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On two stability types for a multicriteria integer linear programming problem

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Abstract. We consider a multicriteria integer linear programming problem with a parametrized optimality principle which is implemented by means of partitioning the partial criteria set into non-empty subsets, inside which relations on the set of solutions are based on the Pareto minimum. The introduction of this principle allows us to connect such classical selection functions as Pareto and aggregative-extremal. A quantitative analysis of two types of stability of the problem to perturbations of the parameters of objective functions is given under the assumption that an arbitrary l_p-Hölder norm, $1 \leq p \leq \infty$, is given in the solution space, and the Chebyshev norm is given in the criteria space. The formulas for the radii of quasistability and strong quasi-stability are obtained. Criteria of these types of stability are given as corollaries.

Mathematics subject classification: 90C09, 90C29, 90C31.
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1 Problem formulation and basic definitions

Consider a multicriteria integer linear programming problem (ILP) in the following formulation. Let $C = [c_{ij}] \in \mathbf{R}^{m\times n}$ be a matrix whose rows are denoted by $C_i = (c_{i1}, c_{i2}, ..., c_{in}) \in \mathbf{R}^n$, $i \in N_m = \{1, 2, ..., m\}$, $m \geq 1$. Let $x = (x_1, x_2, ..., x_n)^T \in X \subset \mathbf{Z}^n$, $n \geq 2$, and the number of elements of the set X is finite and greater than one. On the set of (admissible) solutions X, we define a vector linear criterion

$$Cx = (C_1x, C_2x, ..., C_mx)^T \to \min_{x\in X}. \tag{1}$$

In the space $\mathbf{R}^k$ of arbitrary dimension $k \in \mathbf{N}$ we introduce a binary relation that generates the Pareto optimality principle [1]

$$y \succ y' \;\Leftrightarrow\; y \geq y' \;\&\; y \neq y',$$

where $y = (y_1, y_2, ..., y_k)^T \in \mathbf{R}^k$, $y' = (y'_1, y'_2, ..., y'_k)^T \in \mathbf{R}^k$.

The symbol $\overline{\succ}$ will, as usual, denote the negation of the relation $\succ$.

Let $\emptyset \neq I \subseteq N_m$. By C_I we denote the submatrix of the matrix $C \in \mathbf{R}^{m\times n}$, consisting of rows of this matrix with the numbers of the subset I, i.e.

$$C_I = (C_{i_1}, C_{i_2}, ..., C_{i_h})^T,\; I = \{i_1, i_2, ..., i_h\},\; 1 \leq i_1 < i_2 < ... < i_h \leq m,\; C_I \in \mathbf{R}^{h\times n}.$$

Let $s \in N_m$ and $N_m = \bigcup_{k \in N_s} I_k$ be a partition of the set N_m into s nonempty sets, i.e. $I_k \neq \emptyset$, $k \in N_s$, and $i \neq j \Rightarrow I_i \cap I_j = \emptyset$. For this partition, we introduce a set of generalized-effective, or else $(I_1, I_2, ..., I_s)$-effective solutions according to the formula:

$$G^m(C, I_1, I_2, ..., I_s) = \left\{x \in X \quad : \quad \exists k \in N_s \quad \forall x' \in X \quad \left(C_{I_k}x \overline{\succ} C_{I_k}x'\right)\right\}. \qquad (2)$$

Sometimes for brevity we will denote this set by $G^m(C)$.

Obviously, any N_m-effective solution $x \in G^m(C, N_m)$ $(s = 1)$ is Pareto optimal, i.e. effective solution to problem (1). Therefore, the set $G^m(C, N_m)$ is the Pareto set [1]:

$$P^m(C) = \{x \in X \ : \ \forall x' \in X \quad (Cx \overline{\succ} Cx')\}.$$

In the other extreme case, when $s = m$, $G^m(C, \{1\}, \{2\}, ..., \{m\})$ is a set of extremal solutions [2– 4] generated by the jointly-extremal choice function. This set will be denoted by $E^m(C)$. Thereby,

$$E^m(C) = \{x \in X \ : \ \exists k \in N_m \ \forall x' \in X \ (C_k x \overline{\succ} C_k x')\} =$$
$$= \{x \in X \ : \ \exists k \in N_m \ \forall x' \in X \ (C_k x \leq C_k x')\}.$$

It is easy to see that jointly-extreme choice can be interpreted as finding the best solutions for each of the m criteria and combining them into a single one.

