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On self-adjoint and invertible linear relations generated by integral equations

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Abstract. We define a minimal operator L_0 generated by an integral equation with an operator measure and prove necessary and sufficient conditions for the operator L_0 to be densely defined. In general, L_0^* is a linear relation. We give a description of L_0^* and establish that there exists a one-to-one correspondence between relations $\widehat{L}$ with the property $L_0 \subset \widehat{L} \subset L_0^*$ and relations θ entering in boundary conditions. In this case we denote $\widehat{L} = L_\theta$. We establish conditions under which linear relations L_θ and θ together have the following properties: a linear relation ($l.r$) is self-adjoint; $l.r$ is closed; $l.r$ is invertible, i.e., the inverse relation is an operator; $l.r$ has the finite-dimensional kernel; $l.r$ is well-defined; the range of $l.r$ is closed; the range of $l.r$ is a closed subspace of the finite codimension; the range of $l.r$ coincides with the space wholly; $l.r$ is continuously invertible. We describe the spectrum of L_θ and prove that families of linear relations $L_{\theta(\lambda)}$ and $\theta(\lambda)$ are holomorphic together.

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1 Introduction

In the study of linear operators and relations generated by differential or integral equations with boundary conditions, a problem often arises: to find such boundary conditions that determine an operator or a relation with preassigned properties. In this paper, we consider the integral equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}(s)y(s) - iJ\int_a^t f(s)ds, \tag{1}$$

where y is an unknown function; $f \in L_2(H; a, b)$; J is an operator in a separable Hilbert space H, $J = J^*$, $J^2 = E$ (E is the identical operator); $\mathbf{p}$ is an operator-valued measure defined on Borel sets $\Delta \subset [a, b]$ and taking values in the set of linear bounded operators acting in H; $\int_{t_0}^t$ stands for $\int_{[t_0 t)}$ if $t_0 < t$, for $-\int_{[t,t_0)}$ if $t_0 > t$, and for 0 if $t_0 = t$. We assume that the measure $\mathbf{p}$ is self-adjoint and $\mathbf{p}$ has a bounded variation.

Equation (1) was considered in the paper [11] under the condition that the set $\mathcal{S}_{\mathbf{p}}$ of single-point atoms of measure $\mathbf{p}$ can be arranged in the form of an increasing sequence and this sequence converges to the point b. In this case the operator L_0 is densely defined, where L_0 is the minimal operator generated by equation (1) in the space $L_2(H;a,b)$. This implies that L_0^* is an operator.

In this paper, we do not impose any conditions on the set $\mathcal{S}_{\mathbf{P}}$. We prove that the operator L_0 is densely defined if and only if $\mu(\overline{\mathcal{S}}_p) = 0$, where μ is the "usual" Lebesque measure on $[a,b]$ (i.e., $\mu([\alpha,\beta)) = \beta - \alpha$ for all $\alpha, \beta \in [a,b], \alpha < \beta$). Hence L_0^* is a linear relation (a multi-valued operator), in general. We give a description of the relation L_0^*.

We use different boundary value spaces for L_0^* and establish that there exists a one-to-one correspondence between relations $\widehat{L}$ with the property $L_0 \subset \widehat{L} \subset L_0^*$ and relations θ entering in boundary conditions. In this case we denote $\widehat{L} = L_\theta$. We establish conditions under which linear relations L_θ and θ (or $\theta - \Phi_\delta$, where Φ_δ is a bounded operator defined below in the paper) together have the following properties: 0) a linear relation ($l.r$) is self-adjoint; 1) $l.r$ is closed; 2) $l.r$ is invertible, i.e., the inverse relation is an operator; 3) $l.r$ has the finite-dimensional kernel; 4) $l.r$ is well-defined; 5) the range of $l.r$ is closed; 6) the range of $l.r$ is a closed subspace of the finite codimension; 7) the range of $l.r$ coincides with the space wholly; 8) $l.r$ is continuously invertible. The properties 1) – 8) are borrowed from [1, 2].

We describe the spectrum of the linear relation L_θ and prove that families of linear relations $L_{\theta(\lambda)}$ and $\theta(\lambda)$ are holomorphic together.

We note that linear relations were first employed in work [16] (see also [17]) for the description of self-adjoint extensions of differential operators in terms of boundary conditions.

2 Preliminary assertions

Let H be a separable Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We consider a function $\Delta \to \mathbf{P}(\Delta)$ defined on Borel sets $\Delta \subset [a,b]$ and taking values in the set of bounded linear operators acting in H. The function $\mathbf{P}$ is called an operator measure on $[a,b]$ (see, for example, [3, ch. 5]) if it is zero on the empty set and the equality $\mathbf{P}\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} \mathbf{P}(\Delta_n)$ holds for disjoint Borel sets Δ_n, where series converges weakly. Further, we extend to a segment $[a,b_0] \supset [a,b_0) \supset [a,b]$ any measure $\mathbf{P}$ on $[a,b]$, letting $\mathbf{P}(\Delta) = 0$ for each Borel sets $\Delta \subset (b, b_0]$.

By $\mathbf{V}_\Delta(\mathbf{P})$ we denote $\mathbf{V}_\Delta(\mathbf{P}) = \rho(\Delta) = \sup \sum_k \|\mathbf{P}(\Delta_k)\|$, where the supremum is taken over all finite sums of disjoint Borel sets $\Delta_k \subset \Delta$. The number $\mathbf{V}_\Delta(\mathbf{P})$ is called the variation of the measure $\mathbf{P}$ on the Borel set Δ. Suppose that the measure $\mathbf{P}$ has the bounded variation on $[a,b]$. Then for ρ-almost all $\xi \in [a,b]$ there exists an operator function $\xi \to \Psi_{\mathbf{P}}(\xi)$ such that $\Psi_{\mathbf{P}}$ possesses the values in the set of bounded linear operators acting in H, $\|\Psi_{\mathbf{P}}(\xi)\| = 1$, and the equality $\mathbf{P}(\Delta) = \int_\Delta \Psi_{\mathbf{P}}(\xi) d\rho$ holds for each Borel set $\Delta \subset [a,b]$. This integral converges with respect to the usual operator norm [3, ch. 5].

A function h is integrable with respect to the measure $\mathbf{P}$ on a set Δ if there exists the Bochner integral $\int_\Delta \Psi_{\mathbf{P}}(t) h(t) d\rho = \int_\Delta (d\mathbf{P}) h(t)$. Then the function $y(t) = \int_{t_0}^t (d\mathbf{P}) h(s)$ is continuous from the left.

By $\mathcal{S}_{\mathbf{P}}$ denote a set of single-point atoms of the measure $\mathbf{P}$ (i.e., a set $t \in [a,b]$ such that $\mathbf{P}(\{t\}) \neq 0$). The set $\mathcal{S}_{\mathbf{P}}$ is at most countable. The measure $\mathbf{P}$ is continuous if $\mathcal{S}_{\mathbf{P}} = \varnothing$; it is self-adjoint if $(\mathbf{P}(\Delta))^* = \mathbf{P}(\Delta)$ for each Borel set $\Delta \subset [a,b]$; it is non-negative if $(\mathbf{P}(\Delta)x, x) \geqslant 0$ for all Borel sets $\Delta \subset [a,b]$ and all elements $x \in H$.

In following Lemma 1, $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}$ are operator measures having bounded variations on $[a,b]$ and taking values in the set of linear bounded operators acting in H. Suppose that the measure $\mathbf{q}$ is self-adjoint. We assume that these measures are extended on the segment $[a,b_0] \supset [a,b_0) \supset [a,b]$ in the manner described above.

