly heredity and idempotency, studied above, now we add two other important properties: *maximality* and *minimality*. These types of operators are investigated in parallel, though they are not dual in the sense of duality of the previous two conditions.

For convenience we remind the main necessary definitions (see Section 1). A closure operator $C \in \mathbb{CO}(R)$ is called *maximal* if

$$C_M(N)/N = C_{M/N}(\bar{0}) \tag{4.1}$$

for every $N \subseteq M$. This condition is equivalent to the property that for every submodules $K \subseteq N \subseteq M$ the relation

$$C_M(N)/K = C_{M/K}(N/K) \tag{4.2}$$

is true.

A closure operator $C \in \mathbb{CO}(R)$ is called *minimal* if

$$C_M(N) = C_M(0) + N \tag{4.3}$$

for every $N \subseteq M$, which is equivalent to the relation

$$C_M(N) = C_M(K) + N \tag{4.4}$$

for every $K \subseteq N \subseteq M$.

The other form of these types of operators can be obtained by the mappings Φ, Ψ_1 and Ψ_2 between the classes $\mathbb{CO}(R)$ and $\mathbb{PR}(R)$, defined in Section 1. Namely, an operator $C \in \mathbb{CO}(R)$ is maximal if and only if it is of the form C^r for some preradical $r \in \mathbb{PR}(R)$. Similarly, an operator C is minimal if and only if $C = C_r$ for some $r \in \mathbb{PR}(R)$.

The aim of this section is to describe by the functions $\mathcal{F}_1^C$ or (and) $\mathcal{F}_2^C$ all types of closure operators, related to maximality and minimality. On the base of previous results we can affirm that such characterizations are possible if and only if the studied types of operators are contained in the class of weakly hereditary operators (then $\mathcal{F}_1^C$ is used), or in the class of idempotent operators (using $\mathcal{F}_2^C$).

Taking into consideration these facts, in continuation we will study the cases when an operator C is: a) weakly hereditary and maximal; b) idempotent and maximal; c) weakly hereditary, idempotent and maximal; d) minimal; e) weakly hereditary and minimal (see Figure 1). The case when an operator C is maximal and minimal is investigated in Section 5, where the cohereditary operators are described (see Lemma 1.3).

a) *Weakly hereditary and maximal closure operators*

Let $C \in \mathbb{CO}(R)$ be a weakly hereditary and maximal closure operator. The first condition implies that C can be described by the function $\mathcal{F}_1^C$, which is of the type $\mathcal{F}_1$ (Theorem 3.5). So it remains to express by the function $\mathcal{F}_1^C$ the *maximality* of C. By this purpose we use the following known condition (see Section 2):

$\mathrm{C}\,3^*$. If $K \subseteq N \subseteq M$ and $N/K \in \mathcal{F}(M/K)$, then $N \in \mathcal{F}(M)$.

Proposition 4.1. *Let $C \in \mathbb{CO}(R)$ be a weakly hereditary closure operator. Then C is maximal if and only if the associated function $\mathcal{F}_1^C$ satisfies the condition $\mathrm{C}\,3^*$.*

Proof. ($\Rightarrow$) Suppose that a weakly hereditary operator C is maximal and consider the situation: $K \subseteq N \subseteq M$, $N/K \in \mathcal{F}_1^C(M/K)$. Since C is maximal, we have $C_M(N)/K = C_{M/K}(N/K)$ (see (4.2)). From the condition $N/K \in \mathcal{F}_1^C(M/K)$ we have $C_{M/K}(N/K) = M/K$, therefore $C_M(N)/K = M/K$. This means that $C_M(N) = M$, i.e. $N \in \mathcal{F}_1^C(M)$ and $\mathcal{F}_1^C$ satisfies the condition $\mathrm{C}\,3^*$.

($\Leftarrow$) Let C be a weakly hereditary closure operator such that $\mathcal{F}_1^C$ satisfies the condition $\mathrm{C}\,3^*$. Then in the situation $K \subseteq N \subseteq M$ we have:

$$C_M(N) = \sum_{\alpha \in \mathfrak{A}} \{M_\alpha \subseteq M \mid N \subseteq M_\alpha\,,\ N \in \mathcal{F}_1^C(M_\alpha)\},$$

$$C_{M/K}(N/K) = \sum_{\beta \in \mathfrak{B}} \{M'_\beta/K \subseteq M/K \mid N/K \subseteq M'_\beta/K\,,\ N/K \in \mathcal{F}_1^C(M'_\beta/K)\}.$$

Using the condition $\mathrm{C}\,3^*$, from the relations $N/K \in \mathcal{F}_1^C(M'_\beta/K)$ we obtain $N \in \mathcal{F}_1^C(M'_\beta)$ for every $\beta \in \mathfrak{B}$. Now from the property $\mathrm{C}\,1$ of $\mathcal{F}_1^C$ we conclude that $N \in \mathcal{F}_1^C\big(\sum_{\beta \in \mathfrak{B}} M'_\beta\big)$. This means that $\sum_{\beta \in \mathfrak{B}} M'_\beta$ is one of submodules M_α from the definition of $C_M(N)$. Therefore $C_M(N) \supseteq \sum_{\beta \in \mathfrak{B}} M'_\beta$ and

$$C_M(N)/K \supseteq \big(\sum_{\beta \in \mathfrak{B}} M'_\beta\big)/K = \sum_{\beta \in \mathfrak{B}} (M'_\beta/K) = C_{M/K}(N/K),$$

i.e. $C_M(N)/K \supseteq C_{M/K}(N/K)$. The inverse inclusion follows from (c_3) of Definition 1.1, so we obtain the relation (4.2), i.e. C is maximal. □

Now we can use Theorem 3.5, proved for weakly hereditary closure operators and, restricting the bijection of this theorem, by Proposition 4.1 we obtain a new bijection, which describes the weakly hereditary and maximal closure operators.

Theorem 4.2. *The mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ define a monotone bijection between the **weakly hereditary and maximal** closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ which satisfy the condition $\mathrm{C}\,3^*$.* □

b) *Idempotent and maximal closure operators*

Let $C \in \mathbb{CO}(R)$ be an idempotent and maximal closure operator. Since C is idempotent, it can be described by the function $\mathcal{F}_2^C$, which in this case satisfies the conditions $\mathrm{C}\,1^*$–$\mathrm{C}\,3^*$ (Theorem 3.10). Thus for the characterization of C by $\mathcal{F}_2^C$ it is necessary to express the *maximality* of C by the function $\mathcal{F}_2^C$. For that we will use the known condition, formulated in Section 2:

$\mathrm{C}\,3$. If $K \subseteq N \subseteq M$ and $N \in \mathcal{F}(M)$, then $N/K \in \mathcal{F}(M/K)$.

