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On the structural theory and theory of radicals

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Abstract. This is an overview published in Russian in the book devoted to Academician V. A. Andrunakievich *Academicianul Vladimir Andrunachievichi: Bibliografie*, Institutul de Matematică şi Informatică, Chişinău, 2009.

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Results of the works of V. A. Andrunakievich are very diverse, but they are mainly related to the structural theory of associative algebras (or rings) and the theory of radicals as one of the instruments of structural theory. Most of the main results are published in such journals as "Izvestiya AN SSSR" and "Reports of the Academy of Sciences of the USSR", "Uspekhi Matematicheskikh Nauk" and "Matematicheskii Sbornik", "Izvestiya AN MSSR" and are well known to specialists working in the indicated fields of modern algebra and close to them. Moreover, already in the first papers of V. A. Andrunakievich, the close interconnection between the theory of radicals and the structural theory is revealed, and the theory of special radicals developed by him allowed us to generalize almost all known (by the appropriate time) structural theorems and to prove a number of new theorems, which reflect specific features of rings or algebras being examined.

In fact, even a fluent review or analysis of the works of V. A. Andrunakievich shows that the results obtained by him reveal different possibilities of the devel-

[*] **In memory of Yurii Ryabukhin (08.02.1939 – 07.03.2019).** *"We publish one of the latest articles by algebraist Yurii Ryabukhin, full member of the Academy of Sciences of Moldova and one of the most brilliant disciples of academician Vladimir Andrunachievici (in transcription from Russian: Andrunakievich). This survey was published, in Russian, in the book "Academicianul Vladimir Andrunachievichi: Bibliografie", Institutul de Matematică şi Informatică, Chişinău, 2009.*
At the 4th Conference of Mathematical Society of the Republic of Moldova, held in 2017 and dedicated to the centenary of V. Andrunachievici, Yurii Ryabukhin devoted a plenar lecture (based on the mentioned article) to the major contribution of academician Vladimir Andunachievici to the foundation of the Institute of Mathematics and Computer Science (IMCS) and to the creation of Schools of Contemporary Mathematics of Moldova.
In 2018 this overview, translated in English, was posted on the IMCS website. We consider that this text should be brought back into the circuit of valuable mathematical works. We publish it as a homage to the algebraist Yurii Ryabukhin, which we failed to do during his last year of life, when he managed to celebrate his 80th birthday. We are grateful to Victor Shcherbacov and Elena Cojuhari, who translated in English this survey.
Yurii Ryabukhin has been a member of the Editorial Board of BASM since its inception. His passing away is an irretrievable loss of mathematical community in the Republic of Moldova and in the whole space of the former USSR. Yu. Ryabukhin was buried in Tiraspol."
Editorial Board

opment of the structural theory of algebras, starting with the foundations of this theory and continuing with the study of the varieties of close algebraic systems.

Moreover, the ideas even of the first works of V. A. Andrunakievich continue to work even now, and many of the results obtained by him are reflected in monographs, including the most recent ones.

All this allows us to assert that V. A. Andrunakievich was one of the world's leading experts in the structural theory and theory of radicals. At the same time, many working statements and theorems proved by V. A. Andrunakievich, have already become classical and understandable to all those interested in construction or description of algebras, which we try to "prove" or explain. We will consider associative algebras over some field F. Such algebras as the algebra $F[t]$ of polynomials and algebras of series, algebras of linear transformations of spaces (over the field F), algebra $[F]_n$ of matrices we consider as the well-known ones. In fact, algebras are very diverse and we can, for example, take into account more specialized algebras of triangular or Block-triangular matrices, and in addition to fields, allow skew-fields to be included (for example, the skew-field of quaternions) in which nonzero elements form a group of invertible elements, but multiplication is not commutative.

1. Structural theory and theory of radicals of algebras

In describing a finite-dimensional algebra A, one can apply its embeddings or homomorphisms into the algebras $[\Phi]_n$ of matrices. In addition to ideals (as kernels of homomorphisms), one-sided ideals also arise: right ideals as sums of principal right ideals of the form $aA = \{ax \mid x \in A\}$ and left ideals.

Due to the finite dimensionality, in the algebra A, the **classical radical** $rad(A)$ is constructed as the largest of ideals N that are nilpotent algebras, that is, such that $x_1x_2\dots x_n = 0$ for all $x_i \in N$ and some natural number $n \geq 2$. Passing to the quotient algebra $\overline{A} = A/rad(A)$ we get that $rad\overline{A} = 0$, and, on the other hand, if $R = rad(A)$, then, of course, $rad(R) = R$.

In classical structural theory of finite-dimensional algebras (Wedderburn, Molin, A. I. Mal'tsev and many others), the semisimple algebras Q for which $rad(Q) = 0$ are completely described, and **simple** algebras with unity element (in which the only non-zero ideal is the whole algebra) turned out to be the leading special case. Namely, simple algebras have the form of algebras $[T]_n$ of matrices over skew-fields T (and if the ground field Φ is algebraically closed as is the field of complex numbers, then $T = \Phi$). After this it turns out that for $rad(Q) = 0 \neq Q$, $Q = \bigoplus_1^m Q_i$ is a finite direct sum of simple algebras $Q_i = [T_i]_{n_i}$. This finishes **a description of all classical semisimple algebras** (up to the description of algebras that are skew-fields).