Lemma 1. [10] *Let f, g be functions integrable on $[a,b_0]$ with respect to the measure $\mathbf{q}$; $y_0, z_0 \in H$. Then any functions*

$$y(t) = y_0 - iJ\int_{t_0}^{t} d\mathbf{p}_1(s)y(s) - iJ\int_{t_0}^{t} d\mathbf{q}(s)f(s),$$

$$z(t) = z_0 - iJ\int_{t_0}^{t} d\mathbf{p}_2(s)z(s) - iJ\int_{t_0}^{t} d\mathbf{q}(s)g(s) \quad (a \leqslant t_0 < b_0,\ \ t_0 \leqslant t \leqslant b_0)$$

satisfy the following formula (*analogous to the Lagrange one*):

$$\begin{aligned}\int_{c_1}^{c_2}(d\mathbf{q}(t)f(t), z(t)) - \int_{c_1}^{c_2}(y(t), d\mathbf{q}(t)g(t)) = (iJy(c_2), z(c_2)) - (iJy(c_1), z(c_1)) + \\ + \int_{c_1}^{c_2}(y(t), d\mathbf{p}_2(t)z(t)) - \int_{c_1}^{c_2}(d\mathbf{p}_1(t)y(t), z(t)) - \\ - \sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)} (iJ\mathbf{p}_1(\{t\})y(t), \mathbf{p}_2(\{t\})z(t)) - \\ - \sum_{t\in\mathcal{S}_{\mathbf{q}}\cap\mathcal{S}_{\mathbf{p}_2}\cap[c_1,c_2)} (iJ\mathbf{q}(\{t\})f(t), \mathbf{p}_2(\{t\})z(t)) - \\ - \sum_{t\in\mathcal{S}_{\mathbf{p}_1}\cap\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)} (iJ\mathbf{p}_1(\{t\})y(t), \mathbf{q}(\{t\})g(t)) - \\ - \sum_{t\in\mathcal{S}_{\mathbf{q}}\cap[c_1,c_2)} (iJ\mathbf{q}(\{t\})f(t), \mathbf{q}(\{t\})g(t)), \quad t_0 \leqslant c_1 \leqslant c_2 \leqslant b_0. \end{aligned} \tag{2}$$

Further suppose that $\mathbf{p}$ is a self-adjoint measure with the bounded variation. We consider the equation

$$y(t) = x_0 - iJ\int_{a}^{t} d\mathbf{p}(s)y(s) - iJ\int_{a}^{t} f(s)d\mu(s), \tag{3}$$

where μ is the "usual" Lebesque measure on $[a,b]$ extended to $[a,b_0]$ by the equality $\mu(\Delta) = 0$ for each Borel set $\Delta \subset (b,b_0]$; $x_0 \in H$; $f \in L_2(H;a,b)$ and $f = 0$ on $(b,b_0]$.

From the measure $\mathbf{p}$ we construct a continuous measure $\mathbf{p}_0$ in the following way. We set $\mathbf{p}_0(\{\alpha\}) = 0$ for $\alpha \in \mathcal{S}_{\mathbf{p}}$ and we set $\mathbf{p}_0(\Delta) = \mathbf{p}(\Delta)$ for all Borel sets such that $\Delta \cap \mathcal{S}_{\mathbf{p}} = \varnothing$. The measure $\mathbf{p}_0$ is self-adjoint. We replace $\mathbf{p}$ by $\mathbf{p}_0$ in (3). Then we obtain the equation

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\int_a^t f(s)d\mu(s). \tag{4}$$

Equations (3), (4) have unique solutions (see [9],[10]).

By W denote the operator solution of the equation

$$W(t,\lambda)x_0 = x_0 - iJ\int_a^t d\mathbf{p}_0(s)W(s,\lambda)x_0 - iJ\lambda\int_a^t W(s,\lambda)x_0 d\mu(s),$$

where $x_0 \in H$, $\lambda \in \mathbb{C}$ (the set of complex numbers). Using (2), we get

$$W^*(t,\overline{\lambda})JW(t,\lambda) = J$$

by the standard method (see [11]). The functions $t \to W(t,\lambda)$ and $t \to W^{-1}(t,\lambda) = JW^*(t,\overline{\lambda})J$ are continuous with respect to the uniform operator topology. Consequently there exist constants $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that the inequality

$$\varepsilon_1 \|x\|^2 \leqslant \|W(t,\lambda)x\|^2 \leqslant \varepsilon_2 \|x\|^2 \tag{5}$$

holds for all $x \in H$, $t \in [a, b_0]$, $\lambda \in C \subset \mathbb{C}$ (C is a compact set). The function $\lambda \to W(t,\lambda)x$ is holomorphic for fixed t.

Lemma 2. [9, 10] *A function y is a solution of the equation*

$$y(t) = x_0 - iJ\int_a^t d\mathbf{p}_0(s)y(s) - iJ\lambda\int_a^t y(s)d\mu(s) - iJ\int_a^t f(s)d\mu(s)$$

if and only if y has the form

$$y(t) = W(t,\lambda)x_0 - W(t,\lambda)iJ\int_a^t W^*(s,\overline{\lambda})f(s)d\mu(s),$$

where $x_0 \in H$, $\lambda \in \mathbb{C}$, $a \leqslant t \leqslant b_0$.

3 Linear operators and relations generated by the integral equation

In this section, we introduce a minimal operator L_0 generated by equation (3) and give a description of the adjoint relation L_0^*.

Let $\mathbf{B}_1$, $\mathbf{B}_2$ be Banach spaces. A linear relation T is understood as any linear manifold $T \subset \mathbf{B}_1 \times \mathbf{B}_2$. The terminology on the linear relations can be found, for example, in [1, 2, 12, 13]. Linear operators are treated as linear relations, this is why the notation $\{x_1, x_2\} \in T$ is used also for an operator T. Since all considered relations are linear, we shall often omit the word "linear". In what follows we make use of the following notations: $\{\cdot,\cdot\}$ is an ordered pair; $\mathcal{D}(T)$ is the domain of T; $\mathcal{R}(T)$ is the range of T; $\ker T$ is a set of elements $x \in \mathbf{B}_1$ such that $\{x, 0\} \in T$; Ker$T$ is a set of ordered pairs of the form $\{x, 0\} \in T$; T^{-1} is the relation inverse for T, i.e., the relation formed by the pairs $\{x', x\}$, where $\{x, x'\} \in T$. A relation T is called surjective if $\mathcal{R}(T) = \mathbf{B}_2$. A relation T is called invertible or injective if

$\ker T = \{0\}$ (i.e., the relation T^{-1} is an operator); it is called continuously invertible if it is closed, invertible, and surjective (i.e., T^{-1} is a bounded everywhere defined operator).

Suppose $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}$ and T is a closed relation, $T \subset \mathbf{B} \times \mathbf{B}$. The following notations are used: $\rho(T)$ is a resolvent set of T, i.e., a set of points $\lambda \in \mathbb{C}$ such that the relation $T - \lambda E$ is continuously invertible; $\sigma_p(T)$ is a point spectrum of T, i.e. a set of $\lambda \in \mathbb{C}$ such that $\ker(T - \lambda E) \neq \{0\}$; $\sigma_c(T)$ ($\sigma_r(T)$) is a continuous spectrum (a residual spectrum) of T, i.e., a set of $\lambda \in \mathbb{C}$ such that the relation $T - \lambda E$ is invertible, $\mathcal{R}(T - \lambda E) \neq \mathbf{B}$, and $\overline{\mathcal{R}(T - \lambda E)} = \mathbf{B}$ ($T - \lambda E$ is invertible, $\overline{\mathcal{R}(T - \lambda E)} \neq \mathbf{B}$, respectively).

Let $\mathbf{H}$ be a Hilbert space, $T \subset \mathbf{H} \times \mathbf{H}$ a linear relation. A relation T^* is called adjoint for T if T^* consists of all pairs $\{y, y'\}$ such that equality $(x', y) = (x, y')$ holds for all pairs $\{x, x'\} \in T$. A relation T is called symmetric if $T \subset T^*$ and self-adjoint if $T = T^*$.

By $L_2(H, \mu; a, b_0)$ denote the space of μ-measurable functions y with values in H such that $\int_a^{b_0} \|y(t)\|^2 \, d\mu(t) < \infty$. This space coincides with the space $\mathfrak{H} = L_2(H; a, b)$ since $\mu(\Delta) = 0$ for each Borel set $\Delta \subset (b, b_0]$.

Let us introduce the minimal operator L_0 in the following way. The domain $\mathcal{D}(L_0)$ consists of all functions $y \in \mathfrak{H}$ for each of which there exists a function $f \in \mathfrak{H}$ such that (3) holds and y satisfies conditions

$$y(a) = y(b_0) = y(\alpha) = 0 \tag{6}$$

for all $\alpha \in \mathcal{S}_{\mathbf{p}}$. Then we set $L_0 y = f$. By Lemma 1, it follows that the operator L_0 is symmetric.

Lemma 3. *Equalities* (3), (4) *hold together for any functions* $y \in \mathcal{D}(L_0)$, $f = L_0 y$.