Proposition 4.3. *Let $C \in \mathbb{CO}(R)$ be an idempotent closure operator. Then C is **maximal** if and only if the function $\mathcal{F}_2^C$ satisfies the condition* C 3.

Proof. ($\Rightarrow$) Suppose that the idempotent closure operator C is maximal and consider the situation: $K \subseteq N \subseteq M$, $N \in \mathcal{F}_2^C(M)$. Then $C_M(N) = N$ and from the maximality of C we have $C_M(N)/K = C_{M/K}(N/K)$. Therefore $N/K = C_{M/K}(N/K)$, i.e. $N/K \in \mathcal{F}_2^C(M/K)$ and so $\mathcal{F}_2^C$ satisfies the condition C 3.

($\Leftarrow$) Let C be an idempotent closure operator such that the function $\mathcal{F}_2^C$ satisfies the condition C 3. Then in the situation $K \subseteq N \subseteq M$ we have:

$$C_M(N) = \bigcap_{\alpha \in \mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\ N_\alpha \in \mathcal{F}_2^C(M)\},$$
$$C_{M/K}(N/K) = \bigcap_{\beta \in \mathfrak{B}} \{L_\beta/K \subseteq M/K \mid N/K \subseteq L_\beta/K\,,\ L_\beta/K \in \mathcal{F}_2^C(M/K)\}.$$

Using the condition C 3 for $\mathcal{F}_2^C$, from $N_\alpha \in \mathcal{F}_2^C(M)$ we obtain $N_\alpha/K \in \mathcal{F}_2^C(M/K)$. Therefore N_α/K is one of the submodules L_β/K. Then $\{N_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \{L_\beta \mid \beta \in \mathfrak{B}\}$ and $\bigcap_{\alpha \in \mathfrak{A}} N_\alpha \supseteq \bigcap_{\beta \in \mathfrak{B}} N_\beta$, therefore $\big(\bigcap_{\alpha \in \mathfrak{A}} N_\alpha\big)/K \supseteq \big(\bigcap_{\beta \in \mathfrak{B}} N_\beta\big)/K = \bigcap_{\beta \in \mathfrak{B}} (N_\beta/K)$. This means that $C_M(N)/K \supseteq C_{M/K}(N/K)$ and the inverse inclusion is trivial. So we have the relation (4.2), i.e. C is maximal. □

This result together with Theorem 3.10 leads to the following description of the idempotent and maximal closure operators by the function $\mathcal{F}_2^C$.

Theorem 4.4. *The mappings $C \rightsquigarrow \mathcal{F}_2^C$ and $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ define an antimonotone bijection between the **idempotent and maximal** closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_2$ which satisfy the condition* C 3. □

c) *Weakly hereditary, idempotent and maximal closure operators*

Now we consider the case which combines together the previous two cases: suppose that an operator $C \in \mathbb{CO}(R)$ is *weakly hereditary, idempotent and maximal.* Then it is clear that C can be described both by $\mathcal{F}_1^C$ and by $\mathcal{F}_2^C$.

Namely, since C is weakly hereditary and maximal, the function $\mathcal{F}_1^C$ is of the type $\mathcal{F}_1$ and satisfies the condition C 3^* (Theorem 4.2). In this case the idempotency of C implies the condition C 6 for $\mathcal{F}_1^C$ (Theorem 3.13), so the function $\mathcal{F}_1^C$ of the type $\mathcal{F}_1$ accumulates the conditions C 3^* and C 6.

Similarly we can show the properties of the function $\mathcal{F}_2^C$ in this case. By Theorem 4.4 $\mathcal{F}_2^C$ is a function of the type $\mathcal{F}_2$, which satisfies the condition C 3. It remains to add the effect of the weakly heredity of C to $\mathcal{F}_2^C$, which is expressed by the condition C 6 (Theorem 3.16). Therefore in this case $\mathcal{F}_2^C$ is a function of the type $\mathcal{F}_2$ with the conditions C 3 and C 6. In such a way the previous results give us the following characterizations.

Corollary 4.5. a) *The mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ define a monotone bijections between the **weakly hereditary, idempotent and maximal** closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ which satisfy the conditions* C 3^* *and* C 6.

b) *The mappings* $C \rightsquigarrow \mathcal{F}_2^C$ *and* $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ *define an antimonotone bijection between the* ***weakly hereditary, idempotent and maximal*** *closure operators of R-Mod and the abstract functions of the type* $\mathcal{F}_2$ *which satisfy the conditions* C 3 *and* C 6. □

d) *Minimal closure operators*

Let $C \in \mathbb{CO}(R)$ be a minimal closure operator. By Lemma 1.2 C is idempotent, therefore it can be described by the associated function $\mathcal{F}_2^C$ which satisfies the conditions C 1*– C 3* (Theorem 3.10). Now we will express the *minimality* of C by the function $\mathcal{F}_2^C$. For that we will use the following known condition (Section 2):

C 4. If $N \subseteq L \subseteq M$ and $N \in \mathcal{F}(M)$, then $L \in \mathcal{F}(M)$.

Proposition 4.6. *Let* $C \in \mathbb{CO}(R)$ *be an idempotent closure operator. Then* C *is* ***minimal*** *if and only if the associated function* $\mathcal{F}_2^C$ *satisfies the condition* C 4.

Proof. (⇒) Suppose that the operator $C \in \mathbb{CO}(R)$ is minimal and consider the situation: $N \subseteq L \subseteq M$, $N \in \mathcal{F}_2^C(M)$. Then $C_M(N) = N$ and the minimality of C implies $C_M(N) = C_M(0) + N$ (see (4.3)). Therefore $N = C_M(0) + N$ and $C_M(0) \subseteq N$. Using again the minimality of C we obtain: $C_M(L) = C_M(0) + L \subseteq N + L = L$. Then $C_M(L) \subseteq L$ and $C_M(L) = L$, i.e. $L \in \mathcal{F}_2^C(M)$, which means that $\mathcal{F}_2^C$ satisfies the condition C 4.

(⇐) Let C be an idempotent closure operator such that $\mathcal{F}_2^C$ satisfies the condition C 4. Then C can be reestablished by $\mathcal{F}_2^C$ and in the situation $K \subseteq N \subseteq M$ we have:

$$C_M(N) = \bigcap_{\alpha \in \mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\ N_\alpha \in \mathcal{F}_2^C(M)\},$$

$$C_M(K) = \bigcap_{\beta \in \mathfrak{B}} \{K_\beta \subseteq M \mid K \subseteq K_\beta\,,\ K_\beta \in \mathcal{F}_2^C(M)\}.$$

Since $\mathcal{F}_2^C$ satisfies C 1*, from the relations $K_\beta \in \mathcal{F}_2^C(M)$ $(\beta \in \mathfrak{B})$ it follows that $\bigcap_{\beta \in \mathfrak{B}} K_\beta \in \mathcal{F}_2^C(M)$, i.e. $C_M(K) \in \mathcal{F}_2^C(M)$.