The finite-dimensional radical algebras $R = rad(R)$ receive only "some description" - in such algebras all elements of $r \in R$ **are nilpotent**, i. e., $r^n = 0$, and therefore nilalgebras are obtained. But from the finite dimensionality it follows that all these nilalgebras are nilpotent, and therefore coincide with their radical. Applying matrix representations, we obtain corresponding descriptions in the form of algebras of block-triangular matrices "with zeros on the main diagonal ". All this is visually portrayed

$$\begin{pmatrix} [T_1]_{n_1} & 0 & 0 \\ 0 & [T_2]_{n_2} & 0 \\ 0 & 0 & \ddots \end{pmatrix} \qquad \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ \cdots & 0 & 0 \end{pmatrix}$$

$$Q = \bigoplus_{1}^{m} Q_i,\ rad(Q) = 0 \qquad R = rad(R),\ R^m = 0$$

in the form of written matrix representations, where the diagonal blocks of the semisimple algebra Q are its minimal ideals, which are always simple algebras. In this case, with some refinement of the specifics of the ground field, the splitting of any algebra A is obtained - it turns out that $A = S + R$, where $R = rad(A)$, the algebra S is a semisimple subalgebra, $S \approx A/R$ (and this is refined in the corresponding classical structural theorems of Wedderburn-Molin-Mal'tsev).

In the course of development of the structural theory, algebraists began to apply weaker restrictions instead of the finite-dimensionality condition. In particular, after Artin and Emma Noether algebraists began to consider the **minimality condition** for left ideals, when strictly decreasing chains of left ideals break off at a finite step (and similarly for right ideals).

It turned out that classical nilpotent radical always exists, and the above matrix representations are obtained, that is, semisimple algebras are again described as **finite** direct sums of matrix algebras over skew fields (i.e., simple algebras), and for radical algebras, as the nilpotent ones, we again obtain representations in the form of algebras of triangular matrices (and so on), but the skew-fields are not necessarily finite-dimensional. At the same time, the corresponding minimal left (or right) ideals began to be actively applied.

We have already noted that the most important case of left ideals is **the principal left ideals** - they have the form $Aa = \{xa \mid x \in A\}$. On the other hand, the algebras of matrices arise as algebras of linear transformations of the corresponding spaces P, but if the space P is infinite-dimensional, then we have to specify the type of the emerging matrices, since their size can turn out to be infinite. All this was taken into account in the series of papers [5-12, 16, 18] of V. A. Andrunakievich, and it turned out that to prove the structural theorems "almost the same as the classical ones", the conditions of minimality for the principal left ideals are sufficient.

Theorem. *Let the algebra A satisfy the chain termination condition*

$$Aa_1 \supset Aa_2 \supset \dots$$

of principal left ideals (that is, the minimality condition only for such "very special" left ideals). Then the following statements hold:

a. *If there are nonzero nilpotent ideals in the algebra A, then its non-zero* ***classical radical*** *$rad(A)$ (or nilradical) also arises , and it is nilpotent, and therefore* *the algebra $R = rad(A)$ has a corresponding "triangular representation" with a finite number of blocks and nonzero diagonal blocks. Moreover, if $A \neq rad(A)$, then $rad(A/rad(A)) = 0$ and the corresponding non-zero algebra $\overline{A} = A/rad(A)$ has no nilpotent ideals except the zero ideal 0.*

***b**. If A is a simple algebra (i.e., $xy \neq 0$ for some elements $x, y \in A$ and the only ideal in A is the algebra A), then A is the algebra of matrices of linear transformations of **finite** rank of the corresponding space over some skew-field T, naturally related to algebra A (and all this is refined), but the size of the matrices is not necessarily finite (as for the algebra of linear transformations of finite rank, but for an infinite-dimensional space $P =_\Phi P$). If the algebra A has a unity element, then $A \approx [T]_n$ for the corresponding skew-field T and some natural number $n \geq 1$.*

***c**. If the algebra A **is semisimple** (that is, it does not have nilpotent ideals), but not simple, then $A = \bigoplus_i Q_i$ is the direct sum of simple algebras (described in **b**) and these algebras Q_i run through **all** minimal ideals of the algebra A, but the number of these minimal ideals Q_i can be infinitely large (i.e., in the corresponding matrix representation it can turn out to be "infinitely many" diagonal blocks - compare with the scheme outlined above). However, if the algebra A has a unity element then $A = \bigoplus_1^m Q_i$ for some natural number $m \geq 2$ and in this situation all $Q_i = [T_i]_{n_i}$ for some $n_i \geq 1$ and T_i.*

When proving these and even more general structural theorems (see [6-12, 15, 16, 18]) V. A. Andrunakievich used, of course, many of the results of other authors, which is proved by the analysis that he had carried out beginning in [5] and continued in the following papers. This is what allowed him to generalize the well-known structural theorems. In particular, the role of **idempotents** - elements $e = e^2 \neq 0$ (see [18]) has been revealed, since the minimal left ideal L of a semisimple algebra Q always has the form $L = Ae$ for some idempotent $e \in L$, and eAe turns out to be skew-field, which the "classics" already noticed as well.

One of the obvious corollaries of this theorem is obtained for the case when in the algebra A **there are no nilpotent elements**, i.e., when $a^2 \neq 0$ also follows from $a \neq 0$. In this situation, of course, the radical is equal to 0, and under the condition of minimality for the principal left ideals, it turns out that A is a direct sum of not necessarily finite number of skew-fields.

The best-known particular case is Dedekind's classical theorem on finite-dimensional commutative algebras, which turn out to be "finite" direct sums of fields (extensions of the fundamental one).

In fact, V. A. Andrunakievich has proved many other structural theorems, since algebraists by this time (the 50s of last century) had already begun to study also the algebras, in which "the classical radical no longer exists", since the nilalgebras (where all the elements are nilpotent) are not necessarily nilpotent.

In connection with this situation, various generalizations of the classical radical arose, and sometimes "radicals opposite to the classical". Therefore, there was a need for a "general theory of radicals", which was created in the works of A.G. Kurosh, the scientific supervisor of the first investigations of V. A. Andrunakievich.

Thanks to the research of V. A. Andrunakievich, the **theory of hereditary radicals** was developed, among which special radicals were allocated by him - these radicals are most often used when proving structural theorems. More about this (and the history of the development of the structural theory of rings and algebras) is given in monographs [44, 62, 65] and in [4-17, 22] by V. A. Andrunakievich, but we note some details and basic ideas that led to special radicals of associative algebras.

