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Commutator subgroup of Sylow 2-subgroups of alternating group and the commutator width in the wreath product

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Abstract. It is proved that the commutator length of an arbitrary element of the iterated wreath product of cyclic groups C_{p_i}, $p_i \in \mathbb{N}$, is equal to 1. The commutator width of direct limit of wreath product of cyclic groups is found. This paper gives upper bounds of the commutator width ($cw(G)$) [1] of a wreath product of groups. A presentation in the form of wreath recursion [6] of Sylow 2-subgroups $Syl_2 A_{2^k}$ of A_{2^k} is introduced. As a corollary, we obtain a short proof of the result that the commutator width is equal to 1 for Sylow 2-subgroups of the alternating group A_{2^k}, where $k > 2$, permutation group S_{2^k} and for Sylow p-subgroups $Syl_2 A_{p^k}$ and $Syl_2 S_{p^k}$. The commutator width of permutational wreath product $B \wr C_n$ is investigated. An upper bound of the commutator width of permutational wreath product $B \wr C_n$ for an arbitrary group B is found.

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1 Introduction

This work continues previous investigations of author [12, 13, 15, 16, 18], where minimal generating sets of Sylow 2-subgroups of alternating groups were found.

As is well known the first example of a group G with commutator width greater than 1 ($cw(G) > 1$) was given by Fite [4]. The smallest finite examples of such groups are groups of order 96, two such, nonisomorphic groups were given by Guralnick [20].

We deduce an estimation for commutator width of wreath product $C_n \wr B$, where C_n is a cyclic group of order n, taking into consideration the $cw(B)$ of passive group B. The form of commutators of wreath product $A \wr B$ was shortly considered in [2]. The form of commutator presentation [2] is proposed by us as wreath recursion [9] and the commutator width of it is studied. We impose weaker condition on the presentation of wreath product commutator than it was proposed by J. Meldrum.

In this paper we continue the investigations started in [16,17]. We find a minimal generating set and the structure for commutator subgroup of $Syl_2 A_{2^k}$.

The study of commutator subgroup serves to the solution of inclusion problem [5] for elements of $Syl_2 A_{2^k}$ in its derived subgroup $(Syl_2 A_{2^k})'$. It was known that the commutator width of iterated wreath products of nonabelian finite simple groups is bounded by an absolute constant [3,4]. But it was not proven that a commutator subgroup of $\mathop{\wr}\limits_{i=1}^{k} \mathcal{C}_{p_i}$ consists of commutators. We generalize the passive group of this wreath product to any group B instead of only wreath product of cyclic groups and obtain an exact commutator width.

Also we are going to prove that the commutator width of Sylow p-subgroups of symmetric and alternating groups for $p \geq 2$ is 1.

2 Preliminaries

Let G be a group acting (from the right) by permutations on a set X and let H be an arbitrary group. Then the (permutational) wreath product $H \wr G$ is the semidirect product $H^X \leftthreetimes G$, where G acts on the direct power H^X by the respective permutations of the direct factors. The cyclic group C_p or (C_p, X) is equipped with a natural action by the left shift on $X = \{1, \dots, p\}$, $p \in \mathbb{N}$. It is well known that a wreath product of permutation groups is an associative construction [2].

The multiplication rule of automorphisms g, h which are presented in the form of the wreath recursion [6] $g = (g_{(1)}, g_{(2)}, \dots, g_{(d)})\sigma_g,\ h = (h_{(1)}, h_{(2)}, \dots, h_{(d)})\sigma_h$, is given by the formula:

$$g \cdot h = (g_{(1)}h_{(\sigma_g(1))}, g_{(2)}h_{(\sigma_g(2))}, \dots, g_{(d)}h_{(\sigma_g(d))})\sigma_g\sigma_h.$$

We define σ as $(1, 2, \dots, p)$ where p is determined by context.

The set X^* is naturally a vertex set of a regular rooted tree, i.e. a connected graph without cycles and a designated vertex v_0 is called the root, in which two words are connected by an edge if and only if they are of the form v and vx, where $v \in X^*$, $x \in X$. The set $X^n \subset X^*$ is called the n-th level of the tree X^* and $X^0 = \{v_0\}$. We denote by v_{ji} the vertex of X^j which has the number i, where $1 \leq i \leq X^{2^j}$ and the numeration starts from 1. Note that a unique vertex $v_{k,i}$ corresponds to a unique word v in alphabet X. For every automorphism $g \in AutX^*$ and every word $v \in X^*$ determine the section (state) $g_{(v)} \in AutX^*$ of g at v by the rule: $g_{(v)}(x) = y$ for $x, y \in X^*$ if and only if $g(vx) = g(v)y$. The subtree of X^* induced by the set of vertices $\cup_{i=0}^{k} X^i$ is denoted by $X^{[k]}$. The restriction of the action of an automorphism $g \in AutX^*$ on the subtree $X^{[l]}$ is denoted by $g_{(v)}|_{X^{[l]}}$ as in [7,8]. The restriction $g_{(v_{ij})}|_{X^{[1]}}$ is called the vertex permutation (v.p.) of g at a vertex v_{ij} and is denoted by g_{ij}. For example, if $|X| = 2$ then we just have to distinguish active vertices, i.e., the vertices for which g_{ij} is non-trivial [6].