Now we will use the property C 4 of $\mathcal{F}_2^C$ in the situation: $C_M(K) \subseteq C_M(K) + N \subseteq M$, where $C_M(K) \in \mathcal{F}_2^C(M)$. Then C 4 implies $C_M(K) + N \in \mathcal{F}_2^C(M)$ and this means that $C_M(K) + N$ is one of submodules N_α from the definition of $C_M(N)$. Therefore $C_M(N) \subseteq C_M(K) + N$ and the inverse inclusion is trivial. So we have $C_M(N) = C_M(K) + N$, i.e. the operator C is minimal. □

This proposition combined with Theorem 3.10 leads to the characterization of the minimal closure operators C by the function $\mathcal{F}_2^C$.

Theorem 4.7. *The mappings* $C \rightsquigarrow \mathcal{F}_2^C$ *and* $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ *define an antimonotone bijection between the* ***minimal*** *closure operators of R-Mod and the abstract functions of the type* $\mathcal{F}_2$ *which satisfy the condition* C 4. □

e) *Weakly hereditary and minimal closure operators*

Let $C \in \mathbb{CO}(R)$ be a weakly hereditary and minimal closure operator. The first condition implies that C can be described by $\mathcal{F}_1^C$ (Theorem 3.5). From the other hand, every minimal closure operator C can be characterized by $\mathcal{F}_2^C$ (Theorem 4.7). Therefore in the studied case C can be described both by $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$.

We begin with the characterization of C by $\mathcal{F}_1^C$. By Theorem 3.5 $\mathcal{F}_1^C$ is a function of the type $\mathcal{F}_1$ (i.e. with C1 – C3). Now we must express by $\mathcal{F}_1^C$ the *minimality* of C. With this aim we will formulate a new condition for an abstract function $\mathcal{F}$ of R-Mod:

C7. If $K \subseteq N \subseteq M$, then for every submodule N_α with $N \subseteq N_\alpha \subseteq M$ and $N \in \mathcal{F}(N_\alpha)$, there exists a submodule K_β with $K \subseteq K_\beta \subseteq M$ and $K \in \mathcal{F}(K_\beta)$ such that $N_\alpha = K_\beta + N$.

Proposition 4.8. *Let $C \in \mathbb{CO}(R)$ be a weakly hereditary closure operator. Then C is* ***minimal*** *if and only if the function $\mathcal{F}_1^C$ satisfies the condition* C7.

Proof. ($\Rightarrow$) Suppose that a weakly hereditary closure operator C is minimal. Then in the situation $K \subseteq N \subseteq M$ we have $C_M(N) = C_M(K) + N$ $\big($see (4.4)$\big)$. In this case the operator C can be reestablished by $\mathcal{F}_1^C$ and we obtain:

$$C_M(N) = \sum_{\alpha\in\mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\ N \in \mathcal{F}_1^C(N_\alpha)\},$$

$$C_M(K) = \sum_{\beta\in\mathfrak{B}} \{K_\beta \subseteq M \mid K \subseteq K_\beta\,,\ K \in \mathcal{F}_1^C(K_\beta)\}.$$

The minimality of C implies the relation: $\sum\limits_{\alpha\in\mathfrak{A}} N_\alpha = \big(\sum\limits_{\beta\in\mathfrak{B}} K_\beta\big) + N = \sum\limits_{\beta\in\mathfrak{B}} (K_\beta + N)$.

Now we will verify that $\mathcal{F}_1^C$ satisfies C7. Let $K \subseteq N \subseteq M$ and N_α be a submodule of M such that $N \subseteq N_\alpha \subseteq M$ and $N \in \mathcal{F}_1^C(N_\alpha)$. By the minimality of C in the situation $K \subseteq N \subseteq N_\alpha$ we have $C_{N_\alpha}(N) = C_{N_\alpha}(K) + N$. From the relation $N \in \mathcal{F}_1^C(N_\alpha)$ it follows that $C_{N_\alpha}(N) = N_\alpha$, therefore $N_\alpha = C_{N_\alpha}(K) + N$. Denoting $K_\beta = C_{N_\alpha}(K)$, we have $N_\alpha = K_\beta + N$.

Since C is weakly hereditary, for the pair $K \subseteq N_\alpha$ we have $C_{C_{N_\alpha}(K)}(K) = C_{N_\alpha}(K)$, i.e. $K \in \mathcal{F}_1^C\big(C_{N_\alpha}(K)\big) = \mathcal{F}_1^C(K_\beta)$. Therefore K_β is a submodule required in the condition C7.

($\Leftarrow$) Now we suppose that for a weakly hereditary closure operator C the function $\mathcal{F}_2^C$ satisfies the condition C7. We must verify that C is minimal, i.e. in the situation $K \subseteq N \subseteq M$ the relation $C_M(N) = C_M(K) + N$ is true, which in the above notations means that $\sum\limits_{\alpha\in\mathfrak{A}} N_\alpha = \big(\sum\limits_{\beta\in\mathfrak{B}} K_\beta\big) + N = \sum\limits_{\beta\in\mathfrak{B}} (K_\beta + N)$.

From the condition C7 it follows that every submodule N_α of the definition of $C_M(N)$ is of the form $K_\beta + N$ for a suitable submodule K_β. Therefore we have the relation $\sum\limits_{\alpha\in\mathfrak{A}} N_\alpha \subseteq \sum\limits_{\beta\in\mathfrak{B}} (K_\beta + N)$, i.e. $C_M(N) \subseteq C_M(K) + N$, where the inverse inclusion is trivial. So $C_M(N) = C_M(K) + N$ and the operator C is minimal. $\square$

From this result it follows that we can restrict the bijection of Theorem 3.5, obtained for weakly hereditary closure operators, to a new bijection requiring the minimality for C and the condition C7 for the function $\mathcal{F}_1^C$.

Theorem 4.9. *The mappings* $C \rightsquigarrow \mathcal{F}_1^C$ *and* $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ *define a monotone bijection between the* ***weakly hereditary and minimal*** *closure operators of R-Mod and the abstract functions of the type* $\mathcal{F}_1$ *which satisfy the condition* C7. □

The characterization of weakly hereditary and minimal closure operators C by the associated functions $\mathcal{F}_2^C$ is clear from the previous results and consists in the following. The minimality of C implies that $\mathcal{F}_2^C$ is of the type $\mathcal{F}_2$ and satisfies the condition C 4 (Theorem 4.7). Moreover, the weakly heredity of C implies the condition C 6 for $\mathcal{F}_2^C$ (Theorem 3.16), so $\mathcal{F}_2^C$ accumulates the conditions C 4 and C 6.