Proof. We denote $\mathbf{p}_1 = \mathbf{p} - \mathbf{p}_0$. Then $\mathbf{p}_1(\{\alpha\}) = \mathbf{p}(\{\alpha\})$ if $\alpha \in \mathcal{S}_{\mathbf{p}}$ and $\mathbf{p}_1(\Delta) = 0$ for any Borel set Δ such that $\Delta \cap \mathcal{S}_{\mathbf{p}} = \varnothing$. By (3), it follows that

$$y(t) = x_0 - iJ\int_a^t (d\mathbf{p}_0) y(s) - iJ\int_a^t (d\mathbf{p}_1) y(s) - iJ\int_a^t f(s) d\mu(s).$$

Now equalities (6) imply the desired assertion. □

It follows from Lemma 3 that any function $y \in \mathcal{D}(L_0)$ is continuous. Moreover, using Lemma 3, the equalities $\mu(\{a\}) = \mu([b, b_0]) = 0$, and (6), we obtain that the operator L_0 is independent of whether the measure $\mathbf{p}$ has single-point atoms at the points a, b. Therefore, without loss of generality, it can be assumed that the $b_0 = b$, and $\mathbf{p}(\{a\}) = \mathbf{p}(\{b\}) = 0$ (i.e., $a, b \notin \mathcal{S}_{\mathbf{p}}$), and μ is the "usual" Lebesque measure on $[a, b]$. Further we write ds instead of $d\mu(s)$.

Lemma 4. [10] *The operator* L_0 *is closed. The function* y *belongs to* $\mathcal{D}(L_0 - \lambda E)$ *if and only if the equalities*

$$y(t) = W(t, \lambda) iJ \int_a^t W^*(s, \overline{\lambda}) f(s) ds,$$

$$y(\alpha)=W(\alpha,\lambda)iJ\int_a^\alpha W^*(s,\overline{\lambda})f(s)ds=0$$

hold, where $\alpha\in\mathcal{S}_{\mathbf{p}}\cup\{b\}$, $f=(L_0-\lambda E)y$.

Corollary 1. *The function* $f\in\mathfrak{H}$ *belongs to the range* $\mathcal{R}(L_0-\lambda E)$ *if and only if the function* f *satisfies condition*

$$\int_a^\alpha W^*(s,\overline{\lambda})f(s)ds=0 \tag{7}$$

for all $\alpha\in\mathcal{S}_{\mathbf{p}}\cup\{b\}$.

Remark 1. Condition (7) is equivalent to the following

$$\int_\alpha^\beta W^*(s,\overline{\lambda})f(s)ds=0,\quad \alpha,\beta\in\mathcal{S}_{\mathbf{p}}\cup\{a,b\}. \tag{8}$$

Let $\overline{\mathcal{S}}_{\mathbf{p}}$ be a closure of the set $\mathcal{S}_{\mathbf{p}}$. Then a set $\mathcal{T}_{\mathbf{p}}=(a,b)\setminus\overline{\mathcal{S}}_{\mathbf{p}}$ is open and $\mathcal{T}_{\mathbf{p}}$ is a union of at most a countable number of disjoint open intervals G_k, i.e., $\mathcal{T}_{\mathbf{p}}=\bigcup_{k=1}^{\Bbbk}G_k$, $G_k\cap G_j=\varnothing$ for $k\neq j$, where $\Bbbk$ is a natural number (equal to the number of intervals if this number is finite) or the symbol ∞ (if the number of intervals is infinite). Let $\mathcal{G}$ be the set of the intervals G_k.

Further, by χ_A denote the characteristic function of a set A.

Lemma 5. *The operator* L_0 *is densely defined if and only if* $\mu(\overline{\mathcal{S}}_{\mathbf{p}})=0$.

Proof. Suppose $z\in\mathcal{D}(L_0)$. By (6), it follows that $z(\alpha)=0$ if $\alpha\in\mathcal{S}_{\mathbf{p}}$. Since z is continuous, we have $z(\alpha)=0$ if $\alpha\in\overline{\mathcal{S}}_{\mathbf{p}}$. Assume that there exists a function $f\in\mathfrak{H}$ such that the equality $(f,z)_{\mathfrak{H}}=0$ holds for all $z\in\mathcal{D}(L_0)$. By y denote a solution of equation (4). Suppose that $G_k=(\alpha_k,\beta_k)\in\mathcal{G}$ and $z\in\mathcal{D}(L_0)$. By Lemma 4, it follows that $z_k=\chi_{[\alpha_k,\beta_k]}z\in\mathcal{D}(L_0)$. We apply Lagrange's formula (2) to the functions y, f and z_k, L_0z_k for $c_1=\alpha_k$, $c_2=\beta_k$, $\mathbf{p}_1=\mathbf{p}_2=\mathbf{p}_0$, $\mathbf{q}=\mu$. Then we obtain $(y,L_0z_k)_{\mathfrak{H}}=(f,z_k)_{\mathfrak{H}}=0$. Hence,

$$(y,L_0z_k)_{\mathfrak{H}}=\int_{\alpha_k}^{\beta_k}(y(s),(L_0z_k)(s))ds=(f,z_k)_{\mathfrak{H}}=0$$

for each function $z\in\mathcal{D}(L_0)$. By (5), it follows that a set of functions $t\to W(t,0)x$ is closed in the space $L_2(H;[\alpha_k,\beta_k])$, where $x\in H$. Using Corollary 1 and equality (8), we obtain that there exists $c_k\in H$ such that $y(t)=W(t,0)c_k$ ($\alpha_k\leqslant t\leqslant\beta_k$). Lemma 2 implies that

$$W(t,0)c_k=W(t,0)c_k-W(t,0)iJ\int_{\alpha_k}^t W^*(s,0)f(s)ds,\quad \alpha_k\leqslant t\leqslant\beta_k.$$

Taking into account the invertibility of the operator $W(t,0)$, we obtain $f(t)=0$ for almost all $t\in G_k$. Here k is arbitrary ($1\leqslant k\leqslant\Bbbk$ if $\Bbbk$ is finite and $k\in\mathbb{N}$ if $\Bbbk=\infty$, $\mathbb{N}$ is the set of natural numbers). Hence $f(t)=0$ for almost all $t\in\cup_kG_k$.

Suppose that $\mu(\overline{\mathcal{S}}_{\mathbf{p}})=0$. Then $f(t)=0$ almost everywhere on $[a,b]$. Thus $\mathcal{D}(L_0)$ is dense in $\mathfrak{H}$.

Now assume that $\mu(\overline{\mathcal{S}}_{\mathbf{p}})>0$. It is established above that $z(s)=0$ for any $z\in\mathcal{D}(L_0)$ if $s\in\overline{\mathcal{S}}_{\mathbf{p}}$. Then $(z,v)_{\mathfrak{H}}=0$ for any $z\in\mathcal{D}(L_0)$ if $v\in\mathfrak{H}$ and $v(t)=0$ for $t\in[a,b]\setminus\overline{\mathcal{S}}_{\mathbf{p}}$. We take v such that $v(t)\neq 0$ for $t\in\overline{\mathcal{S}}_{\mathbf{p}}$. Then obtain that the operator L_0 is not densely defined. The lemma is proved. □

Lemma 6. *Suppose $y\in\mathcal{D}(L_0)$, $L_0y=f$. Then $f(t)=0$ for almost all $t\in\overline{\mathcal{S}}_{\mathbf{p}}$.*

Proof. The statement of the lemma is obvious if $\mu(\overline{\mathcal{S}}_{\mathbf{p}})=0$. Suppose $\mu(\overline{\mathcal{S}}_{\mathbf{p}})>0$. By $\mathcal{J}_{\mathbf{p}}$ denote a set of isolated points of the set $\overline{\mathcal{S}}_{\mathbf{p}}$. Clearly, $\mu(\overline{\mathcal{S}}_{\mathbf{p}}\backslash\mathcal{J}_{\mathbf{p}})>0$. Let $y\in\mathcal{D}(L_0)$. Then $y(\alpha)=0$ for all $\alpha\in\overline{\mathcal{S}}_{\mathbf{p}}$ (see the proof of Lemma 5). Using Lemma 4 and the invertibility of $W(t,0)$, we get

$$\int_a^{\alpha} W^*(s,0)f(s)ds=0,\quad \alpha\in\overline{\mathcal{S}}_{\mathbf{p}}. \tag{9}$$

Let $t_0\in\overline{\mathcal{S}}_{\mathbf{p}}\backslash\mathcal{J}_{\mathbf{p}}$. Then there exists a sequence $\{t_n\}$ such that $t_n\in\mathcal{S}_{\mathbf{p}}$, $t_n\neq t_0$ and $\{t_n\}$ converges to t_0. By (9), it follows that

$$(t_n-t_0)^{-1}\int_{t_0}^{t_n} W^*(s,0)f(s)ds=0.$$

Using the invertibility of $W^*(t,0)$, we obtain $f(t_0)=0$ for almost all $t_0\in\overline{\mathcal{S}}_{\mathbf{p}}\backslash\mathcal{J}_{\mathbf{p}}$. The lemma is proved. □

Let $\mathfrak{H}_0\subset\mathfrak{H}$ be a subspace consisting of functions vanishing on $[a,b]\setminus\overline{\mathcal{S}}_{\mathbf{p}}$ and let $\mathfrak{H}_1\subset\mathfrak{H}$ be a subspace consisting of functions vanishing on $\overline{\mathcal{S}}_{\mathbf{p}}$. Then $\mathfrak{H}=\mathfrak{H}_0\oplus\mathfrak{H}_1$. We note that $\mathfrak{H}_0=\{0\}$ if and only if $\mu(\overline{\mathcal{S}}_{\mathbf{p}})=0$. By L_{10} denote restriction of L_0 to $\mathfrak{H}_1$. It follows from the proof of Lemma 5 that the operator L_{10} is densely defined in $\mathfrak{H}_1$. Lemma 6 implies that $\mathcal{R}(L_{10})\subset\mathfrak{H}_1$. Therefore, L_{10}^* is an operator, $L_{10}^*\subset\mathfrak{H}_1\times\mathfrak{H}_1$. Moreover, $\mathcal{D}(L_0)\cap\mathfrak{H}_0=\{0\}$.