Let us label every vertex of X^l, $0 \leq l < k$, by sign 0 or 1 in dependence on the state of v.p. in it, i.e. if g_{ij} is non-trivial then it is labeled by 1 for the case $|X| = 2$. The vertex-labeled regular tree obtained by such a way is an element of $AutX^{[k]}$. All undeclared terms are from [7,8].

Let us fix some notations. For brevity, in the form of wreath recursion we write the commutator as $[a, b] = aba^{-1}b^{-1}$ that is inverse to $a^{-1}b^{-1}ab$. That does not reduce the generality of our reasoning. For convenience the commutator of two group elements a and b is denoted by $[a,b] = aba^{-1}b^{-1}$, conjugation by an element b as

$$a^b = bab^{-1},$$

We define G_k and B_k recursively, i.e.

$$B_1 = C_2,\ B_k = B_{k-1} \wr C_2 \text{ for } k > 1,$$
$$G_1 = \langle e \rangle,\ G_k = \{(g_1, g_2)\pi \in B_k \mid g_1 g_2 \in G_{k-1}\} \text{ for } k > 1.$$

Note that $B_k = \underset{i=1}{\overset{k}{\wr}} C_2$.

The commutator length of an element g of the derived subgroup of a group G, which is denoted *clG(g)*, is the minimal n such that there exist elements $x_1, \dots, x_n, y_1, \dots, y_n$ in G such that $g = [x_1, y_1] \dots [x_n, y_n]$. The commutator length of the identity element is 0. The commutator width of a group G, denoted $cw(G)$, is the maximum of the commutator lengths of the elements of its derived subgroup $[G,G]$. We denote by $d(G)$ the minimal number of generators of the group G.

3 Commutator width of Sylow 2-subgroups of A_{2^k} and S_{2^k}

The following lemma improves Corollary 4.9 of [2] and it will be deduced from Corollary 4.9.

Lemma 1. *An element of the form $(r_1, \dots, r_{p-1}, r_p)$ belongs to $W' = (B \wr C_p)'$ if and only if the product of all r_i (in any order) belongs to B', where $p \in \mathrm{N}$, $p \geq 2$.*

Proof. More details of our argument may be given as follows. If we multiply elements from a tuple $(r_1, \dots, r_{p-1}, r_p) = w$, where $r_i = h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)}$, h_i, $g_i \in B$ and $a, b \in C_p$, then we get the product

$$x = \prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} \in B', \tag{1}$$

so x is the product of appropriate commutators. Therefore, we can represent $r_p = r^{-1}_{p-1} \dots r^{-1}_1 x$. We can rewrite element $x \in B'$ as the product $x = \prod_{j=1}^{m} [h_j, g_j]$, $m \leq cw(B)$.

Note that we impose weaker condition on the product of all r_i to belong to B' than in Definition 4.5. of form $P(L)$ in [2], where the product of all r_i belongs to a subgroup L of B such that $L > B'$.

In more detail deducing of our representation constructing can be reported in the following way. If we multiply elements which have the form of a tuple $(r_1, \dots, r_{p-1}, r_p)$, where $r_i = h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)}$, h_i, $g_i \in B$ and $a, b \in C_p$, then we obtain a product

$$\prod_{i=1}^{p} r_i = \prod_{i=1}^{p} h_i g_{a(i)} h^{-1}_{ab(i)} g^{-1}_{aba^{-1}(i)} \in B'. \tag{2}$$

Note that if we rearrange the elements in (1) as $h_1h_1^{-1}g_1g_2^{-1}h_2h_2^{-1}g_1g_2^{-1}...h_ph_p^{-1}g_pg_p^{-1}$ then by the reason of such permutations we obtain a product of appropriate commutators. Therefore, the following equality holds

$$\prod_{i=1}^{p} h_i g_{a(i)} h_{ab(i)}^{-1} g_{aba^{-1}(i)}^{-1} = \prod_{i=1}^{p} h_i g_i h_i^{-1} g_i^{-1} x_0 = \prod_{i=1}^{p} h_i h_i^{-1} g_i g_i^{-1} x \in B', \tag{3}$$

where x_0, x are products of appropriate commutators. Therefore,

$$(r_1, \dots, r_{p-1}, r_p) \in W' \text{ iff } r_{p-1} \cdot \ldots \cdot r_1 \cdot r_p = x \in B'. \tag{4}$$