Corollary 4.10. *The mappings* $C \rightsquigarrow \mathcal{F}_2^C$ *and* $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ *define an antimonotone bijection between the* ***weakly hereditary and minimal*** *closure operators of R-Mod and the abstract functions of the type* $\mathcal{F}_2$ *which satisfy the conditions* C 4 *and* C 6. □

5 Hereditary and cohereditary closure operators

In this section we will study the hereditary and cohereditary closure operators of R-Mod, as well as diverse combinations with the types of operators investigated above. The purpose is to describe all types of operators related with these new conditions by the associated functions $\mathcal{F}_1^C$ or (and) $\mathcal{F}_2^C$. Firstly we remind the necessary definitions (see Section 1).

A closure operator $C \in \mathbb{CO}(R)$ is called *hereditary* if for every submodules $K \subseteq N \subseteq M$ the relation

$$C_N(K) = C_M(K) \bigcap N \tag{5.1}$$

is true. Dually, a closure operator C is called *cohereditary* if for every submodules $K, N \in \mathbb{L}(M)$ the equality

$$\big(C_M(N) + K\big)/K = C_{M/K}\big((N+K)/K\big) \tag{5.2}$$

holds. This condition can be expressed in other form: for every R-morphism $f : M \to M'$ and every submodule $N \subseteq M$ we have:

$$f\big(C_M(N)\big) = C_{f(M)}\big(f(N)\big). \tag{5.3}$$

It is useful to remind the relations between these new types of operators and the types studied above. Namely, every hereditary closure operator is weakly hereditary, every cohereditary operator is idempotent; moreover, an operator $C \in \mathbb{CO}(R)$ is cohereditary if and only if it is maximal and minimal (see Lemmas 1.1, 1.2, 1.3). Taking into account these facts and using the proved above results, in this section we will characterize by the functions $\mathcal{F}_1^C$ or (and) $\mathcal{F}_2^C$ the following types of closure operators: a) hereditary; b) hereditary and maximal; c) hereditary and idempotent; d) hereditary, maximal and idempotent; e) hereditary and minimal; f) cohereditary; g) weakly hereditary and cohereditary; h) hereditary and cohereditary (see Figure 1).

a) *Hereditary closure operators*

Let $C \in \mathbb{CO}(R)$ be a hereditary closure operator. Then it is weakly hereditary (Lemma 1.1), therefore C can be characterized by the function $\mathcal{F}_1^C$ (Theorem 3.5), which in this case is a function if the type $\mathcal{F}_1$. So it remains to express the *heredity* of C by the function $\mathcal{F}_1^C$. By this aim we will use the following known condition (see Section 2):

C 4^*. If $N \subseteq L \subseteq M$ and $N \in \mathcal{F}(M)$, then $N \in \mathcal{F}(L)$.

Proposition 5.1. *Let $C \in \mathbb{CO}(R)$ be a weakly hereditary closure operator. Then C is* ***hereditary*** *if and only if the associated function $\mathcal{F}_1^C$ satisfies the condition* C 4^*.

Proof. ($\Rightarrow$) Suppose that an operator $C \in \mathbb{CO}(R)$ is hereditary and consider the situation: $N \subseteq L \subseteq M$, $N \in \mathcal{F}_1^C(M)$. Then $C_M(N) = M$ and since C is hereditary we have: $C_L(N) = C_M(N) \bigcap L = M \bigcap L = L$. Therefore $C_L(N) = L$ and $N \in \mathcal{F}_1^C(L)$, which means that $\mathcal{F}_1^C$ satisfies the condition C 4^*.

($\Leftarrow$) Let $C \in \mathbb{CO}(R)$ be a weakly hereditary closure operator such that the function $\mathcal{F}_1^C$ satisfies the condition C 4^*. Then C can be expressed by $\mathcal{F}_1^C$ and for submodules $K \subseteq N \subseteq M$ we have:

$$C_M(K) = \sum_{\alpha \in \mathfrak{A}} \{K_\alpha \subseteq M \mid K \subseteq K_\alpha\,,\ K \in \mathcal{F}_1^C(K_\alpha)\},$$

$$C_N(K) = \sum_{\beta \in \mathfrak{B}} \{N_\beta \subseteq N \mid K \subseteq N_\beta\,,\ K \in \mathcal{F}_1^C(N_\beta)\}.$$

Since $\mathcal{F}_1^C$ satisfies the condition C 1, from the relations $K \in \mathcal{F}_1^C(K_\alpha)$ $(\alpha \in \mathfrak{A})$ it follows that $K \in \mathcal{F}_1^C(\sum_{\alpha \in \mathfrak{A}} K_\alpha) = \mathcal{F}_1^C\big(C_M(K)\big)$.

Now we apply the condition C 4^* of $\mathcal{F}_1^C$ in the situation: $K \subseteq C_M(K) \bigcap N \subseteq C_M(K)$. Having the relation $K \in \mathcal{F}_1^C\big(C_M(K)\big)$, from C 4^* it follows that $K \in \mathcal{F}_1^C\big(C_M(K) \bigcap N\big)$. Therefore $C_M(K) \bigcap N$ is one of the submodules N_β from the definition of $C_N(K)$. Then $C_M(K) \bigcap N \subseteq C_N(K)$ and the inverse inclusion is trivial. So we have $C_M(K) \bigcap N = C_N(K)$, which means that C is hereditary. □

Combining this result with Theorem 3.5, we obtain the characterization of hereditary closure operators C by the functions $\mathcal{F}_1^C$.

Theorem 5.2. *The mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ define a monotone bijection between the* ***hereditary*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ which satisfy the condition* C 4^*. □

b) *Hereditary and maximal closure operators*

If a closure operator $C \in \mathbb{CO}(R)$ is hereditary and maximal, then it can be described by the function $\mathcal{F}_1^C$ which is of the type $\mathcal{F}_1$ and satisfies the condition

$C\,4^*$ (Theorem 5.2). Moreover, in this situation (since C is weakly hereditary) the maximality of C can be expressed by the condition $C\,3^*$ for $\mathcal{F}_1^C$ (Proposition 4.1). Therefore in this case $\mathcal{F}_1^C$ is a function of the type $\mathcal{F}_1$ with the properties $C\,3^*$ and $C\,4^*$.

