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Loops with invariant flexibility under the isostrophy

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Abstract. The question "Are the loops with universal (i.e. invariant under the isotopy of loops) flexibility law $xy \cdot x = x \cdot yx$, middle Bol loops?" is open in the theory of loops. If this conjecture is true then the loops for which all isostrophic loops are flexible are Moufang loops. In the present paper we prove that commutative loops with invariant flexibility under the isostrophy of loops are Moufang loops. In particular, we obtain that commutative IP-loops with universal flexibility are Moufang loops.

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A groupoid $(Q, \cdot)$ is called a quasigroup if, for each $a, b \in Q$, there exist unique $x, y \in Q$ such that $a \cdot x = b$ and $y \cdot a = b$. A quasigroup with a neutral element is called a loop. If (Q, A) is a quasigroup and $\sigma \in S_3$, then the operation ${}^{\sigma}A$, defined by the equivalence

$${}^{\sigma}A(x_{\sigma(1)}, x_{\sigma(2)}) = x_{\sigma(3)} \Leftrightarrow A(x_1, x_2) = x_3,$$

is called a σ-parastrophe of A (or simply, a parastrophe of A) [5]. If the operation A is denoted by $(\cdot)$ then usually the (13)-parastrophe and (23)-parastrophe of $(\cdot)$ are denoted, respectively, by $(/)$ and $(\backslash)$.

If $(Q, \cdot)$ and $(Q', \circ)$ are two quasigroups then $(Q', \circ)$ is called an isotope of $(Q, \cdot)$ if there exist three bijections $\alpha, \beta, \gamma : Q \mapsto Q'$ such that $\gamma(x \circ y) = \alpha(x) \cdot \beta(y), \forall x, y \in Q$. In this case the triple (α, β, γ) is called an isotopy. Isotopy for loops and quasigroups was introduced by Albert [1]. Every isotope of a parastrophe of a quasigroup operation $(\cdot)$ may be represented as a paratrophe of an isotope of $(\cdot)$. If one of two quasigroups is isotopic to a parastrophe of the other, the quasigroups are said to be isostrophic to each other [2]. It is clear that all three types of transformations: parastrophy, isotopy and isostrophy, are equivalence relations.

Invariant properties, in particular invariant identities, under the isotopy of loops are called universal properties, respectively universal identities. For example, universal properties can be obtained from closure conditions in algebraic nets [3]. Known examples of universal identities are: the associative law, Moufang identities $(x(y \cdot xz) = (xy \cdot x)z, (zx \cdot y)x = z(x \cdot yx), (z \cdot xy)z = zx \cdot yz$, which are equivalent in loops) and (left, right, middle) Bol identities $(x(y \cdot xz) = (x \cdot yx)z$, $(zx \cdot y)x = z(xy \cdot x)$, $x(yz\backslash x) = (x/z)(y\backslash x))$.

On the other hand, non-universal identities can be invariant under the isotopy of loops in some subvarieties of loops, given by universal identities. For example,

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the inverse properties: $x^{-1} \cdot xy = y$ (LIP), $yx \cdot x^{-1} = y$ (RIP), $(xy)^{-1} = y^{-1}x^{-1}$ (AAIP, i.e. anti-authomorphic inverse property) are universal in a loop $(Q, \cdot)$ if and only if $(Q, \cdot)$ is, respectively, a left Bol loop, a right Bol loop, a middle Bol loop. In particular, both inverse properties LIP and RIP (i.e. IP) are universal in a loop $(Q, \cdot)$ if and only if $(Q, \cdot)$ is a Moufang loop.

The flexibility law $x \cdot yx = xy \cdot x$ is among the simplest in loops. It is known (see [7,8]) that each local loop of dimension r with universal flexibility is a smooth middle Bol loop of dimension r. This fact suggested the question: is this true for arbitrary loops? Necessary and sufficient conditions when every loop isotopic to a given loop is flexible, and some properties of such loops where given in [9,10]. In particular, it was remarked in the mentioned papers that finite loops with universal flexibility, of order up to six, are middle Bol loops. Also there are known (and can be easily obtained using the computer) examples of finite non-flexible middle Bol loops. The question "Are the loops with universal flexibility middle Bol loops?" is still open. Remark that this problem boosted also the study of middle Bol loops (see, for example, [3,11,12]).

The invariance of flexibility under the isostrophy of loops is studied in the present paper. Necessary and sufficient conditions for the invariance of flexibility are found. Our main result states that commutative loops with invariant flexibility under the isostrophy of loops, are Moufang loops. In particular, we prove that commutative IP-loops with universal flexibility are Moufang loops.