It follows that one element of the wreath recursion $(r_1, \dots, r_{p-1}, r_p)$ depends on the rest of r_i. This implies that the product $\prod\limits_{j=1}^{p} r_j$ for an arbitrary sequence $\{r_j\}_{j=1}^{p}$ belongs to B'. Thus, r_p can be expressed as:

$$r_p = r_1^{-1} \cdot \ldots \cdot r_{p-1}^{-1} x.$$

Denote a j-th tuple, consisting of wreath recursion elements by $(r_{j_1}, r_{j_2}, ..., r_{j_p})$. The fact that the set of forms $(r_1, \dots, r_{p-1}, r_p) \in W = (B \wr C_p)'$ is closed under multiplication follows from the identity

$$\prod_{j=1}^{k} (r_{j1} \dots r_{jp-1} r_{jp}) = \prod_{j=1}^{k} \prod_{i=1}^{p} r_{j_i} = R_1 R_2 ... R_k \in B', \tag{5}$$

where r_{ji} is i-th element of the j-th tuple, $R_j = \prod\limits_{i=1}^{p} r_{ji}$, $1 \le j \le k$. As it was shown above $R_j = \prod\limits_{i=1}^{p-1} r_{ji} \in B'$. Therefore, the product (5) of R_j, $j \in \{1, ..., k\}$, which is similar to the product mentioned in [2], has the property $R_1 R_2 ... R_k \in B'$, because B' is a subgroup. Thus, we get a product of the form (1) and the same argument as above are applies.

Let us prove the sufficiency condition. Let K be the set of elements satisfying the condition of this theorem. That is, all products of all r_i, where each i occurs once, belong to B'. Then using elements of the form

$$(r_1, e, ..., e, r_1^{-1}), ..., (e, e, ..., e, r_i, e, r_i^{-1}), ..., (e, e, ..., e, r_{p-1}, r_{p-1}^{-1}), (e, e, ..., e, r_1 r_2 \cdot \ldots \cdot r_{p-1})$$

we can express any element in the form $(r_1, \dots, r_{p-1}, r_p) \in W = (B \wr C_p)'$. We need to prove that we can express all element from W and only elements of W in such a way. The fact that all elements can be generated by elements of K follows from randomness of choice of every r_i, $i < p$ and the fact that equality (1) holds, so r_p is well defined. $\square$

Lemma 2. *Assume a group B and an integer $p \geq 2$ are given. If $w \in (B \wr C_p)'$, then we can represent w as the following wreath recursion:*

$$w = (r_1, r_2, \dots, r_{p-1}, r_1^{-1} \dots r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j]),$$

where $r_1, \dots, r_{p-1}, f_j, g_j \in B$ and $k \leq cw(B)$.

Proof. According to Lemma 1 we have the following wreath recursion

$$w = (r_1, r_2, \dots, r_{p-1}, r_p),$$

where $r_i \in B$ and $r_{p-1}r_{p-2} \dots r_2 r_1 r_p = x \in B'$. Therefore we can write $r_p = r_1^{-1} \dots r_{p-1}^{-1} x$. We can also rewrite an element $x \in B'$ as the product of commutators $x = \prod_{j=1}^{k} [f_j, g_j]$ where $k \leq cw(B)$. $\square$

Lemma 3. *For any group B and integer $p \geq 2$, suppose $w \in (B \wr C_p)'$ is defined by the following wreath recursion:*

$$w = (r_1, r_2, \dots, r_{p-1}, r_1^{-1} \dots r_{p-1}^{-1}[f, g]),$$

where $r_1, \dots, r_{p-1}, f, g \in B$. Then we can represent w as the following commutator

$$w = [(a_{1,1}, \dots, a_{1,p})\sigma, (a_{2,1}, \dots, a_{2,p})],$$

where

$$\begin{aligned}
&a_{1,i} = e, \ \textit{for } 1 \leq i \leq p-1 \ , \\
&a_{2,1} = (f^{-1})^{r_1^{-1} \dots r_{p-1}^{-1}}, \\
&a_{2,i} = r_{i-1} a_{2,i-1}, \ \textit{for } 2 \leq i \leq p, \\
&a_{1,p} = g^{a_{2,p}^{-1}}.
\end{aligned}$$

Proof. Consider the following commutator

$$\begin{aligned}
\kappa &= (a_{1,1}, \dots, a_{1,p})\sigma \cdot (a_{2,1}, \dots, a_{2,p}) \cdot (a_{1,p}^{-1}, a_{1,1}^{-1}, \dots, a_{1,p-1}^{-1})\sigma^{-1} \cdot (a_{2,1}^{-1}, \dots, a_{2,p}^{-1}) \\
&= (a_{3,1}, \dots, a_{3,p}),
\end{aligned}$$

where

$$a_{3,i} = a_{1,i} a_{2,1+(i \bmod p)} a_{1,i}^{-1} a_{2,i}^{-1}.$$

At first we compute

$$a_{3,i} = a_{1,i} a_{2,i+1} a_{1,i}^{-1} a_{2,i}^{-1} = a_{2,i+1} a_{2,i}^{-1} = r_i a_{2,i} a_{2,i}^{-1} = r_i, \ \text{for } 1 \leq i \leq p-1.$$