Corollary 5.3. *The mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ define a monotone bijection between the **hereditary and maximal** closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ which satisfy the conditions $C\,3^*$ and $C\,4^*$.* □

Remark 1. The mappings (Φ, Ψ_1) (see Section 1) establish a bijection between the hereditary and maximal closure operators of R-Mod and the *pretorsions* (= hereditary preradicals) of this category [5]. If r is a pretorsion, then C^r can be reduced to a closure operator (in the classical sense) of the lattice $\mathbb{L}(_RR)$ of the left ideals of R [7].

c) *Hereditary and idempotent closure operators*

Let $C \in \mathbb{CO}(R)$ be a hereditary and idempotent closure operator. Then C can be described both by $\mathcal{F}_1^C$ (since C is weakly hereditary) and by $\mathcal{F}_2^C$ (since C is idempotent). The characterization of C by $\mathcal{F}_1^C$ can be obtained from the previous results and consists in the following. The heredity of C implies that $\mathcal{F}_1^C$ is a function of the type $\mathcal{F}_1$ with the condition $C\,4^*$ (Theorem 5.2). Moreover, the idempotency of C in this case is equivalent to the transitivity ($C\,6 = C\,6^*$) of $\mathcal{F}_1^C$ (Theorem 3.13).

Corollary 5.4. *The mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ define a monotone bijection between the **hereditary and idempotent** closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ which satisfy the conditions $C\,4^*$ and $C\,6$.* □

Now we will show the description of the studied type of operators by the functions $\mathcal{F}_2^C$, which is possible because C is idempotent (Theorem 3.10). Then $\mathcal{F}_2^C$ is a function of the type $\mathcal{F}_2$ and now we must express by $\mathcal{F}_2^C$ the *heredity* of C. For that we will use a new condition, which is concordant with heredity (to compare with C7):

C7*. If $N \subseteq L \subseteq M$, then for every submodule L_α with $N \subseteq L_\alpha \subseteq L$ and $L_\alpha \in \mathcal{F}(L)$, there exists a submodule M_β with $N \subseteq M_\beta \subseteq M$ and $M_\beta \in \mathcal{F}(M)$ such that $L_\alpha = M_\beta \cap L$.

Proposition 5.5. *Let $C \in \mathbb{CO}(R)$ be an idempotent closure operator. Then C is **hereditary** if and only if the associated function $\mathcal{F}_2^C$ satisfies the condition C7*.*

Proof. (⇒) Suppose that an idempotent operator C is hereditary. Then in the situation $N \subseteq L \subseteq M$ we have $C_L(N) = C_M(N) \cap L$. Since C is idempotent, it can be reestablished by $\mathcal{F}_2^C$ and we have:

$$C_L(N) = \bigcap_{\alpha \in \mathfrak{A}} \{L_\alpha \subseteq L \mid N \subseteq L_\alpha\,,\ L_\alpha \in \mathcal{F}_2^C(L)\},$$
$$C_M(N) = \bigcap_{\beta \in \mathfrak{B}} \{M_\beta \subseteq M \mid N \subseteq M_\beta\,,\ M_\beta \in \mathcal{F}_2^C(M)\}.$$

Then the above relation obtained by the heredity of C has the form: $\bigcap\limits_{\alpha\in\mathfrak{A}} L_\alpha = (\bigcap\limits_{\beta\in\mathfrak{B}} M_\beta) \cap L = \bigcap\limits_{\beta\in\mathfrak{B}} (M_\beta \cap L)$.

In the situation $L_\alpha \subseteq L \subseteq M$ the heredity of C implies that $C_L(L_\alpha) = C_M(L_\alpha) \cap L$. Since $L_\alpha \in \mathcal{F}_2^C(L)$, we have $C_L(L_\alpha) = L_\alpha$, therefore $L_\alpha = C_M(L_\alpha) \cap L$. Denote: $M_\beta = C_M(L_\alpha)$. Since C is idempotent, the submodule M_β is C-closed in M, i.e. $M_\beta \in \mathcal{F}_2^C(M)$, where $N \subseteq L_\alpha \subseteq M_\beta$. In such a way, for every submodule L_α of the indicated type we find a submodule M_β such that $M_\beta \in \mathcal{F}_2^C(M)$ and $L_\alpha = M_\beta \cap L$. This proves that $\mathcal{F}_2^C$ satisfies C7*.

($\Leftarrow$) Let $C \in \mathbb{CO}(R)$ be an idempotent closure operator such that the function $\mathcal{F}_2^C$ satisfies the condition C7*. In the preceding notations the heredity of C is expressed in the form: $\bigcap\limits_{\alpha\in\mathfrak{A}} L_\alpha = \bigcap\limits_{\beta\in\mathfrak{B}} (M_\beta \cap L)$, where $N \subseteq L \subseteq M$.

From the condition C7* it follows that every submodule L_α $(\alpha \in \mathfrak{A})$ has the form $M_\beta \cap L$, therefore $\bigcap\limits_{\alpha\in\mathfrak{A}} L_\alpha \supseteq \bigcap\limits_{\beta\in\mathfrak{B}} (M_\beta \cap L)$. This means that $C_L(N) \supseteq C_M(N) \cap L$, where the inverse inclusion is trivial. So we obtain $C_L(N) = C_M(N) \cap L$ for every $N \subseteq L \subseteq M$, i.e. C is hereditary. □

This result gives us the possibility to restrict the bijection of Theorem 3.16, demanding for C to be hereditary and for $\mathcal{F}_2^C$ to satisfy the condition C7*. Therefore in this case $\mathcal{F}_2^C$ is a function of the type $\mathcal{F}_2$ which satisfies the condition C7*.

Theorem 5.6. *The mappings $C \rightsquigarrow \mathcal{F}_2^C$ and $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ define an antimonotone bijection between the* ***hereditary and idempotent*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_2$ which satisfy the condition* C7*. □

We can remark that in the studied case the operator C is weakly hereditary and idempotent, therefore by Theorems 3.13 and 3.16 both the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$ satisfy the condition of transitivity ($\text{C}\,6 = \text{C}\,6^*$).

d) *Hereditary, maximal and idempotent closure operators*

This subsection is a combination of two previous cases (b) and c)) and contains the description of the closure operators with three properties: they are *hereditary, maximal and idempotent*. By the part b) the operator C can be described by $\mathcal{F}_1^C$, which is a function of the type $\mathcal{F}_1$ with the properties C 3* and C 4* (Corollary 5.3). Moreover, the idempotency of C is expressed for $\mathcal{F}_1^C$ by the condition C 6 (Theorem 3.13).

Corollary 5.7. *The mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ define a monotone bijection between the* ***hereditary, maximal and idempotent*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ which satisfy the conditions* C 3*, C 4* *and* C 6. □

Similarly the operators of the studied type can be characterized by the function $\mathcal{F}_2^C$ using Theorem 5.6. Indeed, since C is hereditary and idempotent, it can be described by the function $\mathcal{F}_2^C$, which is of the type $\mathcal{F}_2$ and satisfies the condition C7*. Finally, we must take into account the effect to $\mathcal{F}_2^C$ of the *maximality* of C, which is expressed by the condition C 3 (Proposition 4.3).