By Lemmas 5, 6, it follows that if $\mu(\overline{\mathcal{S}}_{\mathbf{p}})>0$, then L_0^* is a relation and

$$L_0^*=(\mathfrak{H}_0\times\mathfrak{H}_0)\oplus L_{10}^*, \tag{10}$$

i.e., L_0^* consists of all pairs $\{y,f\}$ of the form

$$\{y,f\}=\{u,v\}+\{z,L_{10}^*z\}=\{u+z,v+L_{10}^*z\},$$

where $u,v\in\mathfrak{H}_0$, $z\in\mathcal{D}(L_{10}^*)$.

We denote $w_k(t,\lambda)=\chi_{[\alpha_k;\beta_k)}(t)W(t,\lambda)W^{-1}(\alpha_k,\lambda)$, where $(\alpha_k,\beta_k)=G_k$, $G_k\in\mathcal{G}$. Let $\ker_k(\lambda)$ be a linear space of functions $t\to w_k(t,\lambda)\xi_k$, $\xi_k\in H$. By (5), it follows that the space $\ker_k(\lambda)$ is closed in $\mathfrak{H}$. The spaces $\ker_k(\lambda)$ and $\ker_j(\lambda)$ are orthogonal for $k\neq j$. Denote $\mathcal{K}_n(\lambda)=\ker_1(\lambda)\oplus\ldots\oplus\ker_n(\lambda)$, where $n=1,\ldots,\Bbbk$ if $\Bbbk$ is finite and $n\in\mathbb{N}$ if $\Bbbk=\infty$. Clearly, $\mathcal{K}_n(\lambda)\subset\mathcal{K}_m(\lambda)$ for $n<m$. Let $\mathcal{K}$ be a closure of the set $\cup_n\mathcal{K}_n(\lambda)$.

Lemma 7. *The equality* $\ker(L_0^* - \lambda E) = \mathfrak{H}_0 \oplus \mathcal{K}$ *holds.*

Proof. It follows from Corollary 1 and (8) that the range $\mathcal{R}(L_{10} - \overline{\lambda}E)$ consists of all functions $f \in \mathfrak{H}$ orthogonal to functions of the form $w_k(\cdot,\lambda)\xi_k$, where $\xi_k \in H$. The equality $\ker(L_{10}^* - \lambda E) \oplus \mathcal{R}(L_{10} - \overline{\lambda}E) = \mathfrak{H}_1$ implies that $\ker(L_{10}^* - \lambda E) = \mathcal{K}$. Now the desired statement follows from (10). The lemma is proved. □

Let $\widetilde{W}_n(t,\lambda) = (w_1(t,\lambda),...,w_n(t,\lambda))$ be the operator one-row matrix, where $n = 1,...,\Bbbk$ if $\Bbbk$ is finite and $n \in \mathbb{N}$ if $\Bbbk = \infty$. For fixed t, λ, the operator $\widetilde{W}_n(t,\lambda)$ maps H^n onto H continuously, where H^n is the Cartesian product of n copies of H; it is convenient to treat elements from H^n as one-column matrices and to assume that $\widetilde{W}_n(t,\lambda)\widetilde{\xi}_n = \sum_{k=1}^n w_k(t,\lambda)\xi_k$, where we denote $\widetilde{\xi}_n = \mathrm{col}(\xi_1,...,\xi_n) \in H^n$, $\xi_k \in H$. By $\mathcal{W}_n(\lambda)$ denote the operator $\widetilde{\xi}_n \to \widetilde{W}_n(\cdot,\lambda)\widetilde{\xi}_n$. The operator $\mathcal{W}_n(\lambda)$ maps H^n onto $\mathcal{K}_n(\lambda) \subset \mathfrak{H}$ continuously and one-to-one.

Lemma 8. [11] *There exist* $\varepsilon_1, \varepsilon_2 > 0$ *such that the inequalities*

$$\varepsilon_1 \sum_{k=1}^n \Delta_k \|\tau_k\|^2 \leqslant \|\mathcal{W}_n(\lambda)\widetilde{\tau}_n\|^2_{\mathfrak{H}} \leqslant \varepsilon_2 \sum_{k=1}^n \Delta_k \|\tau_k\|^2, \quad \widetilde{\tau}_n = (\tau_1,...,\tau_n) \in H^n, \tag{11}$$

$$\varepsilon_1 \sum_{k=1}^n \Delta_k^{-1} \|\varphi_k\|^2 \leqslant \|\mathcal{W}_n(\lambda)\widetilde{\tau}_n\|^2_{\mathfrak{H}} \leqslant \varepsilon_2 \sum_{k=1}^n \Delta_k^{-1} \|\varphi_k\|^2$$

hold, where $n \leqslant \Bbbk$ *if* $\Bbbk$ *is finite and* $n \in \mathbb{N}$ *if* $\Bbbk = \infty$,

$$\Delta_k = \beta_k - \alpha_k, \quad \varphi_k = \int_{\alpha_k}^{\beta_k} w_k^*(s,\lambda) w_k(s,\lambda)\tau_k ds, \ \ (\alpha_k,\beta_k) = G_k \in \mathcal{G}.$$

Suppose $\Bbbk = \infty$. In this case, let $\mathcal{H}_-$, $\mathcal{H}_+$, $\mathcal{H}_0$ be linear spaces of sequences, respectively, $\widetilde{\tau} = \{\tau_k\}$, $\widetilde{\varphi} = \{\varphi_k\}$, $\widetilde{\xi} = \{\xi_k\}$ such that the series $\sum_{k=1}^\infty \Delta_k \|\tau_k\|^2$, $\sum_{k=1}^\infty \Delta_k^{-1}\|\varphi_k\|^2$, $\sum_{k=1}^\infty \|\xi_k\|^2$ converge, where $\tau_k, \varphi_k, \xi_k \in H$. These spaces become Hilbert spaces if we introduce scalar products by the formulas

$$(\widetilde{\tau},\widetilde{\eta})_- = \sum_{k=1}^{\Bbbk} (\Delta_k \tau_k, \eta_k), \ \ \widetilde{\tau},\widetilde{\eta} \in \mathcal{H}_-, \ \ (\widetilde{\varphi},\widetilde{\psi})_+ = \sum_{k=1}^{\Bbbk} (\Delta_k^{-1}\varphi_k, \psi_k), \ \ \widetilde{\varphi},\widetilde{\psi} \in \mathcal{H}_+,$$

$$(\widetilde{\xi},\widetilde{\zeta})_0 = (\widetilde{\xi},\widetilde{\zeta}) = \sum_{k=1}^{\Bbbk} (\xi_k,\zeta_k), \ \ \widetilde{\xi},\widetilde{\zeta} \in \mathcal{H}_0. \tag{12}$$

By $\|\cdot\|_-$, $\|\cdot\|_+$, $\|\cdot\|_0 = \|\cdot\|$ denote the norms in $\mathcal{H}_-$, $\mathcal{H}_+$, $\mathcal{H}_0$, respectively.