So, in this context, the parametrization of the optimality principle refers to the introduction of such a characteristic of the binary preference relation, which allows us to connect the well-known choice functions – Pareto and jointly-extremal.

We denote the multicriteria ILP problem consisting in finding the set $G^m(C, I_1, I_2, ..., I_s)$ by $Z^m(C, I_1, I_2, ..., I_s)$. Sometimes for the brevity we will use the notation $Z^m(C)$ for this problem.

It is easy to see that the set $P^1(C) = E^1(C)$ is the set of optimal solutions to the scalar (single-criterion) problem $Z^1(C, N_1)$, where $C \in \mathbf{R}^n$.

For any nonempty subset $I \subseteq N_m$ we introduce the notation

$$P^m(C, I) = \{x \in X \ : \ \forall x' \in X \ (C_I x \overline{\succ} C_I x')\}.$$

Then, by virtue of (2), we obtain

$$G^m(C, I_1, I_2, ..., I_s) = \{x \in X \ : \ \exists k \in N_s \ (x \in P^m(C, I_k))\}, \qquad (3)$$

i.e.

$$G^m(C, I_1, I_2, ..., I_s) = \bigcup_{k \in N_s} P^m(C, I_k).$$

In addition, we have

$$P^m(C, N_m) = P^m(C) = G^m(C, N_m).$$

It is obvious that all the sets given here are nonempty for any matrix $C \in \mathbf{R}^{m \times n}$.

Perturbation of the elements of the matrix C will be effected by adding matrices C' from $\mathbf{R}^{m\times n}$ to it. Thus, the perturbed problem $Z^m(C+C')$ has the form

$$(C+C')x \to \min_{x\in X},$$

and the set of its $(I_1, I_2, ..., I_s)$-effective solutions is $G^m(C+C', I_1, I_2, ..., I_s)$.

In the space of solutions $\mathbf{R}^n$ we define an arbitrary Hölder norm l_p, $p \in [1,\infty]$, i.e. by the norm of the vector $a = (a_1, a_2, ..., a_n)^T \in \mathbf{R}^n$ we mean the number

$$||a||_p = \begin{cases} \left(\sum\limits_{j\in N_n} |a_j|^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{|a_j| \ : \ j \in N_n\} & \text{if } p = \infty, \end{cases}$$

In the criterion space $\mathbf{R}^m$ we define the Chebyshev norm l_∞. By the norm of the matrix $C \in \mathbf{R}^{m\times n}$ with the rows $C_i,\ i \in N_m$, we mean the norm of a vector whose components are the norms of the rows of the matrix. By that

$$||C||_{p\infty} = ||(||C_1||_p, ||C_2||_p, ..., ||C_m||_p)||_\infty.$$

It is easy to see that for any $p \in [1,\infty]$ the inequalities hold

$$||C_i||_p \le ||C||_{p\infty},\ i \in N_m. \tag{4}$$

For an arbitrary number $\varepsilon > 0$, we define the set of perturbing matrices

$$\Omega(\varepsilon) = \{C' \in \mathbf{R}^{m\times n} \ : \ ||C'||_{p\infty} < \varepsilon\}.$$

Following [5–9], quasistability radius (in the terminology of [10, 11] – radius of T_4-stability) of the ILP problem $Z^m(C, I_1, I_2, ..., I_s)$, $m \in \mathbf{N}$, is the number

$$\rho_1 = \rho_1^{m,p}(C, I_1, I_2, ..., I_s) = \begin{cases} \sup \Xi_1 & \text{if } \Xi_1 \ne \emptyset, \\ 0 & \text{if } \Xi_1 = \emptyset, \end{cases}$$

where

$$\Xi_1 = \{\varepsilon > 0 \ : \ \forall C' \in \Omega(\varepsilon) \quad (G^m(C) \subseteq G^m(C+C'))\}.$$

Thus, the quasi-stability radius of the problem $Z^m(C)$ determines the limit level of perturbations of the elements of the matrix C, that preserve optimality of all the decisions of the set $G^m(C)$ of the original problem and the possibility of the appearance of new generalized-effective solutions is allowed.