This result gives a partial answer to the question mentioned above. Indeed, if it is true that loops with universal flexibility are middle Bol loops, then loops with invariant flexibility under the isostrophy of loops have to be Moufang, as they are (in this case) isostrophes of flexible left (right) Bol loops [4], which are Moufang loops.

It is known [9] that the flexibility law is universal in a loop $(Q, \cdot)$ if and only if $(Q, \cdot)$ satisfies the identity

$$x \setminus [(xy/z) \cdot (b \setminus xz)] = b \setminus [(by/z) \cdot (b \setminus xz)]$$

or, equivalently, the identity

$$[(bx/z) \cdot (b \setminus yx)]/x = [(bx/z) \cdot (b \setminus yz)]/z.$$

In particular, the flexibility is universal in an IP-loop $(Q, \cdot)$ (i.e. every loop isotopic to the IP-loop $(Q, \cdot)$ is flexible) if and only if $(Q, \cdot)$ satisfies the identity

$$(xy \cdot zu)y \cdot zx = xy \cdot z(uy \cdot zx). \tag{1}$$

Proposition 1. *All loops isostrophic to a given loop $(Q, \cdot)$ are flexible if and only if $(Q, \cdot)$ satisfies the following three identities:*

$$(xb)/[((yb)/(x \setminus a)) \setminus a] = [((xb)/(y \setminus a))b]/(x \setminus a), \tag{2}$$

$$[a/((a/x) \setminus (by))] \setminus (bx) = (a/x) \setminus [b((a/y) \setminus (bx))], \tag{3}$$

$$[((x/a)\cdot(b\setminus y))/a]\cdot(b\setminus x)=(x/a)\cdot[b\setminus((y/a)\cdot(b\setminus x))]. \qquad (4)$$

Proof. Let $(Q,\circ)$ be a loop, isostrophic to a given flexible loop $(Q,\cdot)$. Then there exist three bijections $\gamma,\varphi,\psi: Q\mapsto Q$ such that the operation $(\circ)$ has one of the following six forms:

(i) $x\circ y=\gamma^{-1}(\varphi(x)/\psi(y))$, (ii) $x\circ y=\gamma^{-1}(\varphi(x)\setminus\psi(y))$,
(iii) $x\circ y=\gamma^{-1}(\psi(y)\setminus\varphi(x))$, (iv) $x\circ y=\gamma^{-1}(\psi(y)/\varphi(x))$,
(v) $x\circ y=\gamma^{-1}(\psi(y)\cdot\varphi(x))$, (vi) $x\circ y=\gamma^{-1}(\varphi(x)\cdot\psi(y))$.

For $a\in Q$, we will denote by R_a, L_a and I_a, the right, left and middle translations with the element a, respectively, i.e. $R_a(x)=xa$, $L_a(x)=ax$, $I_a(x)=x\backslash a$, $\forall x\in Q$.

Let consider the isostrophic loop $(Q,\circ)$ whose operation is given by (i) and let e be its unit. Taking $x=e$ in (i) and denoting $\varphi(e)=a$, we get $\gamma(y)=a/\psi(y)=I_a^{-1}\psi(y),\forall y\in Q$, so

$$\psi=I_a\gamma. \qquad (5)$$

Analogously, taking $y=e$ in (i) and denoting $\psi(e)=b$, we have $\gamma(x)=\varphi(x)/b=R_b^{-1}\varphi(x),\forall x\in Q$, i.e.

$$\varphi=R_b\gamma. \qquad (6)$$

Using (5) and (6), the equality (i) implies $(\circ)=(/)^{(R_b\gamma,I_a\gamma,\gamma)}$, i.e. $(Q,\circ)\cong(Q,*)$, where

$$x*y=R_b(x)/I_a(y). \qquad (7)$$

Using (7) in the flexibility law $x*(y*x)=(x*y)*x$, we get

$$R_b(x)/I_a(R_b(y)/I_a(x))=R_b(R_b(x)/I_a(y))/I_a(x),$$

which is equivalent to

$$(xb)/[((yb)/(x\setminus a))\setminus a]=[((xb)/(y\setminus a))b]/(x\setminus a),$$

so we obtained the identity (2).

Analogously, considering the cases (ii) and (vi), we get the identities (3) and (4), respectively. The remaining three cases (iii), (iv) and (v) give, respectively, the identities (2), (3) and (4), which we already obtained above. □

The following two examples show that the identities (2), (3) and (4) are not pairwise equivalent in loops. The loop given in Example 1 satisfies (2) and does not satisfy (3) and (4), and the loop from Example 2 satisfies (3) and does not satisfy (2) and (4).

Example 1.