Corollary 5.8. *The mappings $C \rightsquigarrow \mathcal{F}_2^C$ and $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ define an antimonotone bijection between the* ***hereditary, maximal and idempotent*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_2$ which satisfy the conditions* C 3 *and* C7*. □

Remark 2. As in the case b), the mappings (Φ, Ψ_1) define a bijection between the operators studied in this subsection and the *torsions* (= hereditary radicals) of R-Mod. Moreover, such an operator (hereditary, maximal and idempotent) can be reduced to a closure operator of $\mathbb{L}({}_R R)$ [7].

e) *Hereditary and minimal closure operators*

Let $C \in \mathbb{CO}(R)$ be a hereditary and minimal closure operator. The relations between the classes of closure operators (Lemmas 1.1, 1.2, 1.3) show that in this case C can be described both by $\mathcal{F}_1^C$ (since C is weakly hereditary) and by $\mathcal{F}_2^C$ (since C is idempotent). The characterizations of C in this case can be obtained from the previous results and consist in the following.

Since C is hereditary, Theorem 5.2 shows that $\mathcal{F}_1^C$ is a function of the type $\mathcal{F}_1$ with the condition C 4*. Moreover, the minimality of C in this case (because C is weakly hereditary) is equivalent to the condition C7 for $\mathcal{F}_1^C$ (Proposition 4.8). So we obtain the following characterization.

Corollary 5.9. *The mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ define a monotone bijection between the* ***hereditary and minimal*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ which satisfy the conditions* C 4* *and* C7. □

Similarly the hereditary and minimal closure operators C can be described by the functions $\mathcal{F}_2^C$. Namely, by Theorem 4.7 C is minimal if and only if $\mathcal{F}_2^C$ is a function of the type $\mathcal{F}_2$ with the condition C 4. Further, we can apply Proposition 5.5 : since C is minimal, it is idempotent and in this case C is hereditary if and only if $\mathcal{F}_2^C$ satisfies the condition C7*.

Corollary 5.10. *The mappings $C \rightsquigarrow \mathcal{F}_2^C$ and $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ define an antimonotone bijection between the* ***hereditary and minimal*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_2$ which satisfy the conditions* C 4 *and* C7*. □

The rest of this section is dedicated to the cases related to *coheredity* and consists of three parts, studying the operators which are : f) cohereditary; g) weakly hereditary and cohereditary; h) hereditary and cohereditary (see Figure 1).

f) *Cohereditary closure operators*

Let $C \in \mathbb{CO}(R)$ be a cohereditary closure operator. Then C is minimal (Lemma 1.3), therefore it is idempotent (Lemma 1.2). Therefore C can be described by $\mathcal{F}_2^C$, which is of the type $\mathcal{F}_2$ (Theorem 3.10). Now we must find the property of $\mathcal{F}_2^C$ which is equivalent to the coheredity of C. With this aim we will use the following condition (see Section 2) :

C 5. If $f : M \to M'$ is an R-morphism and $N \in \mathcal{F}(M)$, then $f(N) \in \mathcal{F}\big(f(M)\big)$.

Proposition 5.11. *Let $C \in \mathbb{CO}(R)$ be an idempotent closure operator. Then C is **cohereditary** if and only if the function $\mathcal{F}_2^C$ satisfies the condition* C 5.

Proof. ($\Rightarrow$) Let $C \in \mathbb{CO}(R)$ be a cohereditary closure operator, $f : M \to M'$ be an R-morphism, $N \subseteq M$ and $N \in \mathcal{F}_2^C(M)$. Then $C_M(N) = N$ and $f\big(C_M(N)\big) = C_{f(M)}\big(f(N)\big)$, i.e. $f(N) = C_{f(M)}\big(f(N)\big)$. Therefore $f(N) \in \mathcal{F}_2^C\big(f(M)\big)$ and $\mathcal{F}_2^C$ satisfies the condition C 5.

($\Leftarrow$) Suppose that $C \in \mathbb{CO}(R)$ is idempotent and $\mathcal{F}_2^C$ satisfies C 5. Then C can be expressed by $\mathcal{F}_2^C$ and in the situation $f : M \to M'$ and $N \subseteq M$ we have:

$$C_M(N) = \bigcap_{\alpha \in \mathfrak{A}} \{N_\alpha \subseteq M \mid N \subseteq N_\alpha\,,\ N_\alpha \in \mathcal{F}_2^C(M)\},$$
$$C_{f(M)}\big(f(N)\big) = \bigcap_{\beta \in \mathfrak{B}} \{N'_\beta \subseteq f(M) \mid f(N) \subseteq N'_\beta\,,\ N'_\beta \in \mathcal{F}_2^C\big(f(M)\big)\}.$$

Having the relations $N_\alpha \in \mathcal{F}_2^C(M)$ $(\alpha \in \mathfrak{A})$, by C 1^* we conclude that $\bigcap\limits_{\alpha \in \mathfrak{A}} N\alpha \in \mathcal{F}_2^C(M)$. Now by the property C 5 of $\mathcal{F}_2^C$ we obtain $f\big(\bigcap\limits_{\alpha \in \mathfrak{A}} N\alpha\big) \in \mathcal{F}_2^C\big(f(M)\big)$. Therefore the submodule $f\big(\bigcap\limits_{\alpha \in \mathfrak{A}} N\alpha\big) \subseteq f(M)$, which contains $f(N)$, is one of the submodules N'_β. Then $C_{f(M)}\big(f(N)\big) \subseteq f\big(\bigcap\limits_{\alpha \in \mathfrak{A}} N_\alpha\big) = f\big(C_M(N)\big)$ and the inverse inclusion follows from the condition (c_3) (Definition 1.1). So we have $f\big(C_M(N)\big) = C_{f(M)}\big(f(N)\big)$, i.e. C is cohereditary. □

Using this result, now we can restrict the bijection of Theorem 3.10 (which is proved for idempotent operators), demanding for C to be cohereditary and for $\mathcal{F}_2^C$ to satisfy the condition C 5.

Theorem 5.12. *The mappings $C \rightsquigarrow \mathcal{F}_2^C$ and $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ define an antimonotone bijection between the **cohereditary** closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_2$ which satisfy the condition* C 5. □

g) *Weakly hereditary and cohereditary closure operators*

Let $C \in \mathbb{CO}(R)$ be a weakly hereditary and cohereditary closure operator. From the first condition it follows that C can be described by the function $\mathcal{F}_1^C$ (Theorem 3.5), and from the second one it is clear (as in the previous case) that C can be characterized by the function $\mathcal{F}_2^C$. Now we will formulate these characterizations, combining the foregoing results.

