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Polynomial differential systems with explicit expression for limit cycles

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Abstract. In this paper we give an explicit expression of invariant algebraic curves of multi-parameter planar polynomial differential systems of degree nine, then we prove that these systems are integrable and we introduce an explicit expression of a first integral. Moreover, we determine sufficient conditions for these systems to possess two limit cycles: one of them is algebraic and the other one is shown to be non-algebraic, explicitly given. Concrete examples exhibiting the applicability of our result are introduced.

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1 Introduction

An important problem of the qualitative theory of differential equations [6, 12] is to determine the limit cycles of a system of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = P(x,y), \\ \\ y' = \dfrac{dy}{dt} = Q(x,y), \end{cases} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials in the variables x and y. Here, the degree of system (1) is denoted by $n = \max\{\deg P, \deg Q\}$. In the literature equivalent mathematical objects to refer to these planar differential systems appear as a vector field

$$\chi = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

A limit cycle of system (1) is an isolated periodic solution in the set of all its periodic solutions of system (1), and it is said to be algebraic if it is contained in the zero level set of a polynomial function [1, 10]. In 1900 Hilbert [9] in the second part of his 16th problem proposed to find an estimation of the uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree, and also to study their distribution or configuration in the plane $\mathbb{R}^2$, even more difficult problem is to give an explicit expression of them [2, 4]. This was one of the main problems in the qualitative theory of planar differential equations in the 20th century. In [3, 8] examples of explicit limit cycles which are not algebraic are given. To distinguish when a limit cycle is algebraic or not, usually, it is not easy. Thus, the well known limit cycle of the van der Pol differential system exhibited in 1926, was not proved until 1995 by Odani [11] that it was not algebraic. The van der Pol system can be written as a polynomial system (1) of degree 3, but its limit cycle is not known explicitly.

2 Some useful notions

Let us recall some useful notions.

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non-constant C^1 function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω, which is constant on the trajectories of the system (1) contained in Ω, i.e. if

$$\frac{dH\left(x,y\right)}{dt}=\frac{\partial H\left(x,y\right)}{\partial x}P\left(x,y\right)+\frac{\partial H\left(x,y\right)}{\partial y}Q\left(x,y\right)\equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant.

Since for such vector fields the notion of integrability is based on the existence of a first integral [5, 7], the following question arises: Given the polynomial differential system (1), how to recognize if this polynomial differential system has a first Integral? and how to compute it when it exists?

A curve $U\left(x,y\right)=0$, where $U\left(x,y\right)$ is a polynomial with real coefficients, is an invariant algebraic curve of system (1) if and only if there exists a polynomial $K=K\left(x,y\right)$ of degree at most $n-1$ satisfying

$$\frac{\partial U\left(x,y\right)}{\partial x}P\left(x,y\right)+\frac{\partial U\left(x,y\right)}{\partial y}Q\left(x,y\right)=K\left(x,y\right)U\left(x,y\right). \tag{2}$$

The polynomial $K\left(x,y\right)$ is called the cofactor of $U\left(x,y\right)=0$, if the cofactor is identically zero, then $U\left(x,y\right)$ is a polynomial first integral for system (1). The corresponding cofactor of $U\left(x,y\right)$ is always polynomial whether $U\left(x,y\right)$ is algebraic or non-algebraic. If U is real, the curve $U\left(x,y\right)=0$ is an invariant under the flow of differential system (1) and the set $\left\{(x,y)\in\mathbb{R}^2,\ U\left(x,y\right)=0\right\}$ is formed by orbits of system (1). There are strong relationships between the integrability of system (1) and its number of invariant algebraic solutions.

In this paper we give an explicit expression of invariant algebraic curves, then we prove that these systems are integrable and we introduce an explicit expression of a first integral of multi-parameter planar polynomial differential system of degree nine of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = x + P_5(x,y) + xR_8(x,y), \\ \\ y' = \dfrac{dy}{dt} = y + Q_5(x,y) + yR_8(x,y), \end{cases} \tag{3}$$

where

$$P_5(x,y) = -(a+2)\,x^5 + (4+4b)\,x^4y - (2a+4)\,x^3y^2 + (8+4b)\,x^2y^3 - (a+2)\,xy^4 + 4y^5,$$

$$Q_5(x,y) = -4x^5 - (a+2)\,x^4y + (4b-8)\,x^3y^2 - (2a+4)\,x^2y^3 + (4b-4)\,xy^4 - (a+2)\,y^5$$

$$\begin{aligned}\text{and } R_8(x,y) \;=\;& (a+1)\,x^8 - 4bx^7y + (4a+4)\,x^6y^2 - 12bx^5y^3 + (6a+6)\,x^4y^4 \\ & -12bx^3y^5 + (4a+4)\,x^2y^6 - 4bxy^7 + (a+1)\,y^8,\end{aligned}$$

in which a, b are real constants.