Instead of simple algebras, V. A. Andrunakievich proposed to consider the **prime** algebras A in which the inequality $0 \neq xAy$ is always true for nonzero $x, y \in A$. In such algebras **for nonzero ideals** $J(J \triangleleft A)$ **always** $J^2 \neq 0$, since for nonzero ideals B, C we always have $BC = \{\sum b_i c_j \mid b_i \in B, c_j \in C\} \neq 0$. Namely from prime algebras **special classes** of M algebras are constructed such that from $A \in M$ and $J \triangleleft A$ it follows always $J \in M$, and for the prime algebra C that contains the algebra $A \in M$, $A \neq 0$ as an ideal, we always obtain $C \in M$. After this (according to the construction indicated by A.G. Kurosh and Amitsur, who also constructed a general theory of radicals), the **upper radical** S_M defined by the class M is constructed. For the special class M this means that in each algebra A its ideal $S_M(A) = \bigcap\{J \triangleleft A \mid A/J \in M\}$ is constructed as the intersection of all the indicated M-ideals of algebra A. In this case, of course, the indicated algebras A/J are prime, since all algebras from M are prime.

As a result, there arises the **special** (according to V.A. Andrunakievich) **radical** S_M, defined by the given special class M. In this case, always $S_M(A/S_M(A)) = 0$ and always $S_M(S_M(A)) = S_M(A)$ for all radicals in the sense of A.G. Kurosh, but in addition it turns out that when $r = S_M$ for the ideal $J \triangleleft A$ it is always $r(J) = J \bigcap r(A)$. At the same time (according to A.G. Kurosh) for the radical $r = S_M$ we construct the class $\Re(r)$ of all r-radical algebras $R = r(R)$ and the class $\Im(r)$ of all r-semi-simple algebras Q for which $r(Q) = 0$. These classes always determine each other, since for any algebra A the equality

$$\bigcap\{J \triangleleft A \mid S_r(A/J) = 0\} = r(A) = \sum\{R \triangleleft A \mid R = r(R)\}$$

is true. In particular, for a special radical in the S_M-semi-simple algebra Q, the only nilpotent ideal is 0, since in the prime algebra for ideals we always have $J^m = 0 \Rightarrow J = 0$ for $m \geq 2$. Ideals of the algebra Q also turn out to be semisimple algebras, and the ideals of radical algebras are radical, by specifics of S_M.

Remark 1. A non-zero prime algebra Q can have many different non-zero ideals. The most famous example is the algebra $\Phi[t] = A$ of polynomials and all of its nonzero ideals, always having the form $gA = \{gf \mid f \in A\}$ (they all are integral domains, i.e., algebras without divisors of zero - if $x \neq 0 \neq y$, then $xy \neq 0$). However, the prime finite-dimensional algebra $Q \neq 0$ is a simple algebra with unity and has the form $Q = [T]_n$ for some skew-field T and some natural number $n \geq 2$ if Q is not a skew-field. And in the algebra T, which is a skew-field, there are no nonzero one-sided ideals (right or left), since $Tq = T = qT$ for $0 \neq q \in T$.

Remark 2. Among the special radicals there is the smallest one - this is the **lower nilradical** $b = S_\Pi$, constructed from the class Π of all prime algebras. In this case semisimple algebras are exactly algebras without nonzero nilpotent ideals. **The upper nilradical** k is a special one too, for which all nilalgebras are radical, i.e., algebras consisting only of nilpotent elements. At the same time various **nilradicals** arise, i.e., such radicals s in the sense of A.G. Kurosh, that $b(A) \subseteq s(A) \subseteq k(A)$ for all algebras A. In this case many nilradicals are special, i.e., the corresponding class $\Pi \bigcap \Im(s)$ of prime s -semi-simple algebras turns out to be a special class of algebras. In particular, **the locally nilpotent radical** l is a special one too, for which all locally nilpotent algebras are radical (that is, algebras in which all finitely generated subalgebras are nilpotent).

It follows from the above that special radicals are very diverse, and according to the natural order it turns out that $b \leq l \leq k$. At the same time, according to Remarks 1 and 2, for finite-dimensional algebras A we obtain the classical radical $rad(A) = b(A) = k(A)$; i.e., in this case all nilradicals coincide. Moreover, there are other special radicals that coincide in the finite-dimensional case with the classical (nilpotent) radical, many of which are indicated or determined by V. A. Andrunakievich. For example, if a special class M consists of only algebras with unity element, then $M \subseteq \Pi_1$ for the class of all simple algebras with unity element and all classes $M \subseteq \Pi_1$ are always special. Of course, the radical S_M for $M \subseteq \Pi_1$ coincides in finite-dimensional algebras with the classical radical (according to Remark 1), and many of the special radicals have the same property - this was noticed by V. A. Andrunakievich in [14, 17, 22] and continued in the works of many other authors, including the first monograph [62] on the theory of radicals. On the other hand, an algebra Q is **subdirectly irreducible** if it has a smallest nonzero ideal C, called the **heart.**[1] The subdirectly irreducible algebras with idempotent heart C form a special radical $S_{\Pi_0} \leq S_{\Pi_1}$, since the heart C is a simple algebra (but not necessarily having the unity element, see the theorem). It is easy to see that all locally nilpotent algebras S_{Π_0} are radical and therefore it turns out that $L \leq S_{\Pi_0} \leq S_{\Pi_1}$. It is more complicated to see that the following is true:

Proposition. *If R is an S_{Π_0} radical algebra satisfying the maximality condition for ideals (that is, strictly increasing chains of ideals break off at a finite step), then the algebra R is **nilpotent**, i.e., $R^m = 0$ for some natural number $m \geq 2$.*

In particular, under this condition for break, locally nilpotent algebras are nilpotent, and therefore the special radicals S_{Π_0}, l, S_{Π_1} coincide in finite-dimensional algebras with the classical (nilpotent) radical.