$(\cdot)$	0	1	2	3	4	5	6	7
0	2	3	0	1	5	4	7	6
1	3	2	1	0	6	7	4	5
2	0	1	2	3	4	5	6	7
3	5	0	3	7	2	6	1	4
4	1	6	4	2	7	0	5	3
5	4	7	5	6	0	2	3	1
6	7	4	6	5	1	3	2	0
7	6	5	7	4	3	1	0	2

Example 2.

$(\cdot)$	0	1	2	3	4	5	6	7
0	2	3	0	1	5	4	7	6
1	3	2	1	5	6	7	4	0
2	0	1	2	3	4	5	6	7
3	1	0	3	4	7	6	5	2
4	5	6	4	7	2	0	1	3
5	4	7	5	6	0	2	3	1
6	7	4	6	0	1	3	2	5
7	6	5	7	2	3	1	0	4

Remark 1. If $(Q,\cdot)$ is a flexible loop then ${}^{-1}x = x^{-1}$, $\forall x \in Q$, i.e. the left and right inverses of each element coincide, as

$$x = (x \cdot x^{-1}) \cdot x = x \cdot (x^{-1} \cdot x) \Rightarrow e = x^{-1} \cdot x.$$

It is known that loops for which the condition ${}^{-1}x = x^{-1}$ is universal are power associative, i.e. each element of the loop generates an associative subloop (see, for example, [3]). Hence, loops with universal flexibility (in particular, loops with invariant flexibility under the isostrophy of loops) are power associative.

Proposition 2. *Every commutative loop which satisfies the identity (2) is an IP-loop.*

Proof. Let $(Q,\cdot)$ be a commutative loop which satisfies (2) and let e be the unit of $(Q,\cdot)$. Taking $b = e$ in (2) we get

$$x/[(y/(x \setminus a)) \setminus a] = (x/(y \setminus a))/(x \setminus a).$$

Now, making the substitution $a \to xa$ in the last equality and then taking $y = a$, we obtain $x/xa = (x/(a \setminus xa))/a$, which (using the commutativity) is equivalent to $x/xa = a^{-1}$, i.e. to $a^{-1} \cdot ax = x$ and $xa \cdot a^{-1} = x$. □

Proposition 3. *Every loop isostrophic to a commutative loop $(Q,\cdot)$ is flexible if and only if $(Q,\cdot)$ satisfies (1).*

Proof. If $(Q,\cdot)$ is a commutative loop with invariant flexibility under the isostrophy of loops then, according to Propositions 1 and 2, $(Q,\cdot)$ is an IP-loop and satisfies the identities (2), (3) and (4). Using properties of IP-loops, we get that each of these three identities coincides with (1).

Conversely, let $(Q,\cdot)$ be a loop with the identity (1). Taking $u = y^{-1}$ in (1) we have

$$(xy \cdot zy^{-1})y \cdot zx = xy \cdot (z \cdot zx),$$

which, for $z = e$ (the unit of the loop), implies $(xy \cdot y^{-1})y \cdot x = xy \cdot x$, hence $xy \cdot y^{-1} = x, \forall x, y \in Q$. As the considered loop is commutative, we get that it is an IP-loop, so (1) implies (2), (3) and (4), i.e. $(Q,\cdot)$ is a loop with invariant flexibility under the isostrophy of loops. □

It was shown in [9] that the following properties are equivalent in loops with universal flexibility: LIP, RIP, left alternativity $x \cdot xy = (xx)y$, right alternativity $yx \cdot x = y(xx)$.

Theorem 1. *Commutative loops with invariant flexibility under the isostrophy of loops are Moufang loops.*

Proof. Let $(Q,\cdot)$ be a commutative loop such that every loop, isostrophic to $(Q,\cdot)$, is flexible. Then, according to Propositions 2 and 3, $(Q,\cdot)$ is an IP-loop and satisfies the identity (1). Making the substitutions $y \mapsto x^{-1}y$ and $z \mapsto zx^{-1}$ in (1), and then $x \mapsto x^{-1}$, we get:

$$[y(zx \cdot u) \cdot xy]z = y[zx \cdot (u \cdot xy)z]. \tag{8}$$

Now, taking $u = y^{-1}$ in (8), and using the fact that $(Q,\cdot)$ is a commutative, alternative IP-loop, we have

$$(zx \cdot xy)z = y \cdot (xz)^2, \tag{9}$$

which, for $z \mapsto x^{-1}z$, implies $(z \cdot xy) \cdot x^{-1}z = yz^2$. Finally, replacing $y \mapsto x^{-1}y$ in the last equality, we get $zy \cdot x^{-1}z = (x^{-1}y) \cdot z^2$, which is equivalent to

$$z^2 \cdot xy = zx \cdot zy,$$

i.e. $(Q,\cdot)$ is a (commutative) Moufang loop. □

Remark 2. The identity (1) does not imply Moufang identities in quasigroups which are not loops. Indeed, it is known that a quasigroup satisfying any of three Moufang identities $x(y \cdot xz) = (xy \cdot x)z, (zx \cdot y)x = z(x \cdot yx) or (z \cdot xy)z = zx \cdot yz$, is a loop [6], while there exist quasigroups with the identity (1) which are not loops. For example, the quasigroup $(Z_8, *)$, where $x * y = x + 3y, \forall x, y \in Z_8$, Z_8 is the group of residue classes modulo 8, satisfies (1).