Moreover, we determine sufficient conditions for a polynomial differential system to possess two limit cycles: one of them is algebraic and the other one is shown to be non-algebraic, explicitly given. Concrete examples exhibiting the applicability of our result are introduced.

3 Main result

Our main result is contained in the following theorem.

Theorem 1. *Consider a multi-parameter planar polynomial differential system* (3), *then the following statements hold.*

1) The origin of coordinates $O\,(0,0)$ is the unique critical point at finite distance.

2) The curve $U\,(x,y) = x^4 + y^4 + 2x^2y^2 - 1$, is an invariant algebraic curve of system (3) *with cofactor*

$$K\,(x,y) = (-4)\left(x^2+y^2\right)^2\Big((-a-1)\left(x^2+y^2\right)^2 + 4bxy\left(x^2+y^2\right) + 1\Big).$$

3) The system (3) *has the first integral*

$$H\,(x,y) =$$

$$= \frac{\left(x^2+y^2\right)^2 + \Big(1-\left(x^2+y^2\right)^2\Big)\exp\left(a\arctan\frac{y}{x} + b\cos\left(2\arctan\frac{y}{x}\right)\right) f(\arctan\frac{y}{x})}{\Big(\left(x^2+y^2\right)^2 - 1\Big)\exp\left(a\arctan\frac{y}{x} + b\cos\left(2\arctan\frac{y}{x}\right)\right)},$$

where $f(\arctan\frac{y}{x}) = \int_0^{\arctan\frac{y}{x}} \exp(-as - b\cos 2s)ds$.

4) The system (3) *has an explicit limit cycle, given in Cartesian coordinates by* (Γ_1): $x^4 + y^4 + 2x^2y^2 - 1 = 0$.

5) If $a > 0$ and $b \in \mathbb{R} - \{0\}$, then system (3) has non-algebraic limit cycle (Γ_2), explicitly given in polar coordinates (r, θ) by the equation

$$r(\theta, r_*) = \left(\frac{\exp(a\theta + b\cos 2\theta)\left(\frac{e^{2\pi a}}{1 - e^{2\pi a}} f(2\pi) + f(\theta)\right)}{-1 + \exp(a\theta + b\cos 2\theta)\left(\frac{e^{2\pi a}}{1 - e^{2\pi a}} f(2\pi) + f(\theta)\right)} \right)^{\frac{1}{4}},$$

where $f(\theta) = \int_0^\theta \exp(-as - b\cos 2s)ds$.

Moreover, the non-algebraic limit cycle (Γ_2) lies inside the algebraic limit cycle (Γ_1).

Proof. **Proof of statement (1).**

By definition, $A(x_0, y_0) \in \mathbb{R}^2$ is a critical point of system (3) if

$$\begin{cases} x_0 + P_5(x_0, y_0) + x_0 R_8(x_0, y_0) = 0, \\ y_0 + Q_5(x_0, y_0) + y_0 R_8(x_0, y_0) = 0, \end{cases}$$

we have $y_0 P_5(x_0, y_0) - x_0 Q_5(x_0, y_0) = 4x_0^6 + 4y_0^6 + 12x_0^2 y_0^4 + 12x_0^4 y_0^2 = 0$, then $x_0 = 0, y_0 = 0$ is the unique of this equation. Thus the origin is the unique critical point at finite distance.

This completes the proof of statement (1) of Theorem 1.

Proof of statement (2).

A computation shows that $U(x, y) = x^4 + y^4 + 2x^2y^2 - 1$ satisfies the linear partial differential equation (2), the associated cofactor being

$$K(x, y) = (-4)\left(x^2 + y^2\right)^2 \left((-a - 1)\left(x^2 + y^2\right)^2 + 4bxy\left(x^2 + y^2\right) + 1\right),$$

then the curve $U(x, y) = 0$ is an invariant algebraic curve of system (3) with cofactor $K(x, y)$.