The spaces $\mathcal{H}_+$, $\mathcal{H}_-$ can be treated as spaces with positive and negative norms with respect to $\mathcal{H}_0$ (see [3, ch.1], [13, ch.2]). So $\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$ and $\varepsilon_3 \|\widetilde{\varphi}\|_- \leqslant \|\widetilde{\varphi}\|_0 \leqslant \varepsilon_4 \|\widetilde{\varphi}\|_+$, where $\widetilde{\varphi} \in \mathcal{H}_+$, $\varepsilon_3, \varepsilon_4 > 0$, i.e., the space $\mathcal{H}_0$ is equipped with the spaces $\mathcal{H}_+$, $\mathcal{H}_-$. The "scalar product" $(\widetilde{\varphi},\widetilde{\tau}) = (\widetilde{\varphi},\widetilde{\tau})_0$ is defined for $\widetilde{\varphi} \in \mathcal{H}_+$, $\widetilde{\tau} \in \mathcal{H}_-$. If $\widetilde{\tau} \in \mathcal{H}_0$, then $(\widetilde{\varphi},\widetilde{\tau})_0$ coincides with the scalar product in $\mathcal{H}_0$.

Suppose $\Bbbk$ is finite. To consider both cases together, we define the scalar products in space $H^{\Bbbk}$ by formulas (12). By $\mathcal{H}_-$, $\mathcal{H}_+$, $\mathcal{H}_0 = H^{\Bbbk}$ denote spaces equipped with the scalar products $(\cdot,\cdot)_-$, $(\cdot,\cdot)_+$, $(\cdot,\cdot)_0$, respectively. We note that if $\Bbbk$ is finite, then the norms $\|\cdot\|_-$, $\|\cdot\|_+$, $\|\cdot\|_0$ are equivalent.

Suppose $\Bbbk=\infty$. Let $\mathcal{M}\subset\mathcal{H}_-$ be a set of sequences vanishing starting from a certain number (its own for each sequence). The set $\mathcal{M}$ is dense in the space $\mathcal{H}_-$. The operator $\mathcal{W}_n(\lambda)$ is a restriction of $\mathcal{W}_{n+1}(\lambda)$ to H^n. By $\mathcal{W}'(\lambda)$ denote an operator defined on $\mathcal{M}$ such that $\mathcal{W}'(\lambda)\widetilde{\tau}=\mathcal{W}_n(\lambda)\widetilde{\tau}_n$ for all $n\in\mathbb{N}$, where $\widetilde{\tau}=(\widetilde{\tau}_n,0,...)$. It follows from (11) that the operator $\mathcal{W}'(\lambda)$ admits an extension by continuity to the space $\mathcal{H}_-$. By $\mathcal{W}(\lambda)$ denote the extended operator. Moreover, we denote $\widetilde{W}(t,\lambda)\widetilde{\tau}=(\mathcal{W}(\lambda)\widetilde{\tau})(t)$, where $\widetilde{\tau}=\{\tau_k\}\in\mathcal{H}_-$. For almost all fixed t, the operator $\widetilde{W}(t,\lambda)$ maps $\mathcal{H}_-$ into H.

Suppose $\Bbbk$ is finite. In this case, we put $\mathcal{W}(\lambda)=\mathcal{W}_{\Bbbk}(\lambda)$.

We find the form of the adjoint operator $\mathcal{W}^*(\lambda)$. This operator maps continuously $\mathfrak{H}$ onto $\mathcal{H}_+$ and $\mathcal{W}^*(\lambda)$ is zero on $\mathfrak{H}_0$. Suppose $f\in\mathfrak{H}$, $\widetilde{\xi}\in\mathcal{M}$, $\widetilde{\xi}=\{\widetilde{\xi}_n,0,...\}$. Then

$$(\widetilde{\xi},\mathcal{W}^*(\lambda)f)=(\mathcal{W}(\lambda)\widetilde{\xi},f)_{\mathfrak{H}}= \\ =\int_{\bigcup_{k=1}^{\Bbbk}\overline{G}_k}(\widetilde{W}(t,\lambda)\widetilde{\xi},f(t))dt=\int_{\bigcup_{k=1}^{\Bbbk}\overline{G}_k}(\widetilde{\xi},\widetilde{W}^*(t,\lambda)f(t))dt,$$

where $G_k\in\mathcal{G}$. Since $\mathcal{W}^*(\lambda)f\in\mathcal{H}_+$ and the set $\mathcal{M}$ is dense in $\mathcal{H}_-$, we get

$$\mathcal{W}^*(\lambda)f=\int_{\bigcup_{k=1}^{\Bbbk}\overline{G}_k}\widetilde{W}^*(t,\lambda)f(t)dt=\int_a^b\widetilde{W}^*(t,\lambda)f(t)dt. \tag{13}$$

Thus we obtain the following statement.

Lemma 9. *The operator* $\mathcal{W}(\lambda)$ *maps* $\mathcal{H}_-$ *onto* $\ker(L_{10}^*-\lambda E)$ *continuously and one-to-one. A function* z *belongs to* $\ker(L_{10}^*-\lambda E)$ *if and only if there exists an element* $\widetilde{\tau}=\{\tau_n\}\in\mathcal{H}_-$ *such that* $z(t)=(\mathcal{W}(\lambda)\widetilde{\tau})(t)=\widetilde{W}(t,\lambda)\widetilde{\tau}$. *The adjoint operator* $\mathcal{W}^*(\lambda)$ *maps* $\mathfrak{H}$ *onto* $\mathcal{H}_+$ *continuously and acts by formula* (13). *Moreover,* $\mathcal{W}^*(\lambda)$ *maps* $\ker(L_{10}^*-\lambda E)$ *onto* $\mathcal{H}_+$ *one-to-one and* $\ker\mathcal{W}^*(\lambda)=\mathfrak{H}_0\oplus\mathcal{R}(L_{10}-\overline{\lambda}E)$.

Theorem 1. *An ordered pair* $\{y,f\}\in\mathfrak{H}\times\mathfrak{H}$ *belongs to* $L_0^*-\lambda E$ *if and only if there exist functions* $u_1,u_2\in\mathfrak{H}_0$, $h\in\mathfrak{H}_1$, *an element* $\tau\in\mathcal{H}_-$ *such that the equalities*

$$y=u_1+v,\ f=u_2+h,\ v(t)=\widetilde{W}(t,\lambda)\widetilde{\tau}-\sum_{k=1}^{\Bbbk}w_k(t,\lambda)iJ\int_a^t w_k^*(s,\lambda)h(s)ds \tag{14}$$

hold. The series in (14) *converges in* $\mathfrak{H}$ *and* $h=(L_{10}^*-\lambda E)v$.

Proof. The first two equalities in (14) follow from (10). The operator L_{10} is densely defined in $\mathfrak{H}_1$. Besides, $w_k(t,\lambda)=0$ for almost all $t\in\overline{\mathcal{S}}_{\mathbf{p}}$. The equality $(L_{10}^*-\lambda E)v=h$ and the third equality in (14) are proved in the same way as the analogous equality in [11]. □

By standard transformations, the third equality in (14) is reduced to the form

$$v(t)=\widetilde{W}(t,\lambda)\widetilde{\xi}-2^{-1}\widetilde{W}(t,\lambda)i\widetilde{J}\int_a^t \widetilde{W}^*(s,\overline{\lambda})h(s)ds+ \\ +2^{-1}\widetilde{W}(t,\lambda)i\widetilde{J}\int_t^b \widetilde{W}^*(s,\overline{\lambda})h(s)ds, \quad (15)$$

where $\widetilde{\xi}=\{\xi_k\}\in\mathcal{H}_-$, $\xi_k = \tau_k - 2^{-1}iJ\int_{\alpha_k}^{\beta_k} w_k^*(s,\overline{\lambda})h(s)ds$, $\widetilde{J}$ is an operator in $\mathcal{H}_-$ acting by the formula $\widetilde{J}\{\xi_k\}=\{J\xi_k\}$.

4 Self-adjoint extensions of the minimal operator

In this section, we construct a boundary triplet for which "the Green formula" is valid and describe self-adjoint extensions of L_0.

We denote $\mathbf{H}_- = \mathfrak{H}_0\times\mathcal{H}_-$, $\mathbf{H}_+ = \mathfrak{H}_0\times\mathcal{H}_+$. It follows from Theorem 1 and (15) that any pair $\{y,f\}\in L_0^*$ has the form

$$y=u_1+v,\ \ f=u_2+h,\ \ h=L_{10}^*v, \quad (16)$$

where v has form (15) for $\lambda=0$. With each pair $\{y,f\}\in L_0^*$ represented by (16), (15) for $\lambda=0$, we associate a pair of boundary values $\{Y,Y'\}\in\mathbf{H}_-\times\mathbf{H}_+$, where

$$Y=\{u_1,\widetilde{\xi}\}\in\mathbf{H}_-=\mathfrak{H}_0\times\mathcal{H}_-,\ \ Y'=\{u_2,\mathcal{W}^*(0)h\}\in\mathbf{H}_+=\mathfrak{H}_0\times\mathcal{H}_+. \quad (17)$$

By γ denote the operator taking each pair $\{y,f\}\in L_0^*$ to the pair $\{Y,Y'\}$, i.e., $\gamma\{y,f\}=\{Y,Y'\}$. We put $\gamma_1\{y,f\}=P_1\gamma\{y,f\}$, $\gamma_2\{y,f\}=P_2\gamma\{y,f\}$. (Here and next, P_j indicates the natural projection onto a set C_j in the Cartesian product $C=C_1\times C_2, j=1,2$).