The quasi-stability radius of the problem $Z^m(C)$ can also be determined using the well-known (see, for example, [10, 11]) concept of the stability kernel of this problem. Indeed, it is easy to see that

$$\rho_1 = \sup\{\varepsilon > 0 \ : \ Ker^m(C,\varepsilon) = G^m(C)\},$$

where

$$Ker^m(C,\varepsilon) = \{x \in G^m(C) \ : \ \forall C' \in \Omega(\varepsilon) \ \ (x \in G^m(C+C'))\}.$$

The last set is called the kernel of the ε-stability of the problem, and the set

$$Ker^m(C) = Ker^m(C, I_1, I_2, ..., I_s) =$$

$$= \{x \in G^m(C, I_1, I_2, ..., I_s) \ : \ \exists \varepsilon > 0 \ \forall C' \in \Omega(\varepsilon) \ (x \in G^m(C+C', I_1, I_2, ..., I_s))\}$$

is called the stability kernel of the problem $Z^m(C, I_1, I_2, ..., I_s)$. Thus, the kernel of the stability of a problem is the set of all generalized-effective solutions that are stable to small perturbations of the parameters of the problem.

Weakening the requirement of preserving the entire set $G^m(C)$ under "small" perturbations (see the definition of the radius ρ_1), we come to the concept of the strong quasistability radius. This type of stability is interpreted as the possibility of such perturbations in which old generalized-effective solutions can disappear, but there must be at least one generalized-effective solution of the original problem that preserves its efficiency under "small" perturbations of the problem parameters. In other words the set of generalized-effective solutions should be nonempty. As a result, we obtain the following definition.

Following [6, 7], strong quasistability radius (in the terminology of [10, 11]– radius of T_2-stability) of the ILP problem $Z^m(C, I_1, I_2, ..., I_s)$, $m \in \mathbf{N}$, is the number

$$\rho_2 = \rho_2^{m,p}(C, I_1, I_2, ..., I_s) = \begin{cases} \sup \Xi_2 & \text{if } \Xi_2 \neq \emptyset, \\ 0 & \text{if } \Xi_2 = \emptyset, \end{cases}$$

where

$$\Xi_2 = \{\varepsilon > 0 \ : \ \exists x \in G^m(C, I_1, I_2, ..., I_s) \, \forall C' \in \Omega(\varepsilon) \quad (x \in G^m(C+C', I_1, I_2, ..., I_s))\}.$$

It is easy to see that

$$\rho_2 = \sup\{\varepsilon > 0 \ : \ Ker^m(C,\varepsilon) \neq \emptyset\}.$$

2 Auxiliary statements

In the solution space $\mathbf{R}^n$ along with the norm l_p, $p \in [1, \infty]$, we will use the conjugate norm l_{p^*}, where the numbers p and p^* are connected, as usual, by the equality

$$\frac{1}{p} + \frac{1}{p^*} = 1,$$

assuming $p^* = 1$ if $p = \infty$, and $p^* = \infty$ if $p = 1$. Therefore, we further suppose that the range of variation of the numbers p and p^* is the interval $[1, \infty]$, and the numbers themselves are connected by the above conditions.

Below, we use the well-known Hölder inequality

$$|a^T b| \leq ||a||_p ||b||_{p^*}, \tag{5}$$

valid for any vectors $a = (a_1, a_2, ..., a_n)^T \in \mathbf{R}^n$ and $b = (b_1, b_2, ..., b_n)^T \in \mathbf{R}^n$.

Now we will find out conditions under which this inequality becomes equality.

First of all, we exclude the trivial case when at least one of the vectors a and b is zero, since then the equality is obvious.

For the case $1 < p < \infty$ it is well known (see, for example, [12]) that Hölder inequality (5) becomes an equality if and only if two vectors obtained from the vectors a and b by raising the absolute values of their components to the power of p and p^*, respectively, are linearly dependent (proportional) and $sign(a_i b_i)$ is independent of the index $i \in N_n$.

In the case $p = 1$ inequality (5) turns into the inequality

$$\left| \sum_{i \in N_n} a_i b_i \right| \leq \max_{i \in N_n} |b_i| \sum_{i \in N_n} |a_i|. \tag{6}$$

Let the index $k \in N_n$ be such that

$$|b_k| = ||b||_\infty > 0.$$

Then, setting $a_k \neq 0$ and $a_i = 0$, $i \in N_n \backslash \{k\}$, we make sure that inequality (6) turns into equality.