This is one of the well-known "working statements" of V. A. Andrunakievich, and if we consider (following V. A. Andrunakievich) **annihilators**, similar results are obtained under weaker restrictions. That is why the radical S_{Π_0} is called the radical of Andrunakievich (see the monographs [63, 65]), and various special radicals lead to various structural theorems under "comparatively weak restrictions". More details can be found in the works of V. A. Andrunakievich [8-19] and in monographs [44, 65], where there are many results, theorems and "working statements"

[1]It was translated as "core" in the English version of [17,22] in American Mathematical Society Translations: Series 2, 1966, v.52, pp.95-149.

of V. A. Andrunakievich. It is very surprising, but many of these "working statements" can be very simply proved and are very often used (even in the works of many other authors, and sometimes, after some refinement, in arbitrary not necessarily associative algebras). The most famous is

Lemma. *(Lemma of V. A. Andrunakievich (see the monograph [65], published in 2004).) Let $J \lhd B \lhd A$, i.e., J is an ideal of the algebra B, and B is an ideal of the algebra A. Then:*

a. *If J_A is the ideal of the algebra A generated by J, then $J_A^3 \subseteq J$.*

***b**. If the quotient algebra B/J is a semi-prime one (i.e. without nonzero nilpotent ideals), then J is an ideal of the algebra A.*

Indeed, $J_A = J + AJ + JA + AJA$ and therefore $J_A^3 \subseteq BJB \subseteq J$, that is, **a** is true. But then **b** is also true because of the specifics of algebras without nilpotent ideals.

This is used to prove the equality $b(B) = B \bigcap b(A)$ for the lower nilradical B (see Remark 2) and for all special or supernilpotent radicals. In the case under consideration (associative algebras over a field) it turns out that the hereditary radical r is either **supernilpotent** i.e., all nilpotent algebras are r-radical, i.e. $r \geq b$, or **sub-idempotent**, i.e., all r-radical algebras $R = R^2$ and this is equivalent to the fact that all nilpotent algebras turn out to be r-semisimple. Moreover, all special radicals are supernilpotent and the sub-idempotent radicals are opposite to supernilpotent ones. At the same time, "there is a duality for hereditary radicals", introduced into consideration by V. A. Andrunakievich (but more on this later), and the corresponding sub-idempotent radicals are also "very often" used together with special radicals to prove structural theorems.

Theorem. *Consider only the hereditary radicals r, i.e., such that $r(B) = B \bigcap r(A)$, when B is an ideal of the algebra A. Then:*

a. *Among the radicals s such that for a given radical r the equalities $r(A) \bigcap s(A) = 0$ hold for all algebras A, there always exists a **largest** radical r'. Moreover, the radical $s = r'$ is dual, that is, $s = s'' = (s')'$. The class $\Re(r')$ of all r'-radical algebras coincides with the class of all strongly r-semi-simple algebras, i.e. the algebras Q such that $r(\overline{Q}) = 0$ for all homomorphic images of $\overline{Q}$ of the corresponding algebra $Q = r'(Q)$. The equality written above can be rewritten in the form $r(s(A)) = s(r(A)) = 0$ by symmetry.*

b. *The largest sub-idempotent radical is the hereditarily idempotent radical f, i.e., algebras $R = f(R)$ are algebras such that $F = F^2$ for all ideals F of the algebra R. Therefore, for a supernilpotent radical s, the dual radical s' is always sub-idempotent, i. e., equality $s' \leq b' - f = f''$ holds (according to **a**), since $s \geq b$ for the lower nilradical $b = S_\Pi$. For sub-idempotent radicals r the dual radical r'' is always supernilpotent and **is special** - the equality $r' = S_{\Pi(r)}$ holds for the special class $\Pi(r)$ of all subdirectly irreducible algebras with an idempotent core $C = r(C)$. In other words, $r' = S_{\Pi(r)} \geq S_{\Pi_0} = a = a'' \geq b'' \geq b$ (but $a \neq b$), i.e., the radical a is the smallest dual supernilpotent (and special) radical, since $a = f'$.*

All of the above is proved in the papers of V. A. Andrunakievich [17, 19, 20, 21, 22] and his doctoral dissertation, and then applied to prove a number of structural theorems related to the corresponding sub-idempotent radicals, which is reflected in the monograph [44]. The results obtained were applied or generalized by many authors, as is shown in the monographs [62-65], where the lattices of radicals were studied and many of the results of V. A. Andrunakievich (and sometimes of his pupils too) are given with many details. Moreover, it turned out that the ideas of V. A. Andrunakievich and his “working statements” also work in “not necessarily associative algebras”.

2. Additive theory of ideals

One of the most famous results is the Fundamental Theorem of Arithmetic - the natural number is always represented in a unique way in the form $n = p_1^{k_1} \dots p_r^{k_r}$ of products of powers of prime (pairwise distinct) numbers. Translating this theorem into the language of ideals $mZ = \{mz \mid z \in Z\}$ of the ring Z of integers, we find that there exist unique representations $nZ = p_1^{k_1} Z \bigcap \dots \bigcap p_r^{k_r} Z$ of the corresponding ideals in the form of intersection of **primary** ideals - the ideals of the form $p^k Z$ (for prime numbers p). Moreover, for the ideal nZ its radical or root $\sqrt{nZ} = \{z \in Z \mid z^m \in nZ \textit{ for } m \geq 1\}$ is constructed, and for the primary ideal $p^k Z$ its radical is the **unique** maximal ideal containing $p^k Z$, and this is the ideal pZ for the corresponding prime number p. It can be seen that it always follows from $xy \in pZ$ that $x \in pZ$ or $y \in pZ$, and if $xy \in p^k Z$ and $y \notin p^k Z$, $x \neq 0$, then $x^s \in p^k Z$ for some $s \geq 1$ - this characterizes the primary ideals and their radicals. It is not less clear that $\sqrt{nZ} = \bigcap_1^r p_i Z$.