Theorem 2. *The range $\mathcal{R}(\gamma)$ of the operator γ coincides with $\mathbf{H}_-\times\mathbf{H}_+$ and "the Green formula"*

$$(f,z)_{\mathfrak{H}}-(y,g)_{\mathfrak{H}}=(Y',Z)-(Y,Z') \quad (18)$$

holds, where $\{y,f\},\{z,g\}\in L_0^*$, $\gamma\{y,f\}=\{Y,Y'\}$, $\gamma\{z,g\}=\{Z,Z'\}$.

Proof. The equality $\mathcal{R}(\gamma)=\mathbf{H}_-\times\mathbf{H}_+$ follows from Lemma 9 and equality (10). Let us prove (18). Suppose that the pair $\{y,f\}$ has form (16) and the pair $\{z,g\}$ has the form $z=x_1+r$, $g=x_2+q$, $q=L_{10}^*r$, where $x_1,x_2\in\mathfrak{H}_0$, the function $r=r(t)$ is obtained if we replace $v(t)$ by $r(t)$, $\widetilde{\xi}$ by $\widetilde{\zeta}$, $h(s)$ by $q(s)$ in (15) for $\lambda=0$. Then

$$(f,z)_{\mathfrak{H}}-(y,g)_{\mathfrak{H}}=(u_2,x_1)_{\mathfrak{H}_0}-(u_1,x_2)_{\mathfrak{H}_0}+(L_{10}^*v,r)_{\mathfrak{H}_1}-(v,L_{10}^*r)_{\mathfrak{H}_1}. \quad (19)$$

The operator L_{10} is densely defined in $\mathfrak{H}_1$. The following equality

$$(L_{10}^*v,r)_{\mathfrak{H}_1}-(v,L_{10}^*r)_{\mathfrak{H}_1}=(\mathcal{W}^*(0)h,\widetilde{\zeta})-(\widetilde{\xi},\mathcal{W}^*(0)q) \quad (20)$$

is proved in the same way as the analogous equality in [11]. Now equality (18) follows from (19), (20), (17). The Theorem is proved. □

We introduce operators $\Upsilon_- : \mathcal{H}_- \to \mathcal{H}_0$, $\Upsilon_+ : \mathcal{H}_+ \to \mathcal{H}_0$ by the formulas $\Upsilon_-\widetilde{\tau} = \{\Delta_k^{1/2}\tau_k\}$, $\Upsilon_+\widetilde{\varphi} = \{\Delta_k^{-1/2}\varphi_k\}$, where $\widetilde{\tau} = \{\tau_k\} \in \mathcal{H}_-$, $\widetilde{\varphi} = \{\varphi_k\} \in \mathcal{H}_+$. The operator Υ_- (Υ_+) maps $\mathcal{H}_-$ onto $\mathcal{H}_0$ ($\mathcal{H}_+$ onto $\mathcal{H}_0$, respectively) continuously and one-to-one. Suppose that $\{y,f\} \in L_0^*$ has the form (16) and equalities (17) hold. We put $\mathcal{Y} = \Upsilon_1\{y,f\} = \{u_1, \Upsilon_-\widetilde{\xi}\} \in \mathfrak{H}_0 \times \mathcal{H}_0$; $\mathcal{Y}' = \Upsilon_2\{y,f\} = \{u_2, \Upsilon_+\mathcal{W}^*(0)h\} \in \mathfrak{H}_0 \times \mathcal{H}_0$ and $\Upsilon\{y,f\} = \{\Upsilon_1\{y,f\}, \Upsilon_2\{y,f\}\}$. Then $\mathcal{R}(\Upsilon) = (\mathfrak{H}_0 \times \mathcal{H}_0) \times (\mathfrak{H}_0 \times \mathcal{H}_0)$. Using (18), we get

$$(f,z)_{\mathfrak{H}} - (y,g)_{\mathfrak{H}} = (\mathcal{Y}', \mathcal{Z}) - (\mathcal{Y}, \mathcal{Z}'), \tag{21}$$

where $\{y,f\}, \{z,g\} \in L_0^*$, $\Upsilon\{y,f\} = \{\mathcal{Y}, \mathcal{Y}'\}$, $\Upsilon\{z,g\} = \{\mathcal{Z}, \mathcal{Z}'\}$.

It follows from (21) that the ordered triple $(\mathfrak{H}_0 \times \mathcal{H}_0, \Upsilon_1, \Upsilon_2)$ is a space of boundary values (a boundary triplet in another terminology) for the operator L_0 in the sense of papers [4, 5, 15] (see also [13, Ch. 3]). Let θ be a linear relation, $\theta \subset (\mathfrak{H}_0 \times \mathcal{H}_0) \times (\mathfrak{H}_0 \times \mathcal{H}_0)$. By L_θ denote a linear relation such that $L_\theta \subset L_0^*$ and $\Upsilon L_\theta = \theta$. By (21), it follows that the linear relations L_θ and θ are self-adjoint or not together. From here, taking into account the description of self-adjoint relations (see [16]), we obtain the following statement.

Corollary 2. *If U is a unitary operator on $\mathfrak{H}_0 \times \mathcal{H}_0$, then the restriction of the relation L_0^* to the set of pairs $\{y,f\} \in L_0^*$ satisfying the condition*

$$(U - E)\Upsilon_2\{y,f\} + i(U+E)\Upsilon_1\{y,f\} = 0 \tag{22}$$

is a self-adjoint extension of L_0. Conversely, for any self-adjoint extension $\widehat{L}$ of L_0 ($\widehat{L} \subset L_0^*$) there exists a unitary operator U such that this extension is the restriction of L_0^* to the set of pairs $\{y,f\} \in L_0^*$ satisfying (22). *A unitary operator U is uniquely determined by an extension.*

Note that dissipative and accumulative extensions of the operator L_0 are described in a similar way.

5 States of restrictions of L_0^*

In this section, we consider restrictions of the relation L_0^* and study their properties connected with the invertibility. It is convenient to use a special space boundary values (SBV) from [8] (also see references therein).

Let $\mathbf{B}_1$, $\mathbf{B}_2$, B_1, B_2 be Banach spaces, $T \subset \mathbf{B}_1 \times \mathbf{B}_2$ be a closed linear relation, $\delta : T \to B_1 \times B_2$ be a linear operator, $\delta_j = P_j\delta$, $j = 1, 2$. A quadruple $(B_1, B_2, \delta_1, \delta_2)$ is called SBV for the relation T if δ maps T onto $B_1 \times B_2$ continuously and the restriction of δ_1 to $\mathrm{Ker}T$ is a one-to-one mapping of $\mathrm{Ker}T$ onto B_1. We define an operator $\Phi_\delta : B_1 \to B_2$ and a relation T_0 by the equalities $\Phi_\delta = \delta_2(\delta_1|_{\mathrm{Ker}T})^{-1}$, $T_0 = \ker\delta$. We note that operator Φ_δ is bounded. We shall say that the relation T_0 is *the minimal relation* generated SBV. It follows from the definition of SBV that there exists a one-to-one correspondence between relations $\widehat{T}$ with the property $T_0 \subset \widehat{T} \subset T$ and relations $\theta \subset B_1 \times B_2$ and this correspondence is determined by the equality $\delta\widehat{T} = \theta$. In this case we denote $\widehat{T} = T_\theta$.

Let S be a linear relation $S \subset B_1' \times B_2'$, where B_1', B_2' are Banach spaces. The following conditions are borrowed from [1, 2]: 1) S is closed; 2) $\ker S = \{0\}$; 3) $\dim \ker S < \infty$; 4) the relation S is well-defined (i.e., S is invertible and the range $\mathcal{R}(S)$ is closed); 5) $\overline{\mathcal{R}(S)} = \mathcal{R}(S)$; 6) $\mathcal{R}(S)$ is a closed subspace in B_2' of the finite codimension; 7) $\mathcal{R}(S) = B_2'$; 8) S is continuously invertible. Following [1, 2], we shall say that the relation S is in *the state* k if it satisfies condition k).