Finally, for $p = \infty$ inequality (5) turns into the inequality

$$\left| \sum_{i \in N_n} a_i b_i \right| \leq \max_{i \in N_n} |a_i| \sum_{i \in N_n} |b_i|,$$

which becomes an equality if for $\sigma > 0$ the following equalities hold

$$a_i = \sigma \, sign(b_i), \;\; i \in N_n.$$

As a result, the following lemma is proved.

Lemma 1. *For any number $p \in [1, \infty]$ the formula*

$$\forall b \in \mathbf{R}^n \quad \forall \sigma > 0 \quad \exists a \in \mathbf{R}^n \quad (|a^T b| = \sigma ||b||_{p^*} \;\&\; ||a||_p = \sigma)$$

holds.

Lemma 2. *Let $x, x^0 \in X$, $x^0 \neq x$, $\varphi \geq 0$, $\emptyset \neq I \subseteq N_m$, $C \in \mathbf{R}^{m \times n}$ and the following inequalities are valid*

$$[C_i(x - x^0)]^+ \leq \varphi ||x - x^0||_{p^*}, \;\; i \in I. \tag{7}$$

Then for any $\varepsilon > \varphi$ there exists such a perturbing matrix $C^0 \in \Omega(\varepsilon)$ that

$$x^0 \notin P^m(C + C^0, I).$$

Hereinafter, $[a]^+$ is a positive cut-off of the number $a \in \mathbf{R}$, i.e.

$$[a]^+ = \max\{0, a\}.$$

Proof. Let us choose the number σ satisfying the condition

$$\varepsilon > \sigma > \varphi. \tag{8}$$

According to Lemma 1, for any index $i \in I$ there exists a vector $B_i \in \mathbf{R}^n$ such that

$$B_i^T(x - x^0) = -\sigma||x - x^0||_{p^*}, \tag{9}$$
$$||B_i||_p = \sigma.$$

Then the perturbing matrix $C^0 \in \mathbf{R}^{m\times n}$ with the rows

$$C_i^0 = \begin{cases} B_i^T & \text{if } i \in I, \\ \mathbf{0}_n^T & \text{if } i \in N_m \backslash I, \end{cases}$$

where $\mathbf{0}_n \in \mathbf{R}^n$ is a vector of zeroes, is such that

$$||C^0||_{p\infty} = \sigma,$$

i.e. $C^0 \in \Omega(\varepsilon)$. Consistently applying relation (9), (7) and (8), for any index $i \in I$ we obtain

$$(C_i + B_i^T)(x - x^0)) = C_i(x - x^0) - \sigma||x - x^0||_{p^*} \leq$$
$$\leq [C_i(x - x^0)]^+ - \sigma||x - x^0||_{p^*} \leq (\varphi - \sigma)||x - x^0||_{p^*} < 0.$$

This means that $X^0 \notin P^m(C + C^0, I)$. □

Lemma 3. *If the solutions $x, x' \in X$ are such that for some index $k \in N_m$ the inequality*

$$C_k(x - x') > 0,$$

holds, then for every vector $b \in \mathbf{R}^n$ such that

$$||b||_p||x - x'||_{p^*} < C_k(x - x'), \tag{10}$$

the following inequality is valid

$$(C_k + b^T)(x - x') > 0.$$

Proof. By the Hölder inequality (5) we have

$$b^T(x - x') \geq -||b||_p||x - x'||_{p^*}.$$

Therefore, taking into account (10), we obtain

$$(C_k + b^T)(x - x') = C_k(x - x') + b^T(x - x') \geq C_k(x - x') - ||b||_p||x - x'||_{p^*} > 0.$$

□

3 Quasistability radius formula

For the multicriteria ILP problem $Z^m(C, I_1, I_2, ..., I_s)$, $m \in \mathbf{N}$, for any $p \in [1, \infty]$ and $s \in N_m$ we define

$$\varphi_1 = \varphi_1^{m,p}(C, I_1, I_2, ..., I_s) = \min_{x' \in G^m(C)} \max_{k \in N_s} \min_{x \in X \setminus \{x'\}} \max_{i \in I_k} \frac{[C_i(x - x')]^+}{||x - x'||_{p^*}}. \quad (11)$$

It is evident that $\varphi_1 \geq 0$.