It turned out that similar results are obtained for commutative rings (or algebras) with the maximality condition for ideals (the best known example, apart from the ring Z, is the algebra of polynomials of a finite number of variables, according to Hilbert's theorem). In this ring A for an ideal B the radical $\sqrt{B}$ consisting of $a \in A$ such that $a^m \in B$ for some $m \geq 1$ is always constructed. If we take into account products of ideals, then, thanks to the maximality condition, it turns out that always $(\sqrt{B})^m \subseteq B$ for some sufficiently large number $m \geq 1$. After this, there arise **prime** ideals P, i.e., such that $B \subseteq P$ or $C \subseteq P$ follows from $BC \subseteq P$ (these are analogues of the prime ideals pZ), and then **primary** ideals of Q, for which $\sqrt{Q}$ is a prime ideal (and this is an analogue of the primary ideals $p^k Z$). According to Emmy Noether the following theorems are true:

Theorem. *(**Existence theorem.**) For an ideal B, there always exists a representation in the form of an intersection $\bigcap_1^r Q_i$ of a finite number of primary ideals Q_i.*

Theorem. *(**Intersection theorem.**) The intersection of primary ideals with the same radical P is a primary ideal Q with the same radical $P = \sqrt{Q}$.*

Theorem. *(**Uniqueness theorem.**) For an ideal B, there exists an irreducible representation $B = \bigcap_1^r Q_i$ in the form of intersection of primary ideals, i.e., such that* *all* $\bigcap_{i\neq j} Q_i \neq B$. *The irreducible representation is unique; therefore, the set of prime ideals* $P_i = \sqrt{Q_i}$ *is also unique, for which the unique and irreducible representation* $\sqrt{B} = \bigcap_i P_i$ *for radicals is also obtained.*

These theorems are fundamental for Noetherian primarity, and in fact, many other beautiful "work" statements about primary and simple ideals are also obtained. After that, a situation appeared that resembled something that happened in the structural theory: the search for "generalizations" of classical Noetherian primarity to the noncommutative case began, but under the condition of maximality for the ideals (or one-sided ideals) of the rings under consideration. However, the necessary generalizations were not obtained for a relatively long time (about fifty years), then "tertiarity" arose (in the works of the French algebraists Léonce Lesieur and Robert Croisot) as one of the possible generalizations, and numerous "almost generalizations" arose either not coincident in commutative rings with Noetherian primarity or such that one of the "defining" theorems mentioned above was violated. V. A. Andrunakievich has joined the search for possible generalizations, and then his disciples (I.M. Goyan was one of the first) too. After clarifying the statement of problems, V. A. Andrunakievich explained the emerging "difficulties" (with the active help of pupils) - a bit unexpectedly it turned out that the following is true.

Theorem. *When considering the generalizations of classical primarity to the non-commutative case, there is only one generalisation – primarity, for which the existence theorem, the intersection theorem and the uniqueness theorem hold. In this case, the ideals* Q *arise as primary ideals (in corresponding already non-commutative rings), that are irreducible with respect to intersection (i.e., such that* $B \supseteq Q$ *or* $C \supseteq Q$ *follows from* $B \cap C \supseteq Q$*).*

This was proved in a series of works by V. A. Andrunakievich [31-35] and "everything explained". Moreover, in the definition of irreducible ideals, only the specificity of the lattice of ideals is just taken into account (and if the lattice satisfies the maximality or minimality condition, then this already allows us to prove the "Existence theorem"). Therefore, in the "final" paper [35] (published in "Izvestia of the Academy of Sciences of the USSR"), an analogous theorem was proved exactly for lattices. In this case the right or left quotients (of ideals or elements of the emerging multiplicative lattices) were the main tool, according to Andrunakievich's formulation of the way of solving the problem. We note that the corresponding "quotient" or conditions for the termination of chains of quotients have already been applied by V. A. Andrunakievich in the proof of structural theorems (see, for example, [15, 18, 26, 27], and in more detail - a monograph [45]).

At the same time, it turned out that an appropriate "primary theory" can be constructed for many algebraic systems (for subgroups of groups, subsemigroups of semigroups, submodules of modules, etc.). On the other hand, restrictions can be weakened; for example, only the ideals of an algebra with the maximality condition for ideals can be considered, and the requirements of "defining theorems"can be weakened. As a result, various generalizations of the "diprimarity" type are obtained only for two-sided ideals (and in the commutative case the classical Noetherian primarity is obtained), and sometimes (for stronger restrictions), generalizations of classical Artin-Rees theorems are obtained. In this area, I. M. Goyan - the pupil of V. A. Andrunakievich (there are also others) worked most actively and works. He considered generalizations that do not necessarily coincide in the commutative case with classical primarity.

3. Modules, one-sided ideals and radicals

It has already been noted that in describing the structure of algebras **representations** are often used, i.e., homomorphisms of algebras into the algebras $\pounds(M)$ of linear transformations of spaces, that is, $\rho : A \to \pounds(M)$. If the space is finite-dimensional, then matrix representations are obtained, since $\rho(A)$ turns out to be a subalgebra of the algebra $[\Phi]_n$ of matrices isomorphic to the algebra $\pounds(M)$ (see the theorems in the first section). Therefore, one can speak simply about "matrix representations", and in the general case, but in this case we obtain matrices that have not necessarily a finite size. Some possibilities and refinements of such representations were considered in the papers of V. A. Andrunakievich [6, 9, 11] and the works of many other authors.

If the representation $\rho : A \to \pounds(M)$ is given, then the space M turns into the **module** M_a (right), according to the rule $xa = x\rho(a) \in M$. This means that the element $xa \in M$ is defined for all $x \in M$, $a \in A$, and the resulting multiplication is connected in the module by the laws $x(ab) = (xa)b$, $x(a + b) = xa + xb$, $(x + y)a = xa + ya$ as for spaces, but with replacing of multiplication by elements of a field Φ by multiplication by elements of algebra A. On the other hand, according to the indicated laws, if the module M_A is given, then according to the rule $x\rho(a) = xa \in M$ the corresponding representation $\rho : A \to \pounds(M)$ is obtained. In other words, the definition of a module is equivalent to specifying a representation, and we can assume that the algebra A has the unit $e \neq 0$ (since it is always possible to "attach" one) - in this case it turns out that $\rho(e) = \varepsilon_M$, i.e., $xe = x$ for all $x \in M$. By symmetry, with the help of multiplication $x \to ax$ the left modules ${}_AM$ are determined and the corresponding representations, too.