The function $\mathcal{F}_1^C$ in this case is of the type $\mathcal{F}_1$ and we must join the effect of coheredity of C to $\mathcal{F}_1^C$. We remind that C is cohereditary if and only if it is maximal and minimal (Lemma 1.3). The maximality of C is equivalent to the condition C 3^* for $\mathcal{F}_1^C$ (Proposition 4.1), while the minimality of C means that $\mathcal{F}_1^C$ satisfies the condition C7 (Proposition 4.8). Therefore by Theorem 3.5 and Propositions 4.1, 4.8 we obtain the following characterization.

Corollary 5.13. *The mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ establish a monotone bijection between the **weakly hereditary and cohereditary** closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ which satisfy the conditions* C 3^* *and* C7. □

The description of the weakly hereditary and cohereditary closure operators by the functions $\mathcal{F}_2^C$ can be shown in a similar way: by Theorem 5.12 the coheredity of C is equivalent to the property C 5 of $\mathcal{F}_2^C$, while the weakly heredity of C is expressed by the condition C 6 of $\mathcal{F}_2^C$ (Theorem 3.16).

Corollary 5.14. *The mappings $C \rightsquigarrow \mathcal{F}_2^C$ and $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ define an antimonotone bijection between the* ***weakly hereditary and cohereditary*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_2$ which satisfy the conditions* C 5 *and* C 6. □

h) *Hereditary and cohereditary closure operators*

In the last case of this work the class of hereditary and cohereditary operators of R-Mod is studied. It is contained in all previous classes and it is obvious that such operators can be described both by the functions $\mathcal{F}_1^C$ and $\mathcal{F}_2^C$.

Indeed, the heredity of C implies that $\mathcal{F}_1^C$ is a function of the type $\mathcal{F}_1$ with the condition C 4^* (Theorem 5.2). At the same time, by the case g) (Corollary 5.13) the coheredity of C implies for $\mathcal{F}_1^C$ the conditions C 3^* and C7.

Corollary 5.15. *The mappings $C \rightsquigarrow \mathcal{F}_1^C$ and $\mathcal{F} \rightsquigarrow C^{\mathcal{F}}$ define a monotone bijection between the* ***hereditary and cohereditary*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_1$ which satisfy the conditions* C 3^*, C 4^* *and* C7. □

It remains to formulate the characterization of the studied closure operators by the functions $\mathcal{F}_2^C$. From the case g) we see that in this case $\mathcal{F}_2^C$ is of the type $\mathcal{F}_2$ with the conditions C 5 and C 6 (Corollary 5.14). Now we join the effect of heredity of C to $\mathcal{F}_2^C$, which is expressed by the condition C7^* (Proposition 5.5).

Corollary 5.16. *The mappings $C \rightsquigarrow \mathcal{F}_2^C$ and $\mathcal{F} \rightsquigarrow C_{\mathcal{F}}$ define an antimonotone bijection between the* ***hereditary and cohereditary*** *closure operators of R-Mod and the abstract functions of the type $\mathcal{F}_2$ which satisfy the conditions* C 5, C 6 *and* C7^*. □

We totalize this work by a review of the exposed above results. The previous investigations show the situation related to the characterization of the principal types of the closure operators C of R-Mod in the language of C-*dense* submodules (i.e. by the function $\mathcal{F}_1^C$) or (and) of C-*closed* submodules (i.e. by the function $\mathcal{F}_2^C$). The description of $C \in \mathbb{CO}(R)$ by $\mathcal{F}_1^C$ is possible if and only if C is *weakly hereditary*. Dually, C can be characterized by $\mathcal{F}_2^C$ if and only if C is *idempotent*. These facts determine what types of operators and how can be described by the dense submodules or (and) by the closed submodules.

In this work all cases are analyzed, which are defined by diverse combinations of the basic conditions on the operators: weakly heredity – idempotency, maximality – minimality, heredity – coheredity. Altogether 16 types of operators are described, from which 7 types possess the double characterizations (by $\mathcal{F}_1^C$ and by $\mathcal{F}_2^C$). The general situation with all results is illustrated in Figure 1.