Theorem 1. *For any $m \in \mathbf{N}$, $p \in [1, \infty]$ and $s \in N_m$ the quasistability radius of the multicriteria ILP problem $Z^m(C, I_1, I_2, ..., I_s)$ satisfies the formula*

$$\rho_1 = \rho_1^{m,p}(C, I_1, I_2, ..., I_s) = \varphi_1^{m,p}(C, I_1, I_2, ..., I_s).$$

Proof. First we prove the inequality $\rho_1 \geq \varphi_1$. For $\varphi_1 = 0$ this inequality is obvious. Let $\varphi_1 > 0$ and $C' \in \Omega(\varphi_1)$ be a perturbing matrix with the rows C'_i, $i \in N_m$. Then according to the definition of the number φ_1 and by virtue of (4) we derive

$$\forall x' \in G^m(C, I_1, I_2, ..., I_s) \quad \exists k \in N_s \quad \forall x \in X \setminus \{x'\} \quad \exists r = r(x) \in I_k$$
$$\left(||C'_r||_p \leq ||C'||_{p\infty} < \varphi_1 \leq \frac{[C_r(x - x')]^+}{||x - x'||_{p^*}} \right). \quad (12)$$

Since $\varphi_1 > 0$ then $C_r(x - x') > 0$. Therefore, in view of (12) we get

$$C_r(x - x') > ||C'_r||_p ||x - x'||_{p^*}.$$

Hence, by Lemma 3, we have

$$(C + C')_r(x - x') > 0.$$

As a result, we derive the formula

$$\forall C' \in \Omega(\varphi_1) \quad \forall x' \in G^m(C) \quad \exists k \in N_s \quad \forall x \in X$$
$$\left((C + C')_{I_k} x' \,\overline{\succ}\, (C + C')_{I_k} x \right)$$

that means that $x' \in P^m(C + C', I_k)$. Therefore, according to (3) $x' \in G^m(C + C', I_1, I_2, ..., I_s)$ for $C' \in \Omega(\varphi_1)$. Thus, we conclude

$$\forall C' \in \Omega(\varphi_1) \ \left(G^m(C, I_1, I_2, ..., I_s) \subseteq G^m(C + C', I_1, I_2, ..., I_s) \right).$$

Therefore, $\rho_1 \geq \varphi_1$.

Next, we prove the inequality $\rho_1 \leq \varphi_1$. In accordance with the definition of the number $\varphi_1 \geq 0$ we get the formula

$$\exists x^0 \in G^m(C) \quad \forall k \in N_s \quad \exists x(k) \in X \setminus \{x^0\} \quad \forall i \in I_k$$
$$\left([C_i(x(k) - x^0)]^+ \leq \varphi_1 ||x(k) - x^0||_{p^*} \right).$$

Therefore, according to Lemma 2, for any number $\varepsilon > \varphi_1$ there exists such a perturbing matrix $C^0 \in \Omega(\varepsilon)$ that for any index $k \in N_s$

$$x^0 \notin P^m(C + C^0, I_k).$$

From here and from (3) we obtain

$$x^0 \notin G^m(C + C^0).$$

Thus, we get the formula

$$\forall \varepsilon > \varphi_1 \quad \exists C^0 \in \Omega(\varepsilon) \quad (G^m(C) \not\subseteq G^m(C + C^0)).$$

Consequently, $\rho_1 \le \varphi_1$. □

The following two corollaries follow directly from Theorem 1.

Corollary 1. *For any $m \in \mathbf{N}$ and $p \in [1, \infty]$ the quasistability radius of the multicriteria ILP problem $Z^m(C, N_m)$, consisting in finding the set of effective solutions, i.e. the Pareto set $P^m(C)$, satisfies the formula*

$$\rho_1^{m,p}(C, N_m) = \min_{x' \in P^m(C)} \min_{x \in X \backslash \{x'\}} \max_{i \in N_m} \frac{C_i(x - x')}{||x - x'||_{p^*}}.$$

Since the right-hand side of this equality is a non-negative number, the positive cut-off contained in (11) is removed here.

Corollary 2 [9]. *For any $m \in \mathbf{N}$ and $p \in [1, \infty]$ the quasistability radius of the multicriteria ILP problem $Z^m(C, \{1\}, \{2\}, ..., \{m\})$, consisting in finding the set of extreme solutions $E^m(C)$, satisfies the formula*

$$\rho_1^{m,p}(C, \{1\}, \{2\}, ..., \{m\}) = \min_{x' \in E^m(C)} \max_{i \in N_m} \min_{x \in X \backslash \{x'\}} \frac{[C_i(x - x')]^+}{||x - x'||_{p^*}}.$$

Remark 1. The formulas for the quasi-stability radii given in Corollaries 1 and 2, in the case of $p = \infty$ turn into the known (see [7] and [13], respectively) results on the quasistability radii of the ILP problem with the Chebyshev metric (l_∞) in the parameter spaces of the problem.