Then we make some transformation of $a_{3,p}$:

$$\begin{aligned} a_{3,p} &= a_{1,p}a_{2,1}a_{1,p}^{-1}a_{2,p}^{-1} \\ &= (a_{2,1}a_{2,1}^{-1})a_{1,p}a_{2,1}a_{1,p}^{-1}a_{2,p}^{-1} \\ &= a_{2,1}[a_{2,1}^{-1}, a_{1,p}]a_{2,p}^{-1} \\ &= a_{2,1}a_{2,p}^{-1}a_{2,p}[a_{2,1}^{-1}, a_{1,p}]a_{2,p}^{-1} \\ &= (a_{2,p}a_{2,1}^{-1})^{-1}[(a_{2,1}^{-1})^{a_{2,p}}, a_{1,p}^{a_{2,p}}] \\ &= (a_{2,p}a_{2,1}^{-1})^{-1}[(a_{2,1}^{-1})^{a_{2,p}a_{2,1}^{-1}}, a_{1,p}^{a_{2,p}}]. \end{aligned}$$

Now we can see that the form of the commutator κ is similar to the form of w.

Introduce the following notation

$$r' = r_{p-1} \dots r_1.$$

We note that from the definition of $a_{2,i}$, for $2 \le i \le p$ it follows that

$$r_i = a_{2,i+1}a_{2,i}^{-1}, \text{ for } 1 \le i \le p-1.$$

Therefore

$$\begin{aligned} r' &= (a_{2,p}a_{2,p-1}^{-1})(a_{2,p-1}a_{2,p-2}^{-1}) \dots (a_{2,3}a_{2,2}^{-1})(a_{2,2}a_{2,1}^{-1}) \\ &= a_{2,p}a_{2,1}^{-1}. \end{aligned}$$

Then

$$(a_{2,p}a_{2,1}^{-1})^{-1} = (r')^{-1} = r_1^{-1} \dots r_{p-1}^{-1}.$$

Now we compute the following

$$\begin{aligned} (a_{2,1}^{-1})^{a_{2,p}a_{2,1}^{-1}} &= (((f^{-1})^{r_1^{-1} \dots r_{p-1}^{-1}})^{-1})^{r'} = (f^{(r')^{-1}})^{r'} = f, \\ a_{1,p}^{a_{2,p}} &= (g^{a_{2,p}^{-1}})^{a_{2,p}} = g. \end{aligned}$$

Finally we conclude that

$$a_{3,p} = r_1^{-1} \dots r_{p-1}^{-1}[f, g].$$

Thus, the commutator κ is presented exactly in a similar form as w has. □

For future using we formulate the previous lemma for the case $p = 2$.

Corollary 1. *For any group B, if $w \in (B \wr C_2)'$ is defined by the following wreath recursion*

$$w = (r_1, r_1^{-1}[f, g]),$$

where $r_1, f, g \in B$ then we can represent w as the commutator

$$w = [(e, a_{1,2})\sigma, (a_{2,1}, a_{2,2})],$$

where

$$a_{2,1} = (f^{-1})^{r_1^{-1}},$$
$$a_{2,2} = r_1 a_{2,1},$$
$$a_{1,2} = g^{a_{2,2}^{-1}}.$$

Lemma 4. *For any group B and integer $p \geq 2$ the inequality*

$$cw(B \wr C_p) \leq \max(1, cw(B))$$

holds.

Proof. We can represent any $w \in (B \wr C_p)'$ by Lemma 1 with the following wreath recursion

$$\begin{aligned} w &= (r_1, r_2, \dots, r_{p-1}, r_1^{-1} \dots, r_{p-1}^{-1} \prod_{j=1}^{k} [f_j, g_j]) \\ &= (r_1, r_2, \dots, r_{p-1}, r_1^{-1} \dots, r_{p-1}^{-1} [f_1, g_1]) \cdot \prod_{j=2}^{k} [(e, \dots, e, f_j), (e, \dots, e, g_j)], \end{aligned}$$

where $r_1, \dots, r_{p-1}, f_j, g_j \in B$ and $k \leq cw(B)$. Now by Lemma 3 we see that w can be represented as a product of $\max(1, cw(B))$ commutators. □

Corollary 2. *If $W = C_{p_k} \wr \dots \wr C_{p_1}$ then $cw(W) = 1$ for $k \geq 2$.*

Proof. If $B = C_{p_k} \wr C_{p_{k-1}}$, then take into consideration that $cw(B) > 0$ (because $C_{p_k} \wr C_{p_{k-1}}$ is not a commutative group). Since Lemma 4 implies that $cw(C_{p_k} \wr C_{p_{k-1}}) = 1$, and using the inequality $cw(C_{p_k} \wr C_{p_{k-1}} \wr C_{p_{k-2}}) \leq \max(1, cw(B))$ from Lemma 4 we obtain $cw(C_{p_k} \wr C_{p_{k-1}} \wr C_{p_{k-2}}) = 1$. Similarly, if $W = C_{p_k} \wr \dots \wr C_{p_1}$ and inductive assumption for $C_{p_k} \wr \dots \wr C_{p_2}$ holds, then using the associativity of a permutational wreath product we obtain from the inequality of Lemma 4 and the equality $cw(C_{p_k} \wr \dots \wr C_{p_2}) = 1$ that $cw(W) = 1$. □