Let $(Q,\cdot)$ be a loop and let consider the 4-ary quasigroups (Q, f_i), $i = 1, 2, 3$, where $f_1(y, b, x, a) = (yb)/(x\backslash a)$, $f_2(a, x, b, y) = (a/x)\backslash(by)$, $f_3(x, a, b, y) = (x/a)(b\backslash y)$. Using the 4-ary operations f_1, f_2 and f_3, the identities (2), (3) and (4) take the form, respectively:

$$f_1(x,b,f_1(y,b,x,a),a) = f_1(f_1(x,b,y,a),b,x,a), \tag{8}$$

$$f_2(a,f_2(a,x,b,y),b,x) = f_2(a,x,b,f_2(a,y,b,x)), \tag{9}$$

$$f_3(f_3(x,a,b,y),a,b,x) = f_3(x,a,b,f_3(y,a,b,x)). \tag{10}$$

So, the identities (2)-(4) lead to three functional equations and the 4-quasigroups $f_1(y,b,x,a) = (yb)/(x\backslash a)$, $f_2(a,x,b,y) = (a/x)\backslash(by)$, $f_3(x,a,b,y) = (x/a)(b\backslash y)$ are solutions of (8), (9) and (10), respectively. Remark that every 4-ary (1,3)-commutative groupoid is a solution of (8), every 4-ary (2,4)-commutative groupoid is a solution of (9) and every 4-ary (1,4)-commutative groupoid is a solution of (10). Nevertheless, the solutions of these functional equations may not always have the corresponding commutativity, as it follow from next proposition.

Proposition 4. *Let $(Q,\cdot)$ be a loop. The following statements hold:*
(i) The 4-quasigroup (Q,f_1), where $f_1(y,b,x,a) = (yb)/(x\backslash a)$, is (1,3)-commutative if and only if $(Q,\cdot)$ is an abelian group;
(ii) The 4-quasigroup (Q,f_2), where $f_2(a,x,b,y) = (a/x)\backslash(by)$, is (2,4)-commutative if and only if $(Q,\cdot)$ is an abelian group;
(iii) The 4-quasigroup (Q,f_3), where $f_3(x,a,b,y) = (x/a)(b\backslash y)$, is (1,4)-commutative if and only if $(Q,\cdot)$ is an abelian group.

Proof. (i) (Q,f_1) is (1,3)-commutative if and only if $f_1(y,b,x,a) = f_1(x,b,y,a)$, i.e. if and only if $(yb)/(x\backslash a) = (xb)/(y\backslash a)$, for every $a,b,x,y \in Q$. Taking $y = e$ and, after that, making the replacements $a \mapsto xa, b \mapsto ba$ in the previous equality, we get $b\cdot xa = x\cdot ba$, which implies (for $a = e$) $bx = xb, \forall x,b \in Q$. From the last two identities it follows that $(Q,\cdot)$ is an abelian group. Conversely, if $(Q,\cdot)$ is an abelian group, then

$$f_1(y,b,x,a) = yba^{-1}x = f_1(x,b,y,a),$$

i.e. (Q,f_1) is (1,3)-commutative.

(ii) (Q,f_2) is (2,4)-commutative if and only if $(a/x)\backslash(by) = (a/y)\backslash(bx)$, for every $a,b,x,y \in Q$. Taking $y = e$ and replacing $a \mapsto ax, b \mapsto ab$ in the last identity, we have $ax\cdot b = ab\cdot x, \forall a,x,b \in Q$, which (for $a = e$) implies $xb = bx$, so the loop $(Q,\cdot)$ is commutative and associative. The converse is clear.

(iii) (Q,f_3) is (2,4)-commutative if and only if $(x/a)(b\backslash y) = (y/a)(b\backslash x), \forall a,b,x,y \in Q$. Making the substitutions $x \mapsto xa, y \mapsto by$ in the last equality, we have $xy = (by/a)(b\backslash xa)$. Now, taking $y = e, b \mapsto ba$ and $x \mapsto bx$, the previous identity implies $ba\cdot x = bx\cdot a$, which (for $b = e$) gives $ax = xa$. Using the last two identities we obtain that $(Q,\cdot)$ is an abelian group. □

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