4 Strong quasistability radius formula

For any $m \in \mathbf{N}$, $p \in [1, \infty]$ and $s \in N_m$ we define

$$\varphi_2 = \varphi_2^{m,p}(C, I_1, I_2, ..., I_s) = \max_{x' \in G^m(C)} \max_{k \in N_s} \min_{x \in X \backslash \{x'\}} \max_{i \in I_k} \frac{[C_i(x - x')]^+}{||x - x'||_{p^*}}. \qquad (13)$$

It is evident that $\varphi_2 \ge 0$.

Theorem 2. *For any $m \in \mathbf{N}$, $p \in [1, \infty]$ and $s \in N_m$ the strong quasistability radius of the multicriteria ILP problem $Z^m(C, I_1, I_2, ..., I_s)$ satisfies the formula*

$$\rho_2 = \rho_2^{m,p}(C, I_1, I_2, ..., I_s) = \varphi_2^{m,p}(C, I_1, I_2, ..., I_s).$$

Proof. First we prove the inequality $\rho_2 \geq \varphi_2$. If $\varphi_2 = 0$ then the inequality is obvious.

Let $\varphi_2 > 0$ and $C' \in \Omega(\varphi_2)$ be a perturbing matrix with the rows C'_i, $i \in N_m$. Then, according to the definition of the number φ_2 there exist a solution $x' \in G^m(C, I_1, I_2, ..., I_s)$ and an index $k \in N_s$ such that for any vector $x \in X \backslash \{x'\}$ there is an index $r = r(x) \in I_k$ for which, in view of (4), the following inequalities hold

$$||C'_r||_p \leq ||C'||_{p\infty} < \varphi_2 \leq \frac{[C_r(x - x')]^+}{||x - x'||_{p^*}}.$$

Now, repeating the reasoning carried out in the proof of Theorem 1, we obtain that

$$x' \in P^m(C + C', I_k), \quad C' \in \Omega(\varphi_2).$$

Therefore, by virtue of (3) $x' \in G^m(C + C', I_1, I_2, ..., I_s)$ for $C' \in \Omega(\varphi_2)$. Consequently, $\rho_2 \geq \varphi_2$.

Further, we show that $\rho_2 \leq \varphi_2$. For this, it suffices to prove the formula

$$\forall \varepsilon > \varphi_2 \quad \forall x' \in G^m(C, I_1, I_2, ..., I_s) \quad \exists C' \in \Omega(\varepsilon) \quad (x' \notin G^m(C + C', I_1, I_2, ..., I_s)). \quad (14)$$

Let $\varepsilon > \varphi_2$, $x^0 \in G^m(C)$. Then according to the definition of the number φ_2, for any index $k \in N_s$ there exists a solution $x \in X \backslash \{x'\}$ such that

$$\varepsilon > \varphi_2 \geq \frac{[C_i(x - x^0)]^+}{||x - x^0||_{p^*}}, \ i \in I_k.$$

Repeating the reasoning carried out in the proof of the inequality $\rho_1 \leq \varphi_1$ in Theorem 1, we conclude that there exists a perturbing matrix $C^0 \in \Omega(\varphi_2)$ such that

$$x^0 \notin G^m(C + C^0).$$

Thus, formula (14) holds. Therefore, $\rho_2 \leq \varphi_2$.

□

The following two corollaries follow directly from Theorem 2.