From the above it turns out that for the algebra A **kernels** $Ker\rho = \{a \in A \mid xa = 0 \; for \; x \in M_A\} = (0 : M)_A$ runs through all ideals of the algebra A. The corresponding factor-algebra $A/(0 : M_A) = \overline{A}$ already is **isomorphically** embedded in the algebra $\pounds(M)$ - the rule $x\overline{a} = xa$ transforms the module M_A to the module $M_{\overline{A}}$ with annihilator $(0 : M)_{\overline{A}} = 0$, i. e., **exact** representation of the algebra $\overline{A}$ is constructed. Considering submodules in the module M_A one can see that for the cyclic submodule $xA = \{xa \mid a \in A\}$, the **annihilator** $(0 : x)_A = \{a \in A \mid xa = 0\}$ is a **right ideal** of the algebra A, and $xA \approx A/(0 : x)_A$ (when algebra A has a unity). By "symmetry"we obtain a relationship between the left ideals of the algebra A and the annihilators of the left A-modules. In addition, one can take into account that the algebra A naturally converts to the modules A_A and ${}_AA$, and the submodules of the module A_A are right ideals of the algebra A, and the submodules of the module

${}_AA$ are left ideals in A (and so on).

Due to these relationships that algebras are often described with the help of appropriate modules (or representations). In particular, one can take into account **simple or irreducible modules** M_A for which M is the only submodule except the zero one, with $xa \neq 0$ for some $x \in M$, $a \in A$, and therefore $M_A = xA$ for each $x \neq 0$. At the same time, it turns out that for an irreducible module its algebra $E = End(M_A)$ of endomorphisms is a skew-field (that is, it coincides with the group of automorphisms) according to the classical Schur's lemma. After that we get a description of the primitive algebras, i.e., the algebras Q which have exact irreducible representation or exact simple module M_Q (this is the Jacobson "density theorem" that gives a description of the algebra Q as a "very special" subalgebra in the algebra of linear transformations of the space $M_Q =_E M$ over the corresponding skew-field E).

As a result, there arises one of the most special radicals – the Jacobson **radical** j, naturally associated with irreducible modules. In this, V. A. Andrunakievich noticed that the class π of primitive algebras is special, and therefore the radical $j = S_\pi$ is **special** too. At the same time, we obtain a well-known description of the radical $j(A)$ as the intersection of the kernels of all irreducible representations of the algebra A or of the annihilators $(0 : M)_A$ of irreducible modules. In fact, it turned out that the class of irreducible modules is one of the special classes of modules (analogues of special classes of algebras), which was shown by V. A. Andrunakievich by introducing **prime** modules – such modules M_A that $xa \neq 0$ for some $x \in M$, $a \in A$ and $(0 : M)_A = (0 : N)_A$ for any nonzero submodule N_A of the module M_A. In this, special classes of modules consist of prime modules (compare with special classes of algebras).

Theorem. *Prime algebras are precisely algebras that have an exact prime module. Therefore, for the lower nilradical of* b *and any algebra* A*, the radical* $b(A)$ *coincides with the intersection of annihilators of the prime modules* M_A*. A similar construction is obtained* ***for all*** *special radicals by means of the corresponding special class of modules (among which there is the class of irreducible modules, as it was already noted). Moreover, analogous representations are obtained for all hereditary radicals (and then for all radicals), using appropriate classes of modules.*

The arising representations of radicals were obtained in a series of works by V. A. Andrunakievich (together with his pupil and co-author) [24, 28, 29, 30]. At the same time, it turned out that radicals of algebras are represented as the intersection of very special one-sided ideals – it suffices to note that $(0 : M)_A = \bigcap\{(0 : x)_A \mid 0 \neq x \in M\}$ for right modules, which is indicated by the corresponding right ideals $(0 : x)_A$. In particular, for the radical $j = S_\pi$, the representation $j(A)$ is obtained as the intersection of the right ideals $(0 : x)_A$ for elements of irreducible A modules. If the algebra A has a unity, then the right ideals $(0 : x)_A$ (from the indicated representation) are all maximal right ideals of the algebra A. Due to this, we obtain structural theorems on primitive algebras, including simple algebras with unity.

Representations of radicals with the help of respective classes of modules are considered in great detail in the monograph [65], where many other results of V. A. Andrunakievich (and his pupils) are mentioned. On the other hand, in the last series of papers of V. A. Andrunakievich [56-60], radicality and primitivity "modulo" a right ideal P have been considered – for $P = 0$, from the proved general theorems of V. A. Andrunakievich the well-known structural theorems on primitive algebras and simple algebras with unity are obtained.

4. Radical algebras and adjoint multiplication

In the proof of structural theorems, as a rule, only semisimple algebras having zero radicals are sufficiently well described (under appropriate restrictions). It is well illustrated for special radicals and described in great detail in the monograph [44] based on the papers of V. A. Andrunakievich. However, even in the first papers, V. A. Andrunakievich also proved a number of theorems about radical algebras in the sense of radical j, that is, algebras $R = j(R)$. Moreover, thanks to the research carried out by V. A. Andrunakievich in his Ph.D. thesis, a certain "parallelism" arose between such radical algebras and skew-fields – algebras Q for which $Q \setminus 0$ is a multiplicative group of invertible elements. At the same time, there arose various radical algebras, and in the course of subsequent studies, the "manifolds of radical algebras" also arose.