We define our partially ordered set M as the set of all finite wreath products of cyclic groups. We make use of directed set $\mathbb{N}$.

$$H_k = \mathop{\wr}_{i=1}^{k} \mathcal{C}_{p_i} \tag{6}$$

Moreover, it has already been proved in Corollary 3 that each group of the form $\mathop{\wr}_{i=1}^{k} \mathcal{C}_{p_i}$ has the commutator width equal to 1, i.e $cw(\mathop{\wr}_{i=1}^{k} \mathcal{C}_{p_i}) = 1$. A partially ordered set of subgroups is ordered by relation of inclusion of group as a subgroup. Define the injective homomorphism $f_{k,k+1}$ from $\mathop{\wr}_{i=1}^{k} \mathcal{C}_{p_i}$ into $\mathop{\wr}_{i=1}^{k+1} \mathcal{C}_{p_i}$ by mapping a generator of active group $\mathcal{C}_{p_i}$ of H_k into a generator of active group $\mathcal{C}_{p_i}$ of H_{k+1}. In more details the injective homomorphism $f_{k,k+1}$ is defined as $g \mapsto g(e,...,e)$, where a generator $g \in \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$, $g(e,...,e) \in \underset{i=1}{\overset{k+1}{\wr}} \mathcal{C}_{p_i}$.

Therefore this is an injective homomorphism of H_k into subgroup $\underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ of H_{k+1}.

The direct limit of the resulting direct system is denoted by $\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ and is defined as the disjoint of the H_k's modulo a certain equivalence relation:

$$\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} = {}^{\coprod_k \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}} /\sim .$$

Corollary 3. *The direct limit $\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ of direct system $\left\langle f_{k,j}, \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} \right\rangle$ has the commutator width 1.*

Proof. We make transition to the direct limit in the direct system $\left\langle f_{k,j}, \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} \right\rangle$ of injective mappings from the chain $e \to \; ... \; \to \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} \to \underset{i=1}{\overset{k+1}{\wr}} \mathcal{C}_{p_i} \to \underset{i=1}{\overset{k+2}{\wr}} \mathcal{C}_{p_i} \to ....$

Since all mappings in chains are injective homomorphisms, it has a trivial kernel. Therefore the transition to a direct limit boundary preserves the property $cw(H) = 1$, because each group H_k from the chain is endowed with $cw(H_k) = 1$.

The direct limit of the direct system is denoted by $\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ and is defined as disjoint union of the H_k's modulo a certain equivalence relation:

$$\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i} = {}^{\coprod_k \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}} /\sim .$$

Since every element g of $\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$ coincides with a corresponding element from some H_k of direct system, then by the injectivity of the mappings for g the property $cw(\underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}) = 1$ also holds. Thus, it holds for the whole $\varinjlim \underset{i=1}{\overset{k}{\wr}} \mathcal{C}_{p_i}$. □

Corollary 4. *For prime p and $k \geq 2$, the commutator width $cw(Syl_p(S_{p^k})) = 1$ and for prime $p > 2$ and $k \geq 2$, the commutator width $cw(Syl_p(A_{p^k})) = 1$.*

Proof. Since $Syl_p(S_{p^k}) \simeq \underset{i=1}{\overset{k}{\wr}} C_p$ (see [10, 11, 13]), then $cw(Syl_p(S_{p^k})) = 1$. As well known in the case $p > 2$ we have $Syl_p S_{p^k} \simeq Syl_p A_{p^k}$ (see [16, 19]), then $cw(Syl_p(A_{p^k})) = 1$. □

Proposition 1. *The inclusion $B_k' < G_k$ holds.*

Proof. Induction on k. For $k = 1$ we have $B'_k = G_k = \{e\}$. Let us fix some $g = (g_1, g_2) \in B'_k$. Then $g_1 g_2 \in B'_{k-1}$ by Lemma 1. As $B'_{k-1} < G_{k-1}$ by induction hypothesis therefore $g_1 g_2 \in G_{k-1}$ and by the definition of G_k it follows that $g \in G_k$. □

Corollary 5. *The set G_k is a subgroup in the group B_k.*

Proof. According to recursive definition of G_k and B_k, where $G_k = \{(g_1, g_2)\pi \in B_k \mid g_1 g_2 \in G_{k-1}\}$ $k > 1$, G_k is a subset of B_k with condition $g_1 g_2 \in G_{k-1}$. It is easy to check the closedness by multiplication of elements of G_k with condition $g_1 g_2, h_1 h_2 \in G_{k-1}$, because G_{k-1} is a subgroup, so $g_1 g_2 h_1 h_2 \in G_{k-1}$, too. The inverses can be verified easily. □