Corollary 3. *For any $m \in \mathbf{N}$ and $p \in [1, \infty]$ the strong quasistability radius of the multicriteria ILP problem $Z^m(C, N_m)$, consisting in finding the set of effective solutions, i.e. the Pareto set $P^m(C)$, satisfies the formula*

$$\rho_2^{m,p}(C, N_m) = \max_{x' \in P^m(C)} \min_{x \in X \backslash \{x'\}} \max_{i \in N_m} \frac{C_i(x - x')}{||x - x'||_{p^*}}. \quad (15)$$

Corollary 4. *For any $m \in \mathbf{N}$ and $p \in [1, \infty]$ the strong quasistability radius of the multicriteria ILP problem $Z^m(C, \{1\}, \{2\}, ..., \{m\})$, consisting in finding the set of extreme solutions $E^m(C)$, satisfies the formula*

$$\rho_2^{m,p}(C, \{1\}, \{2\}, ..., \{m\}) = \max_{x' \in E^m(C)} \max_{i \in N_m} \min_{x \in X \backslash \{x'\}} \frac{[C_i(x - x')]^+}{||x - x'||_{p^*}}.$$

Remark 2. The formula given in Corollary 3, in the case of $p = \infty$ turns into the known result on the strong quasistability radius of the ILP problem with the Chebyshev metric (l_∞) in the parameter spaces of the problem (see, for example, [6,7]).

Remark 3. Since the right-hand side of equality (15) is a non-negative number, the positive cut-off contained in (13) is removed.

5 Quasistability and strong quasistability conditions

In the space $\mathbf{R}^k$ of arbitrary dimension $k \in \mathbf{N}$ we introduce one more binary relation:

$$y \vdash y' \;\Leftrightarrow\; y_i \geq y'_i, \; i \in N_k,$$

where $y = (y_1, y_2, ..., y_k)^T \in \mathbf{R}^k$, $y' = (y'_1, y'_2, ..., y'_k)^T \in \mathbf{R}^k$.

Now we define a set of strictly generalized-effective solutions to the problem $Z^m(C)$ according to the formula:

$$SG^m(C) = SG^m(C, I_1, I_2, ..., I_s) =$$
$$= \{x \in X \;:\; \exists k \in N_s \; \forall x' \in X \backslash \{x\} \quad (C_{I_k} x \overline{\vdash} C_{I_k} x')\}.$$

5.1 Quasistability

We call a multicriteria ILP problem $Z^m(C)$, $m \geq 1$, quasistable (to perturbations of the elements of the matrix C) if there exists a number $\varepsilon > 0$ such that

$$\forall C' \in \Omega(\varepsilon) \quad (G^m(C) \subseteq G^m(C + C')).$$

It is obvious that the quasistability property of a problem is a discrete analogue of the lower semicontinuity property (according to Hausdorff) at the point $C \in \mathbf{R}^{m \times n}$ of the optimal mapping

$$G^m(C) : \; \mathbf{R}^{m \times n} \to 2^X,$$

i.e. of the point-to-set mapping which associates with each set of problem parameters (each matrix C) the set of $(I_1, I_2, ..., I_s)$-effective solutions $G^m(C, I_1, I_2, ..., I_s)$.

Theorem 1 implies the following

Corollary 5. *For any $m \in \mathbf{N}$, $p \in [1, \infty]$ and $s \in N_m$ for the multicriteria ILP problem $Z^m(C, I_1, I_2, ..., I_s)$ the following statements are equivalent:*

(i) the problem $Z^m(C, I_1, I_2, ..., I_s)$ is quasistable;

(ii) $G^m(C, I_1, I_2, ..., I_s) = SG^m(C, I_1, I_2, ..., I_s) = Ker^m(C, I_1, I_2, ..., I_s)$;

(iii) $\varphi_1^{m,p}(C, I_1, I_2, ..., I_s) > 0$.

In the case when $s = 1$ the set $SG^m(C, N_m)$ turns into the well-known Smale set [14], i.e. into the set of strictly effective solutions to the problem $Z^m(C, N_m)$:

$$Sm^m(C) = \{x \in X \ : \ \forall x' \in X \backslash \{x\} \quad (Cx \,\overline{\vdash}\, Cx')\}.$$

Therefore, Corollary 5 implies the following well-known result [5–7, 10, 11].

Corollary 6. *For any $m \in \mathbf{N}$, and $p \in [1, \infty]$ for the multicriteria ILP problem $Z^m(C, N_m)$, consisting in finding the Pareto set $P^m(C)$, the following statements are equivalent:*
(i) the problem $Z^m(C, N_m)$ is quasistable;
(ii) $P^m(C) = Sm^m(C) = Ker^m(C, N_m)$;
(iii) $\varphi_1^{m,p}(C, N_m) > 0$.

From this, in particular, we obtain the following corollary.