By the beginning of these studies (1946-1947), the embeddings of algebras without zero divisors into skew-fields were already actively used (similarly to the embeddings of the algebras $\Phi[t]$ of polynomials into the fields $\Phi(t)$ of fractions of the form fg^{-1}, where f, $g \in \Phi[t]$, and $g \neq 0$) as were embeddings of semigroups into groups. On the other hand, the "circle operation", or **adjoint multiplication**, given by the rule $x \circ y = x + y - xy$ had also been used. It is almost obvious that for any algebra A we obtain an "adjoint" monoid $A(\circ)$, where zero plays the role of the unity according to the equalities $x \circ 0 = x = 0 \circ x$. Moreover, we have already "noticed" that $R = j(R)$ if and only if $R(\circ)$ is a group. It is this operation that V. A. Andrunakievich used to prove theorems on embeddings of algebras into radical algebras, which led to the construction of various and very interesting radical algebras.

According to V. A. Andrunakievich, the element $c \in A$ is radical if $c \circ c^* = 0 = c^* \circ c$ for an element $c^* \in A$ (which is the quasi-inverse to the element c). The element $c \in A$ is **semi-radical** if for all $a, b \in A$ the equalities $a \circ c = b \circ c$, $a = b$, $c \circ a = c \circ b$ are equivalent. An algebra A is semi-radical if all its elements are semi-radical, and according to the already noted, algebra R is radical, i.e., $R = j(R)$, if and only if all elements of R are radical. In particular, it turns out that all radical algebras are semi-radical, and there arises the problem of embedding of semi-radical algebras into the radical ones (analogous to the problem of embedding into the skew-fields).

Theorem. *There exist semiradical algebras that are not subalgebras of radical algebras (a relevant example is analogous to the classical example, due to A.I. Malcev, of* *an algebra without zero divisors that is not embeddable into a skew-field). However, if a semiradical algebra satisfies the condition*

$$(^*) \quad d, g \in R \Rightarrow \exists x, \quad y \in R \mid d \circ x = g \circ y,$$

then the algebra R is a subalgebra of an algebra $\hat{R} = R \circ R^*$ consisting of adjoint fractions of the form $a \circ b^*$ with the equality rule $a \circ d^* = c \circ g^* \Leftrightarrow d \circ x = c \circ y$ (see (*)). In this, operations with adjoint fractions are performed according to the rules:

$$b_1 \circ z = a_2 \circ t \Rightarrow (a_1 \circ b_1^*) \circ (a_2 \circ b_2^*) = (a_1 \circ z) \circ (b_2 \circ t)^*,$$

$$c = d \circ x = g \circ y \Rightarrow a \circ d^* + b \circ g^* = (a \circ x - c + b \circ y) \circ c^*,$$

$$\forall \alpha \in \Phi \mid \alpha(a \circ b^*) = (\alpha(a - b) + b) \circ b^* \in R \circ R^*,$$

then it should be always taken into account that $xy = x + y - x \circ y$. In this situation, the ***algebra*** *$\hat{R} = R \circ R^*$ is* ***radical****, since the equalities $(a \circ b^*)^* = b \circ a^*$ are always true. The embedding is performed according to the rule $r = r \circ 0^* = (r \circ t) \circ t^*$. Moreover, for any radical algebra Q each homomorphism $\varphi : R \to Q$ of algebras* ***always and uniquely*** *extends to the homomorphism $\hat{\varphi} : \hat{R} \to \bigcup$ of radical algebras now. Therefore, the radical algebra $\hat{R}$ is uniquely defined to within an isomorphism which fixes R.*

In the course of the proof of this theorem, V. A. Andrunakievich noted that algebras can be considered as “new” algebraic systems in which, instead of multiplication, the adjoint multiplication is considered, due to which the adjoint monoid $A(\circ)$ is obtained. In this, the distributivity laws are rewritten in the “more complicated” form $x \circ (y - t + z) = x \circ y - x \circ t + x \circ z$ and similarly $(y - t + z) \circ x = y \circ x - t \circ x + z \circ x$. This is what led to the above construction $R \subseteq \hat{R} = R \circ R^*$ for embedding algebras into radical algebras under the indicated restrictions.

As a simple corollary, it turns out that the commutative algebra K without zero divisors and without unity is **always** isomorphically embedded into the radical algebra $\hat{K} = K \circ K^*$, since for K the conditions of (*) and semiradicality are satisfied. The constructed algebra $\hat{K}$ is a subalgebra of the arising field $Q_{cl}(K)$ of fractions of the algebra K. This leads to a variety of radical algebras that are algebras without zero divisors. On the other hand, if N is a **nilalgebra**, i.e., for $a \in N$ we have $a^{n+1} = 0$ for some natural number $n = n(a)$, then the equalities $a \circ a^* = 0 = a^* \circ a$ are also obtained when $-a^* = a + a^2 + \ldots + a^n$. Therefore, all the nilpotent elements are radical and all nilalgebras are radical algebras. After this, it can be noticed that in the finite-dimensional algebras the radical $j = S_\pi$ coincides with the classical nilpotent radical, since all finite-dimensional radical algebras are nilalgebras. Using the construction from the theorem, more facts can be proved.

Proposition. *For the algebra $K = tA = \langle t \rangle$ of polynomials with zero free term, the radical algebra $J = \langle t \rangle^* = \hat{K} = K \circ K^*$ is constructed (according to the theorem). In this:*

a. *An algebra R is radical if and only if for each* $r \in R$ *there exists a homomorphism* $\varphi_r : J \to R$ *for which* $\varphi_r(t) = r$*, which takes into account the specificity of algebras* $K = \langle t \rangle$ *and* $J = \langle t \rangle^*$ *(already radical).*

b. *For algebra J in the field* $Q_c l(K)$ *of fractions the following equality is true:* $J = \{a(e-b)^{-1} \mid a, b \in K\}$*, because* $c \circ b^* = (c-b)(e-b)^{-1}$ *for all* $c, b \in K$ *and unity* $e \notin K = tA$*. Therefore, only the powers* $J^{n+1} = t^n J$ *of the algebra J that form a strictly decreasing chain are nonzero ideals of J.*

c. *Radical algebra R is a nilalgebra if and only if the algebra J* ***is not*** *isomorphically embedded into R. In particular, if the radical algebra R is a subalgebra of a finitely generated algebra, and the main field* Φ *is uncountable (as the field of real numbers is), then R is a nilalgebra, since the dimension is* $dim J \geq \mid \Phi \mid$*.*