Lemma 5. *For any $k \geq 1$ we have $|G_k| = |B_k|/2$.*

Proof. Induction on k. For $k = 1$ we have $|G_1| = 1 = |B_1|/2$. Every element $g \in G_k$ can be uniquely written as the following wreath recursion

$$g = (g_1, g_2)\pi = (g_1, g_1^{-1} x)\pi$$

where $g_1 \in B_{k-1}$, $x \in G_{k-1}$ and $\pi \in C_2$. Elements g_1, x and π are independent, therefore $|G_k| = 2|B_{k-1}| \cdot |G_{k-1}| = 2|B_{k-1}| \cdot |B_{k-1}|/2 = |B_k|/2$. □

Corollary 6. *The group G_k is a normal subgroup in the group B_k, i.e. $G_k \lhd B_k$.*

Proof. The proof follows immediately from Lemma 5. □

Theorem 1. *For any $k \geq 1$ we have $G_k \simeq Syl_2 A_{2^k}$.*

Proof. Group C_2 acts on the set $X = \{1, 2\}$. Therefore we can recursively define sets X^k on which group B_k acts: $X^1 = X$, $X^k = X^{k-1} \times X$ for k¿1. At first we define $S_{2^k} = Sym(X^k)$ and $A_{2^k} = Alt(X^k)$ for all integers $k \geq 1$. Then $G_k < B_k < S_{2^k}$ and $A_{2^k} < S_{2^k}$.

It follows from [16] that $B_k \simeq Syl_2(S_{2^k})$. Since $|A_{2^k}| = |S_{2^k}|/2$, therefore $|Syl_2 A_{2^k}| = |Syl_2 S_{2^k}|/2 = |B_k|/2$. By Lemma 5 it follows that $|Syl_2 A_{2^k}| = |G_k|$. Therefore it remains to show that $G_k < Alt(X^k)$.

Let us fix some $g = (g_1, g_2)\sigma^i$ where $g_1, g_2 \in B_{k-1}$, $i \in \{0, 1\}$ and $g_1 g_2 \in G_{k-1}$. Then we can represent g as follows

$$g = (g_1 g_2, e) \cdot (g_2^{-1}, g_2) \cdot (e, e,)\sigma^i.$$

In order to prove this theorem it is enough to show that $(g_1 g_2, e), (g_2^{-1}, g_2), (e, e,)\sigma \in Alt(X^k)$.

Element $(e, e,)\sigma$ just switches letters x_1 and x_2 for all $x \in X^k$. Therefore $(e, e,)\sigma$ is a product of $|X^{k-1}| = 2^{k-1}$ transpositions and therefore $(e, e,)\sigma \in Alt(X^k)$.

Elements g_2^{-1} and g_2 have the same cycle type. Therefore elements (g_2^{-1}, e) and (e, g_2) also have the same cycle type. Let us fix the following cycle decompositions

$$(g_2^{-1}, e) = \sigma_1 \cdot \ldots \cdot \sigma_n,$$

$$(e, g_2) = \pi_1 \cdot \ldots \cdot \pi_n.$$

Note that element (g_2^{-1}, e) acts only on letters like x_1 and element (e, g_2) acts only on letters like x_2. Therefore we have the following cycle decomposition

$$(g_2^{-1}, g_2) = \sigma_1 \cdot \ldots \cdot \sigma_n \cdot \pi_1 \cdot \ldots \cdot \pi_n.$$

So, element (g_2^{-1}, g_2) has even number of odd permutations and then $(g_2^{-1}, g_2) \in Alt(X^k)$.

Note that $g_1 g_2 \in G_{k-1}$ and $G_{k-1} = Alt(X^{k-1})$ by induction hypothesis. Therefore $g_1 g_2 \in Alt(X^{k-1})$. As elements $g_1 g_2$ and $(g_1 g_2, e)$ have the same cycle type then $(g_1 g_2, e) \in Alt(X^k)$. □

As it was proven by the author in [16] Sylow 2-subgroup of A_{2^k} has the structure $B_{k-1} \ltimes W_{k-1}$, where definition of B_{k-1} is the same that was given in [16].

Recall that we denoted by W_{k-1} the subgroup of $AutX^{[k]}$ such that it has active states only on X^{k-1} and the number of such states is even, i.e. $W_{k-1} \lhd St_{G_k}(k-1)$ [6]. It was proven that the size of W_{k-1} is equal to $2^{2^{k-1}-1}$, $k > 1$, and its structure is $(C_2)^{2^{k-1}-1}$. The following structural theorem characterizing the group G_k was proved by us [16].

Theorem 2. *A maximal 2-subgroup of $AutX^{[k]}$ that acts by even permutations on X^k has the structure of the semidirect product $G_k \simeq B_{k-1} \ltimes W_{k-1}$ and is isomorphic to $Syl_2 A_{2^k}$.*

Note that W_{k-1} is a subgroup of the stabilizer of X^{k-1}, i.e. $W_{k-1} < St_{AutX^{[k]}}(k-1) \lhd AutX^{[k]}$ and $W_{k-1} \lhd AutX^{[k]}$ is normal too, because conjugation keeps the cyclic structure of permutation so an even permutation maps into even one. Therefore such conjugation induces an automorphism of W_{k-1} and $G_k \simeq B_{k-1} \ltimes W_{k-1}$.