Corollary 7. *The single criterion (scalar) ILP problem $Z^1(C)$, $C \in \mathbf{R}^n$, consisting in finding optimal solutions, is quasistable if and only if it has a unique optimal solution.*

In the case when $s = m$ the set $SG^m(C) = SG^m(C, \{1\}, \{2\}, ..., \{m\})$ turns into the set of strictly extremal solutions to the problem $Z^m(C, \{1\}, \{2\}, ..., \{m\})$:

$$SE^m(C) = \{x \in X \ : \ \exists k \in N_m \ \forall x' \in X \backslash \{x\} \ (C_k x < C_k x')\}.$$

Thus, from Corollary 5 we derive the following corollary.

Corollary 8. *For any $m \in \mathbf{N}$ and $p \in [1, \infty]$ for the multicriteria ILP problem $Z^m(C, \{1\}, \{2\}, ..., \{m\})$, consisting in finding the set of extreme solutions $E^m(C)$, the following statements are equivalent:*
(i) the problem $Z^m(C, \{1\}, \{2\}, ..., \{m\})$ is quasistable;
(ii) $E^m(C) = SE^m(C) = Ker^m(C, \{1\}, \{2\}, ..., \{m\})$;
(iii) $\varphi_1^{m,p}(C, \{1\}, \{2\}, ..., \{m\}) > 0$.

6 Strong quasistability

The ILP problem $Z^m(C, I_1, I_2, ..., I_s)$ will be called strongly quasistable if the following formula holds:

$$\exists \varepsilon > 0 \quad \exists x \in G^m(C, I_1, I_2, ..., I_s) \quad \forall C' \in \Omega(\varepsilon) \quad (x \in G^m(C + C', I_1, I_2, ..., I_s)).$$

Theorem 2 implies the following result.

Corollary 9. *For any $m \in \mathbf{N}$, $p \in [1, \infty]$ and $s \in N_m$ for the multicriteria ILP problem $Z^m(C, I_1, I_2, ..., I_s)$ the following statements are equivalent:*
(i) the problem $Z^m(C, I_1, I_2, ..., I_s)$ is strongly quasistable;
(ii) $SG^m(C, I_1, I_2, ..., I_s) = Ker^m(C, I_1, I_2, ..., I_s) \neq \emptyset$;
(iii) $\varphi_2^{m,p}(C, I_1, I_2, ..., I_s) > 0$.

For $s = 1$, as already noted, the set $SG^m(C, N_m)$ turns into the Smale set $Sm^m(C)$. Therefore, Corollary 9 implies the following well-known result [6, 7, 10, 11].

Corollary 10. *For any $m \in \mathbf{N}$ and $p \in [1, \infty]$ for the multicriteria ILP problem $Z^m(C, N_m)$ consisting in finding the Pareto set $P^m(C)$, the following statements are equivalent:*

(i) the problem $Z^m(C, N_m)$ is strongly quasistable;

(ii) $Sm^m(C, N_m) = Ker^m(C, N_m) \neq \emptyset$;

(iii) $\varphi_2^{m,p}(C, N_m) > 0$.

From this, in particular, we obtain the following corollary.

Corollary 11. *The single criterion (scalar) ILP problem $Z^1(C)$, $C \in \mathbf{R}^n$, is strongly quasistable if and only if it has a unique optimum.*

In the case when $s = m$, from Corollary 9, we derive

Corollary 12. *For any $m \in \mathbf{N}$, $p \in [1, \infty]$ and $s \in \mathbf{N}$ for the multicriteria ILP problem $Z^m(C, \{1\}, \{2\}, ..., \{m\})$, consisting in finding the set of extreme solutions $E^m(C)$, the following statements are equivalent:*

(i) the problem $Z^m(C, \{1\}, \{2\}, ..., \{m\})$ is strongly quasistable;

(ii) $SE^m(C) = Ker^m(C, \{1\}, \{2\}, ..., \{m\}) \neq \emptyset$;

(iii) $\varphi_2^{m,p}(C, \{1\}, \{2\}, ..., \{m\}) > 0$.

Remark 4. Taking into account the equivalence of any two norms in a finite-dimensional linear space (see, for example, [15, 16]), all the consequences of Section 5 are valid for any norms in the parameter space of the problem.

In conclusion, we note that in [13, 17–25], similar quantitative characteristics of various types of stability of multicriteria discrete optimization problems and game theory problems with other types of optimality principles were considered.

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