The statements **a, b** are in fact proved in the first papers of V. A. Andrunakievich [1-4] and are a simple consequence of the above theorem. It is not less obvious that the algebra J is infinite dimensional, and applying **a, b**, we see that if the algebra J **is not** embedded isomorphically into the radical algebra R (for example, when the algebra R is a finite dimensional one), then for $r \in R$ and the related homomorphism φ_r the following inclusion is always true: $J^n \subseteq Ker\varphi_r$, and therefore, $r^n = 0$, i.e. R is a nilalgebra. Moreover, applying the well-known basis of the field $Q_c l(K) = \Phi(t)$, we obtain the linearly independent set $\{t(e - \alpha t)^{-l} \mid \alpha \in \Phi\} \subseteq J$, and therefore always $dim J \geq \mid \Phi \mid$. It remains to note that all finitely-generated (and countably generated) algebras have at most countable dimension.

This result was "rediscovered" by other authors 10 years after the work of V. A. Andrunakievich. After 25 years, attention was paid to the fact that radical algebras form a "variety [2] of algebras" – with the additional operation $x \to x^*$ of taking a quasi-inverse. On the other hand, in the joint papers of V. A. Andrunakievich, the construction of the arising theory of the variety of radical algebras was continued, which led to the construction of very interesting radical algebras that are "free in some variety". In fact, we did not notice much else (for example, the theory of algebras without nilpotent elements), but it is already clear that the ideas of even the first works of V. A. Andrunakievich "continue to work".

5. Conclusions and comments

Many of the results of V. A. Andrunakievich have already been included in monographs, beginning with the monograph [62] of Divinsky (published in Canada) and concluding with the last monograph [65] on the theory of radicals. The most well-known are the results of research that are included in the doctoral thesis of V. A. Andrunakievich (defended in 1958 at Moscow State University). According to these researches [5] and the works [5-22], it turned out that it is just special and subidempotent radicals that lead to a variety of structural theorems. Since that time (early 60s of the last century) V. A. Andrunakievich has already become a

[2] Varieties formerly were called manifolds in some of the literature.

leading world specialist in structural theory and the theory of radicals of algebras or rings (associative ones). Moreover, it turned out that the ideas and working statements of V. A. Andrunakievich are very useful in similar domains of algebra. This is reflected, for example, in monographs [63, 65] and in a number of papers of algebraists from Novosibirsk, where a number of structural theorems on alternative and Jordan algebras were proved with the help of the "Andrunakievich lemma", the "Andrunakievich radical" and "Andrunakievich varieties, similar to the associative ones".

In fact, after the investigations of V. A. Andrunakievich and his pupils, the theory of special radicals and torsions has been developed by many authors for such algebraic systems as semigroups with zero, nearrings, or even multi-operator algebras. For semigroups with zero, this was done by the V. A. Andrunakievich pupil R.S. Grigor (Florya), and continued by a number of Hungarian and German algebraists. For more general systems, the representations of radicals with representations in papers [28, 29, 30] were very useful, which is also reflected in the monographs [64, 65].

By this time (early 70s of the last century), due to V. A. Andrunakievich's typical care of people a sufficiently large number of pupils were ensured by ideas and work, and many of the results obtained are reflected in the monograph [44], where the theory of hereditary radicals has been developed "in almost all good enough categories". Thanks to the general theory, it has been shown that exactly special radicals are most naturally connected with M ideals and structural theory (see the first section). This interconnection is a bit weaker for supernilpotent or "weakly special" radicals. This was the continuation of the works of V. A. Andrunakievich [20-23] and allowed constructing theory of special radicals in semigroups with zero (where, of course, it is necessary to take into account the distinctive properties of semigroups too). After the "duality theorems" of V. A. Andrunakievich (see the first section and the works [19-23, 25, 39-44]), the study of "lattices of radicals" continued (by many authors). In addition, in algebras with sufficiently weak "finiteness conditions", the supernilpotent radicals coincide with the special ones, according to V. A. Andrunakievich. However, in the general case "there are a lot" of supernilpotent but not special radicals, which is reflected in the monographs [44, 64, 65] and was the solution to the problem set in the monograph [62] in connection with the works of V. A. Andrunakievich.

The ideas of the first researches of V. A. Andrunakievich about radical algebras (from his Ph.D. dissertation defended in 1947), which are the beginning of the structural theory of radical algebras, continue to work. The interest in this direction increased significantly when it was found that "radical algebras form a variety" (and the free radical algebra was constructed by the English algebraist P. Cohn in view of the first works of V. A. Andrunakievich and his works on embeddings algebras into skew fields). In this way, very interesting "radical algebras that are free in some variety" also appeared together with the concrete variety of radical algebras considered in the joint papers of V. A. Andrunakievich [51-55], which is reflected in [61] as well.

Naturally, the joint works on “radicality relative to right ideals”, where generalizations of classical structural theorems are also obtained, are associated with the above-mentioned as well, as it is noted in the works [41-44, 47, 48, 56-59]. On the other hand, the study of some other special radicals has also been continued – in this way a structural theory of algebras without nilpotent elements has been constructed, where the variety of strictly regular algebras also appeared [36-39, 41, 46-52]. Studies of radical algebras and manifolds are still far from complete and can be continued. A number of problems related to the structural theory of radicals of rings or algebras has been solved in the course of the investigations already carried out. However, problems still exist that are related to the locally nilpotent radical that arose in the investigations of V. A. Andrunakievich (see [15, 25]) and reflected in the monographs [63, 65]. Let’s hope that these problems will be solved by his pupils or pupils of his pupils.

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