Remark 1. As a consequence, the structure found by the author in [16] is fully consistent with the recursive group representation (which is used in this paper) based on the concept of wreath recursion [9].

Theorem 3. *Elements of B'_k have the following form $B'_k = \{[f, l] \mid f \in B_k, l \in G_k\} = \{[l, f] \mid f \in B_k, l \in G_k\}$.*

Proof. It is enough to show either $B'_k = \{[f, l] \mid f \in B_k, l \in G_k\}$ or $B'_k = \{[l, f] \mid f \in B_k, l \in G_k\}$ because if $f = [g, h]$ then $f^{-1} = [h, g]$.

We prove the proposition by induction on k. For the case $k = 1$ we have $B'_1 = \langle e \rangle$.

Consider the case $k > 1$. According to Lemma 2 and Corollary 1 every element $w \in B'_k$ can be represented as

$$w = (r_1, r_1^{-1}[f, g])$$

for some $r_1, f \in B_{k-1}$ and $g \in G_{k-1}$ (by induction hypothesis). By Corollary 1 we can represent w as commutator of

$$(e, a_{1,2})\sigma \in B_k \text{ and } (a_{2,1}, a_{2,2}) \in B_k,$$

where

$$a_{2,1} = (f^{-1})^{r_1^{-1}},$$
$$a_{2,2} = r_1 a_{2,1},$$
$$a_{1,2} = g^{a_{2,2}^{-1}}.$$

If $g \in G_{k-1}$ then by the definition of G_k and Corollary 6 we obtain $(e, a_{1,2})\sigma \in G_k$. □

Remark 2. Theorem 3 improves Corollary 4 for the case $Syl_2 S_{2^k}$.

Proposition 2. *If g is an element of the group B_k then $g^2 \in B'_k$.*

Proof. Induction on k. We note that $B_k = B_{k-1} \wr C_2$. Therefore we fix some element

$$g = (g_1, g_2)\sigma^i \in B_{k-1} \wr C_2,$$

where $g_1, g_2 \in B_{k-1}$ and $i \in \{0, 1\}$. Let us consider g^2, then two cases are possible:

$$g^2 = (g_1^2, g_2^2) \text{ or } g^2 = (g_1 g_2, g_2 g_1).$$

In the second case we consider the product of coordinates $g_1 g_2 \cdot g_2 g_1 = g_1^2 g_2^2 x$. Since according to the induction hypothesis $g_i^2 \in B'_k$, $i \leq 2$, then $g_1 g_2 \cdot g_2 g_1 \in B'_k$, also according to Lemma 1, $x \in B'_k$. Therefore the following inclusion holds $(g_1 g_2, g_2 g_1) = g^2 \in B'_k$. In the first case the proof is even simpler because $g_1^2, g_2^2 \in B'$ by the induction hypothesis. □

Lemma 6. *If an element $g = (g_1, g_2) \in G'_k$ then $g_1, g_2 \in G_{k-1}$ and $g_1 g_2 \in B'_{k-1}$.*

Proof. As $B'_k < G_k$ therefore it is enough to show that $g_1 \in G_{k-1}$ and $g_1 g_2 \in B'_{k-1}$. Let us fix some $g = (g_1, g_2) \in G'_k < B'_k$. Then Lemma 1 implies that $g_1 g_2 \in B'_{k-1}$.

In order to show that $g_1 \in G_{k-1}$ we firstly consider just one commutator of arbitrary elements from G_k

$$f = (f_1, f_2)\sigma, \; h = (h_1, h_2)\pi \in G_k,$$

where $f_1, f_2, h_1, h_2 \in B_{k-1}$, $\sigma, \pi \in C_2$. The definition of G_k implies that $f_1 f_2, h_1 h_2 \in G_{k-1}$.

If $g = (g_1, g_2) = [f, h]$ then

$$g_1 = f_1 h_i f_j^{-1} h_k^{-1}$$

for some $i, j, k \in \{1, 2\}$. Then

$$g_1 = f_1 h_i f_j (f_j^{-1})^2 h_k (h_k^{-1})^2 = (f_1 f_j)(h_i h_k) x (f_j^{-1} h_k^{-1})^2,$$

where x is the product of commutators of f_i, h_j and f_i, h_k, hence $x \in B'_{k-1}$.

It is enough to consider the first product $f_1 f_j$. If $j = 1$ then $f_1^2 \in B'_{k-1}$ by Proposition 2, if $j = 2$ then $f_1 f_2 \in G_{k-1}$ according to the definition of G_k, the same is true for $h_i h_k$. Thus, for any i, j, k, $f_1 f_j, h_i h_k \in G_{k-1}$ holds. Moreover, the square $(f_j^{-1} h_k^{-1})^2 \in B'_k$ according to Proposition 2. Therefore $g_1 \in G_{k-1}$ because of Proposition 2 and Proposition 1, the same is true for g_2.