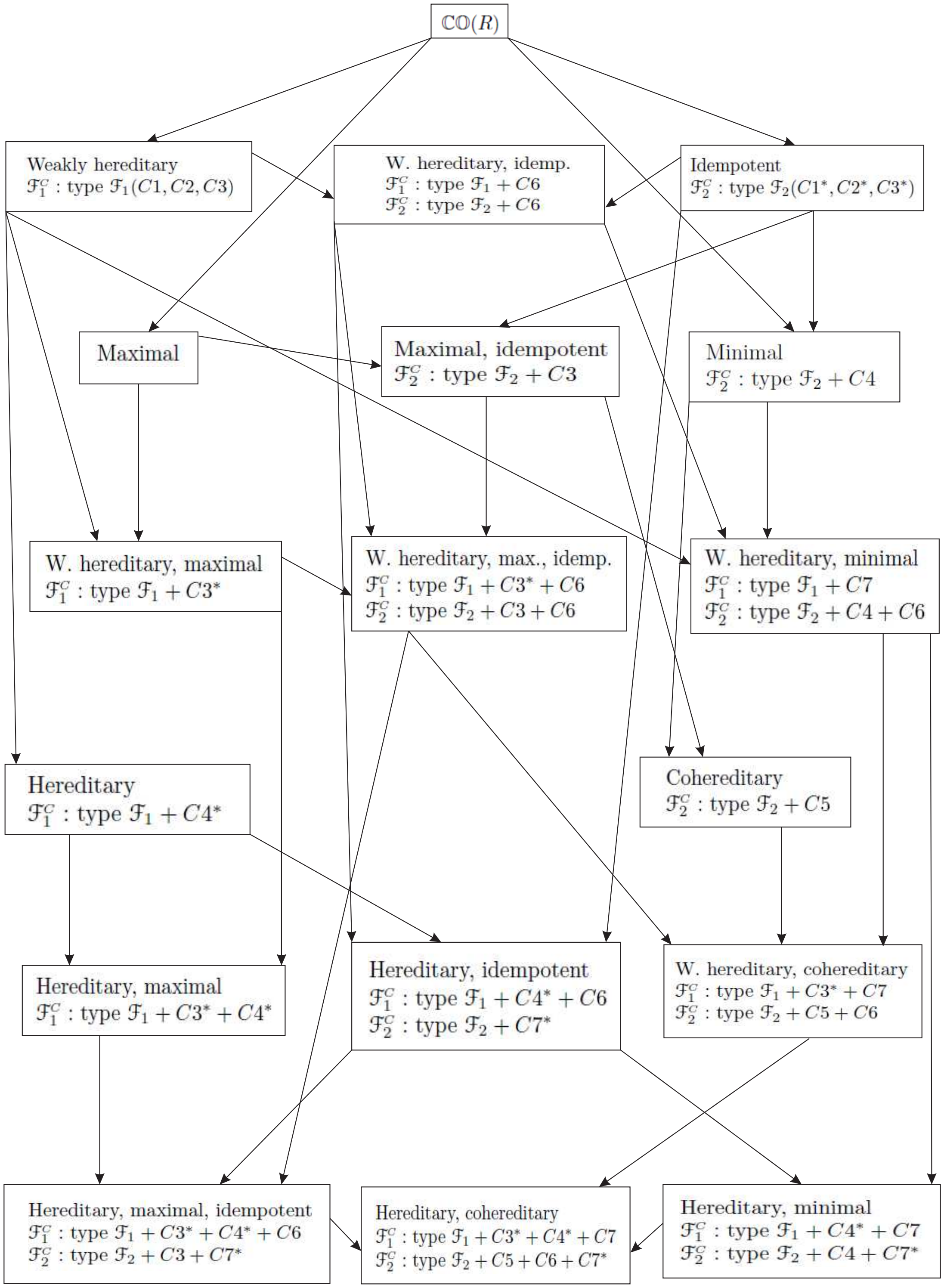

$\mathbb{CO}(R)$
Weakly hereditary
$\mathcal{F}_1^C$: type $\mathcal{F}_1(C1, C2, C3)$
W. hereditary, idemp.
$\mathcal{F}_1^C$: type $\mathcal{F}_1 + C6$
$\mathcal{F}_2^C$: type $\mathcal{F}_2 + C6$
Idempotent
$\mathcal{F}_2^C$: type $\mathcal{F}_2(C1^*, C2^*, C3^*)$
Maximal
Maximal, idempotent
$\mathcal{F}_2^C$: type $\mathcal{F}_2 + C3$
Minimal
$\mathcal{F}_2^C$: type $\mathcal{F}_2 + C4$
W. hereditary, maximal
$\mathcal{F}_1^C$: type $\mathcal{F}_1 + C3^*$
W. hereditary, max., idemp.
$\mathcal{F}_1^C$: type $\mathcal{F}_1 + C3^* + C6$
$\mathcal{F}_2^C$: type $\mathcal{F}_2 + C3 + C6$
W. hereditary, minimal
$\mathcal{F}_1^C$: type $\mathcal{F}_1 + C7$
$\mathcal{F}_2^C$: type $\mathcal{F}_2 + C4 + C6$
Hereditary
$\mathcal{F}_1^C$: type $\mathcal{F}_1 + C4^*$
Cohereditary
$\mathcal{F}_2^C$: type $\mathcal{F}_2 + C5$
Hereditary, maximal
$\mathcal{F}_1^C$: type $\mathcal{F}_1 + C3^* + C4^*$
Hereditary, idempotent
$\mathcal{F}_1^C$: type $\mathcal{F}_1 + C4^* + C6$
$\mathcal{F}_2^C$: type $\mathcal{F}_2 + C7^*$
W. hereditary, cohereditary
$\mathcal{F}_1^C$: type $\mathcal{F}_1 + C3^* + C7$
$\mathcal{F}_2^C$: type $\mathcal{F}_2 + C5 + C6$
Hereditary, maximal, idempotent
$\mathcal{F}_1^C$: type $\mathcal{F}_1 + C3^* + C4^* + C6$
$\mathcal{F}_2^C$: type $\mathcal{F}_2 + C3 + C7^*$
Hereditary, cohereditary
$\mathcal{F}_1^C$: type $\mathcal{F}_1 + C3^* + C4^* + C7$
$\mathcal{F}_2^C$: type $\mathcal{F}_2 + C5 + C6 + C7^*$
Hereditary, minimal
$\mathcal{F}_1^C$: type $\mathcal{F}_1 + C4^* + C7$
$\mathcal{F}_2^C$: type $\mathcal{F}_2 + C4 + C7^*$

Figure 1.

Table 1. The list of conditions

C 1. $M_\alpha \subseteq M\ (\alpha \in \mathfrak{A}),\ N \in \mathcal{F}(M_\alpha) \Rightarrow N \in \mathcal{F}(\sum_{\alpha \in \mathfrak{A}} M_\alpha)$;	C 1*. $N_\alpha \subseteq M\ (\alpha \in \mathfrak{A}),\ N_\alpha \in \mathcal{F}(M) \Rightarrow \bigcap_{\alpha \in \mathfrak{A}} N_\alpha \in \mathcal{F}(M)$;
C 2. $N \subseteq L \subseteq M,\ N \in \mathcal{F}(L) \Rightarrow N + K \in \mathcal{F}(L+K)\ \forall K \subseteq M$;	C 2*. $N \subseteq L \subseteq M,\ N \in \mathcal{F}(L) \Rightarrow N \cap K \in \mathcal{F}(L \cap K)\ \forall K \subseteq M$;
C 3. $K \subseteq N \subseteq M,\ N \in \mathcal{F}(M) \Rightarrow N/K \in \mathcal{F}(M/K)$;	C 3*. $K \subseteq N \subseteq M,\ N/K \in \mathcal{F}(M/K) \Rightarrow N \in \mathcal{F}(M)$;
C 4. $N \subseteq L \subseteq M,\ N \in \mathcal{F}(M) \Rightarrow L \in \mathcal{F}(M)$;	C 4*. $N \subseteq L \subseteq M,\ N \in \mathcal{F}(M) \Rightarrow N \in \mathcal{F}(L)$;
C 5. $f : M \to M',\ N \in \mathcal{F}(M) \Rightarrow f(N) \in \mathcal{F}\big(f(M)\big)$;	C 5*. $g : M \to M',\ N' \in \mathcal{F}\big(g(M)\big) \Rightarrow g^{-1}(N') \in \mathcal{F}(M)$;
C 6 = C 6*. $N \subseteq L \subseteq M,\ N \in \mathcal{F}(L),\ L \in \mathcal{F}(M) \Rightarrow N \in \mathcal{F}(M)$ (*transitivity*)	
C7. $K \subseteq N \subseteq M \Rightarrow \forall N_\alpha \big(N \subseteq N_\alpha \subseteq M,\ N \in \mathcal{F}(N_\alpha)\big)\ \exists K_\beta \big(K \subseteq K_\beta \subseteq M,\ K \in \mathcal{F}(K_\beta)\big)$ such that $N_\alpha = K_\beta + N$.	C7*. $N \subseteq L \subseteq M \Rightarrow \forall L_\alpha \big(N \subseteq L_\alpha \subseteq L,\ L_\alpha \in \mathcal{F}(L)\big)\ \exists M_\beta \big(N \subseteq M_\beta \subseteq M,\ M_\beta \in \mathcal{F}(M)\big)$ such that $L_\alpha = M_\beta \cap L$.

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