Theorem 3. [8, 9] *Let $\mathcal{R}(T) = \mathbf{B}_2$. The relation T_θ is in state k $(1 \leqslant k \leqslant 8)$ if and only if the same is true for the relation $\theta - \Phi_\delta$.*

We put $\mathbf{B}_1 = \mathfrak{H}$, $\mathbf{B}_2 = \mathfrak{H}$, $T = L_0^*$, $B_1 = \mathbf{H}_- = \mathfrak{H}_0 \times \mathcal{H}_-$, $B_2 = \mathbf{H}_+ = \mathfrak{H}_0 \times \mathcal{H}_+$. Suppose that equalities (16) hold for a pair $\{y, f\} \in L_0^*$, but the function v has form (14) for $\lambda = 0$. Define the boundary values $\{\widetilde{Y}, \widetilde{Y}'\} = \widetilde{\delta}\{y, f\}$ by the formulas

$$\widetilde{Y} = \widetilde{\delta}_1\{y, f\} = \{u_1, \widetilde{\tau}\} \in \mathbf{H}_-, \quad \widetilde{Y}' = \widetilde{\delta}_2\{y, f\} = \{u_2, \mathcal{W}^*(0)h\} \in \mathbf{H}_+. \tag{23}$$

Note that $\widetilde{Y} = \{u_1, v(a)\}$, $\widetilde{Y}' = \{u_2, iJW^{-1}(b, 0)(v(b) - v(a))\}$ if $\mathcal{S}_{\mathbf{p}} = \varnothing$. It follows from Lemma 9 that the quadruple $(\mathbf{H}_-, \mathbf{H}_+, \widetilde{\delta}_1, \widetilde{\delta}_2)$ is SBV for L_0^*, $\Phi_{\widetilde{\delta}} = 0$, and $\ker \widetilde{\delta} = L_0$. Let θ be a linear relation, $\theta \subset \mathbf{H}_- \times \mathbf{H}_+$. By L_θ denote a restriction of L_0^* to a set of pairs $\{y, f\} \in L_0^*$ such that $\widetilde{\delta}\{y, f\} \in \theta$. Theorem 3 implies the following statement.

Corollary 3. *The relation L_θ is in state k $(1 \leqslant k \leqslant 8)$ if and only if the same is true for the relation θ.*

A pair $\{y, f\} \in L_0^*$ if and only if the pair $\{y, f - \lambda y\} \in L_0^* - \lambda E$. To each pair $\{y, f - \lambda y\} \in L_0^* - \lambda E$ assign a pair of bounded values by formula $\widetilde{\delta}(\lambda)\{y, f - \lambda y\} = \widetilde{\delta}\{y, f\}$. Then the quadruple $(\mathbf{H}_-, \mathbf{H}_+, \widetilde{\delta}_1(\lambda), \widetilde{\delta}_2(\lambda))$ is SBV for $L_0^* - \lambda E$ and $\Phi_{\widetilde{\delta}(\lambda)} = \begin{pmatrix} \lambda E & 0 \\ 0 & \lambda \mathcal{W}^*(0)\mathcal{W}(\lambda) \end{pmatrix}$. Theorem 3 implies

Corollary 4. *Suppose that the relation θ is closed. A point $\lambda \in \mathbb{C}$ belongs to the point spectrum $\sigma_p(L_\theta)$ of the relation L_θ if and only if $\ker(\theta - \Phi_{\widetilde{\delta}(\lambda)}) \neq \{0\}$. A point λ belongs to the continuous spectrum $\sigma_c(L_\theta)$ (to the residual spectrum $\sigma_r(L_\theta)$) if and only if the relation $(\theta - \Phi_{\widetilde{\delta}(\lambda)})^{-1}$ is a densely defined and unbounded (non-densely defined) operator. A point λ belongs to the resolvent set $\rho(L_\theta)$ if and only if $(\theta - \Phi_{\widetilde{\delta}(\lambda)})^{-1}$ is a bounded everywhere defined operator.*

We note that properties 1) – 8) were considered in [9] for linear relations generated in space $L_2(H, d\mathbf{m}; a, b)$ ($\mathbf{m}$ is a nonnegative operator measure) by an integral equation in which the measure $\mathbf{p}$ is not assumed to be self-adjoint. However, in [9] the relation L_θ satisfies the condition $L_0 \subset L_\theta \subset L$, where L is a closure of a set pairs $\{y, f\} \in L_2(H, d\mathbf{m}; a, b)$ satisfying the integral equation. Then $L \subset L_0^*$ if the measure $\mathbf{p}$ is self-adjoint, but $L \neq L_0^*$, in general.

6 Holomorphic restrictions of L_0^*

In this section, we describe holomorphic restrictions of the relation L_0^*.

Let $\mathbf{B}$ be a Banach space. A family of linear manifolds in $\mathbf{B}$ is understood as a function $\lambda \to \mathcal{L}(\lambda)$, where $\mathcal{L}(\lambda)$ is a linear manifold, $\mathcal{L}(\lambda) \subset \mathbf{B}$, $\lambda \in \mathcal{D} \subset \mathbb{C}$. A family of (closed) subspaces $\mathcal{L}(\lambda)$ is called holomorphic at the point $\lambda_0 \in \mathbb{C}$ if there exist a Banach space $\mathbf{B}_0$ and a family of bounded linear operators $\mathcal{F}(\lambda) : \mathbf{B}_0 \to \mathbf{B}$ such that the operator $\mathcal{F}(\lambda)$ bijectively maps $\mathbf{B}_0$ onto $\mathcal{L}(\lambda)$ for any fixed λ and the function $\lambda \to \mathcal{F}(\lambda)$ is holomorphic in some neighborhood of λ_0. A family of subspaces is called holomorphic on the domain $\mathcal{D}$ if it is holomorphic at all points belonging to $\mathcal{D}$. Since the closed relation $T(\lambda)$ is the subspace in $\mathbf{B}_1 \times \mathbf{B}_2$, the definition of holomorphic families is applied to families of linear relations. This definition generalizes the corresponding definition of holomorphic families of closed operators [14, Ch. 7].

Lemma 10. [6,7] *Let $\lambda \to \mathcal{L}(\lambda)$ be a family of subspaces in a Banach space $\mathbf{B}$ such that the subspace $\mathcal{L}(\lambda_0)$ admits a direct complement in $\mathbf{B}$ at some point λ_0, i.e., there exists subspace (closed) $\mathfrak{N} \subset \mathbf{B}$ such that the decomposition into the direct sum $\mathbf{B} = \mathcal{L}(\lambda_0) \dot{+} \mathfrak{N}$ holds. The family $\lambda \to \mathcal{L}(\lambda)$ is holomorphic at λ_0 if and only if the space $\mathbf{B}$ is decomposed into the direct sum $\mathbf{B} = \mathcal{L}(\lambda) \dot{+} \mathfrak{N}$ for all λ belonging to some neighborhood of λ_0 and the function $\lambda \to \mathcal{P}(\lambda)$ is holomorphic at λ_0, where $\mathcal{P}(\lambda)$ is the projection of the space $\mathbf{B}$ onto $\mathcal{L}(\lambda)$ in parallel to $\mathfrak{N}$.*

Lemma 11. *Suppose $\mathbf{B}_1$, $\mathbf{B}_2$ are Banach spaces, $\lambda \to \mathcal{L}(\lambda)$ is a family of closed linear relations $\mathcal{L}(\lambda) \subset \mathbf{B}_1 \times \mathbf{B}_2$. If this family is holomorphic at the point λ_0 and $\mathcal{L}(\lambda_0)$ is an everywhere defined operator, then there exists a neighborhood of λ_0 such that $\mathcal{L}(\lambda)$ are everywhere defined operators for all λ belonging to this neighborhood.*

Proof. Any pair $\{x_1, x_2\} \in \mathbf{B} = \mathbf{B}_1 \times \mathbf{B}_2$ is uniquely represented in the form $\{x_1, x_2\} = \{x_1, \mathcal{L}(\lambda_0)x_1\} + \{0, x_2 - \mathcal{L}(\lambda_0)x_1\}$. Hence the decomposition into the direct sum $\mathbf{B} = \mathcal{L}(\lambda_0) \dot{+} (\{0\} \times \mathbf{B}_2)$ holds. Using Lemma 10, we obtain $\mathbf{B} = \mathcal{L}(\lambda) \dot{+} (\{0\} \times \mathbf{B}_2)$ for all λ belonging to some neighborhood of λ_0. Let $\mathcal{P}(\lambda)$ be the projection of the space $\mathbf{B}$ onto $\mathcal{L}(\lambda)$ in parallel $\{0\} \times \mathbf{B}_2$. It follows from Lemma 10 that the function $\lambda \to P_1\mathcal{P}(\lambda)$ is holomorphic at λ_0, where P_1 is the natural projection onto $\mathbf{B}_1$ in $\mathbf{B}_1 \times \mathbf{B}_2$. The operator P_1 maps $\mathcal{L}(\lambda_0)$ onto $\mathbf{B}_1$ continuously and one-to-one. Hence there exists a neighborhood of λ_0 such that the operator P_1 maps $\mathcal{L}(\lambda)$ onto $\mathbf{B}_1$ continuously and one-to-one for all λ belonging to this neighborhood of λ_0. Therefore $\mathcal{L}(\lambda)$ are everywhere defined operators for λ from this neighborhood. The lemma is proved. □

Note that under the conditions of Lemma 11 the operator function $\lambda \to \mathcal{L}(\lambda)$ is holomorphic at λ_0 (see [14, ch.7]). In the case where $\mathbf{B}_1$, $\mathbf{B}_2$ are Hilbert spaces, Lemma 11 is proved in [6].