Now it remains to consider the product of some $f = (f_1, f_2), h = (h_1, h_2)$, where $f_1, h_1 \in G_{k-1}$, $f_1 h_1 \in G_{k-1}$ and $f_1 f_2, h_1 h_2 \in B'_{k-1}$,

$$fh = (f_1 h_1, f_2 h_2).$$

Since $f_1 f_2, h_1 h_2 \in B'_{k-1}$ by the imposed condition in this item and taking into account that $f_1 h_1 f_2 h_2 = f_1 f_2 h_1 h_2 x$ for some $x \in B'_{k-1}$, then $f_1 h_1 f_2 h_2 \in B'_{k-1}$ by Lemma 1. In other words the closedness by multiplication holds and so according to Lemma 1, we have an element of commutator G'_k. ☐

In the following theorem we prove two facts at once.

Theorem 4. *The following statements are true.*

1. *An element $g = (g_1, g_2) \in G'_k$ iff $g_1, g_2 \in G_{k-1}$ and $g_1 g_2 \in B'_{k-1}$.*
2. *The commutator subgroup G'_k coincides with the set of all commutators for $k \geq 1$*

$$G'_k = \{[f_1, f_2] \mid f_1 \in G_k, f_2 \in G_k\}.$$

Proof. For the case $k = 1$ we have $G'_1 = \langle e \rangle$. If $k = 2$, then we have $G'_2 = V_4 = \langle e \rangle$, where V_4 is the Klein four-group. As a result $cw(G_2) = 0$. So, further we consider the case $k > 2$.

Sufficiency of the first statement of this theorem follows from Lemma 6. So, in order to prove necessity of the both statements it is enough to show that element

$$w = (r_1, r_1^{-1} x),$$

where $r_1 \in G_{k-1}$ and $x \in B'_{k-1}$, can be represented as a commutator of elements from G_k. By Theorem 3 we have $x = [f, g]$ for some $f \in B_{k-1}$ and $g \in G_{k-1}$. Therefore

$$w = (r_1, r_1^{-1}[f, g]).$$

By Corollary 1 we can represent w as the commutator of

$$(e, a_{1,2})\sigma \in B_k \text{ and } (a_{2,1}, a_{2,2}) \in B_k,$$

where $a_{2,1} = (f^{-1})^{r_1^{-1}}, a_{2,2} = r_1 a_{2,1}, a_{1,2} = g^{a_{2,2}^{-1}}$. It only remains to show that $(e, a_{1,2})\sigma$,
$(a_{2,1}, a_{2,2}) \in G_k$. Note that

$$a_{1,2} = g^{a_{2,2}^{-1}} \in G_{k-1} \text{ by Corollary 6.}$$

$$a_{2,1} a_{2,2} = a_{2,1} r_1 a_{2,1} = r_1 [r_1, a_{2,1}] a_{2,1}^2 \in G_{k-1} \text{ by Proposition 1 and Proposition 2.}$$

So we have $(e, a_{1,2})\sigma \in G_k$ and $(a_{2,1}, a_{2,2}) \in G_k$ by the definition of G_k. ☐

Example. The commutator subgroup of $Syl'_2(A_8)$ consists of elements: $\{e, (13)(24)\times$

$\times(57)(68), (12)(34), (14)(23)(57)(68), (56)(78), (13)(24)(58)(67), (12)(34)(56)(78), (14)\times$

$\times\ (23)(58)(67)\}$. The commutator $Syl'_2(A_8) \simeq C_2^3$ that is an elementary abelian 2-group of order 8. This fact confirms our formula $d(G_k) = 2k-3$, because $k = 3$ and $d(G_k) = 2k-3 = 3$. A minimal generating set of $Syl'_2(A_8)$ consists of 3 generators: $(1,3)(2,4)(5,7)(6,8), (1,2)(3,4), (1,3)(2,4)(5,8)(6,7)$.

Corollary 7. *The commutator width of the group $Syl_2 A_{2^k}$ is equal to* 1 *for $k > 2$.*

Notice that for the case $k = 1$ we have $(Syl_2 A_2)' = \langle e \rangle$. If $k = 2$, then we have $(Syl_2 A_{2^2})' = V_4 = \langle e \rangle$, where V_4 is the Klein four-group. As a result $cw(Syl_2 A_4) = 0$. For rest of the cases the proof immediately follows from item 2 of Theorem 4.

4 Conclusion

A new approach to the presentation of Sylow 2-subgroups of alternating group A_{2^k} is applied. As a result a short proof of the fact that the commutator width of Sylow 2-subgroups of alternating group A_{2^k}, where $k > 2$, permutation group S_{2^k} and Sylow p-subgroups $Syl_2 A_{p^k}$ $(Syl_2 S_{p^k})$ are equal to 1 is obtained. The structure of commutator subgroup of Sylow 2-subgroups of alternating group $Syl_2 A_{2^k}$ is found. The commutator width of permutational wreath product $B \wr C_n$ is investigated.

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