Lemma 12. *Suppose $\mathbf{G}$, $\mathbf{D}$ are Banach spaces, $\delta : \mathbf{D} \to \mathbf{G}$ is a linear, continuous, and surjective operator such that $\ker\delta$ admits a direct complement in $\mathbf{D}$, i.e., $\mathbf{D} =$*

$\ker\delta \dotplus \mathfrak{N}_0$, *where subspace* $\mathfrak{N}_0 \subset \mathbf{D}$. *Let* $\lambda \to \theta(\lambda)$, $\lambda \to \mathcal{L}(\lambda)$ *be families of subspaces* $\theta(\lambda) \subset \mathbf{G}$, $\mathcal{L}(\lambda) \subset \mathbf{D}$ *such that* $\ker\delta \subset \mathcal{L}(\lambda)$ *and* $\delta\mathcal{L}(\lambda) = \theta(\lambda)$. *Assume that the subspace* $\mathcal{L}(\lambda_0)$ *admits a direct complement*

$$\mathbf{D} = \mathcal{L}(\lambda_0) \dotplus \mathfrak{M}_1 \tag{24}$$

for some point $\lambda_0 \in \mathbb{C}$ *or the subspace* $\theta(\lambda_0)$ *admits a direct complement*

$$\mathbf{G} = \theta(\lambda_0) \dotplus \mathfrak{M}_2, \tag{25}$$

where $\mathfrak{M}_1 \subset \mathbf{D}$, $\mathfrak{M}_2 \subset \mathbf{G}$. *Then the family* $\lambda \to \mathcal{L}(\lambda)$ *is holomorphic at the point* λ_0 *if and only if the family* $\lambda \to \theta(\lambda)$ *is holomorphic at* λ_0.

Proof. By δ_0 denote the restriction of δ to $\mathfrak{N}_0$. The operator δ_0 maps $\mathfrak{N}_0$ onto $\mathbf{G}$ continuously and one-to-one. Suppose equality (24) holds. Then for any $y \in \mathbf{D}$ there exist unique elements $z_0 \in \mathcal{L}(\lambda_0)$, $m_1 \in \mathfrak{M}_1$ such that $y = z_0 + m_1$. This implies $\delta y = \delta z_0 + \delta m_1$. If $\delta z_0 + \delta m_1 = 0$, then $z_0 + m_1 \in \ker\delta$. Therefore, $m_1 = 0$ and $\delta z_0 = 0$. So, equality (25) holds, where $\mathfrak{M}_2 = \delta\mathfrak{M}_1$.

Now suppose equality (25) is valid. We claim that (24) holds. Indeed, $\mathbf{D} = \delta_0^{-1}\theta(\lambda_0) \dotplus \ker\delta \dotplus \delta_0^{-1}\mathfrak{M}_2 = \mathcal{L}(\lambda_0) \dotplus \mathfrak{M}_1$, where $\mathfrak{M}_1 = \delta_0^{-1}\mathfrak{M}_2$.

Let $\lambda \to \mathcal{L}(\lambda)$ be the holomorphic family at λ_0. It follows from Lemma 10 that $\mathbf{D} = \mathcal{L}(\lambda) \dotplus \mathfrak{M}_1$ for all λ belonging to some neighborhood of λ_0 and the function $\lambda \to \mathcal{P}(\lambda)$ is holomorphic at λ_0, where $\mathcal{P}(\lambda)$ is the projection of the space $\mathbf{D}$ onto $\mathcal{L}(\lambda)$ in parallel to $\mathfrak{M}_1$. Then $\mathcal{Q}(\lambda) = \delta\mathcal{P}(\lambda)\delta_0^{-1}$ is the projection of the space $\mathbf{G}$ onto $\theta(\lambda)$ in parallel to $\mathfrak{M}_2 = \delta\mathfrak{M}_1$ and the function $\lambda \to \mathcal{Q}(\lambda)$ is holomorphic at λ_0. By Lemma 10, it follows that $\lambda \to \theta(\lambda)$ is the holomorphic family at λ_0.

Conversely, suppose $\lambda \to \theta(\lambda)$ is the holomorphic family at λ_0 and $\mathcal{Q}(\lambda)$ is the projection of the space $\mathbf{G}$ onto $\theta(\lambda)$ in parallel to $\mathfrak{M}_2$. We put $\mathcal{P}(\lambda)g = g$ if $g \in \ker\delta$ and put $\mathcal{P}(\lambda)h = \delta_0^{-1}\mathcal{Q}(\lambda)\delta h$ if $h \in \mathfrak{N}_0$. We extend $\mathcal{P}(\lambda)$ to $\mathbf{D}$ letting $\mathcal{P}(\lambda)(g+h) = \mathcal{P}(\lambda)g + \mathcal{P}(\lambda)h$. Then $\mathcal{P}(\lambda)$ is the projection of the space $\mathbf{D}$ onto $\mathcal{L}(\lambda)$ in parallel to $\mathfrak{M}_1 = \delta_0^{-1}\mathfrak{M}_2$. Arguing as above, we obtain that the family $\lambda \to \mathcal{L}(\lambda)$ is holomorphic at λ_0. The Lemma is proved. □

In Lemma 12, we take $\mathbf{G} = \mathbf{H}_- \times \mathbf{H}_+$; $\mathbf{D} = L_0^*$; $\lambda \to \theta(\lambda)$ is family of linear relations $\theta(\lambda) \subset \mathbf{H}_- \times \mathbf{H}_+$; $\delta = \widetilde{\delta}$ is a linear operator taking each pair $\{y, f\} \in L_0^*$ to a pair of boundary values $\{\widetilde{Y}, \widetilde{Y}'\} \in \mathbf{H}_- \times \mathbf{H}_+$; $\mathcal{L}(\lambda) = L_{\theta(\lambda)}$ is the restriction of L_0^* to a set of pairs $\{y, f\}$ such that $\widetilde{\delta}\{y, f\} \in \theta(\lambda)$. Then $\ker\widetilde{\delta} = L_0$ and $\mathbf{G}$, $\mathbf{D}$ are Hilbert spaces. Lemma 12 implies the following assertion.

Corollary 5. *The family of relations* $L_{\theta(\lambda)}$ *is holomorphic at a point* λ_0 *if and only if the family of relations* $\theta(\lambda)$ *is holomorphic at* λ_0.

The following statement follows directly from Lemma 11, Corollaries 4, 5.

Theorem 4. *Suppose that the relation* $\theta(\lambda_0) - \Phi_{\widetilde{\delta}(\lambda_0)}$ (*or the relation* $L_{\theta(\lambda_0)} - \lambda_0 E$) *is continuously invertible and the family* $\lambda \to \theta(\lambda)$ (*or the family* $L_{\theta(\lambda)}$) *is holomorphic at the point* λ_0. *Then there exists a neighborhood of* λ_0 *such that the relations* $\theta(\lambda) - \Phi_{\widetilde{\delta}(\lambda)}$, $L_{\theta(\lambda)} - \lambda E$ *are continuously invertible for all* λ *belonging to this* *neighborhood and the operator functions* $\lambda \to (\theta(\lambda) - \Phi_{\widetilde{\delta}(\lambda)})^{-1}$, $\lambda \to (L_{\theta(\lambda)} - \lambda E)^{-1}$ *are holomorphic at* λ_0.

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