This completes the proof of statement (2) of Theorem 1.

Proof of statements (3), (4) and (5) of Theorem 1.

In order to prove our results (3), (4) and (5) we write the polynomial differential system (3) in polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then the system becomes

$$\begin{cases} r' = \dfrac{dr}{dt} = r + (-2 - a + 2b\sin 2\theta) r^5 + (1 + a - 2b\sin 2\theta) r^9, \\ \theta' = \dfrac{d\theta}{dt} = -4r^4. \end{cases} \tag{4}$$

Since θ' is negative for all $t \in \mathbb{R}$, the orbits $(r(t), \theta(t))$ of system (4) have the opposite orientation with respect to those $(x(t), y(t))$ of system (3).

Taking θ as an independent variable, we obtain the equation

$$\frac{dr}{d\theta}=\frac{-1}{4r^3}+\left(\frac{1}{4}a+\frac{1}{2}-\frac{1}{2}b\sin 2\theta\right)r+\left(\frac{-1}{4}-\frac{a}{4}+\frac{1}{2}b\sin 2\theta\right)r^5. \tag{5}$$

Via the change of variables $\rho=r^4$, this equation (5) is transformed into the Riccati equation

$$\frac{d\rho}{d\theta}=(-a-1+2b\sin 2\theta)\rho^2+(a+2-2b\sin 2\theta)\rho-1. \tag{6}$$

This equation is integrable, since it possesses the particular solution $\rho=1$.

By introducing the standard change of variables $y=\rho-1$ we obtain the Bernoulli equation

$$\frac{dy}{d\theta}=(-a+2b\sin 2\theta)y+(-1-a+2b\sin 2\theta)y^2. \tag{7}$$

By introducing the standard change of variables $z=\frac{1}{y}$ we obtain the linear equation

$$\frac{dz}{d\theta}=(a-2b\sin 2\theta)\,z+(1+a-2b\sin 2\theta)\,. \tag{8}$$

The general solution of linear equation (8) is

$$\begin{aligned} z(\theta) &= 1,\\ z(\theta) &= \frac{\lambda+\int_0^\theta (1+a-2b\sin 2w)\exp\left(\int_0^w(-a+2b\sin 2s)\,ds\right)dw}{\exp\left(\int_0^\theta(-a+2b\sin 2s)\,ds\right)}, \end{aligned}$$

where $\lambda\in\mathbb{R}$.

Then the general solution of equation (7) is

$$\begin{aligned} y(\theta) &= 1,\\ y(\theta) &= \frac{\exp\left(\int_0^\theta(-a+2b\sin 2s)\,ds\right)}{\lambda+\int_0^\theta (1+a-2b\sin 2w)\exp\left(\int_0^w(-a+2b\sin 2s)\,ds\right)dw}, \end{aligned}$$

where $\lambda\in\mathbb{R}$.

Then the general solution of equation (6) is

$$\begin{aligned} \rho(\theta) &= 1,\\ \rho(\theta) &= \frac{\exp(a\theta+b\cos 2\theta)\,(h+f(\theta))}{-1+\exp(a\theta+b\cos 2\theta)\,(h+f(\theta))}, \end{aligned}$$

where $h=(1+\lambda)\exp(-b)\in\mathbb{R}$.

Consequently, the general solution of (5) is

$$r(\theta,h) = 1,$$

$$r(\theta,h) = \left(\frac{\exp\left(a\theta+b\cos 2\theta\right)\left(h+f(\theta)\right)}{-1+\exp\left(a\theta+b\cos 2\theta\right)\left(h+f(\theta)\right)}\right)^{\frac{1}{4}},$$

where $h\in\mathbb{R}$.

From this solution we obtain a first integral in the variables (x,y) of the form

$$H(x,y) =$$

$$= \frac{\left(x^2+y^2\right)^2+\left(1-\left(x^2+y^2\right)^2\right)\exp\left(a\arctan\frac{y}{x}+b\cos\left(2\arctan\frac{y}{x}\right)\right)f(\arctan\frac{y}{x})}{\left(\left(x^2+y^2\right)^2-1\right)\exp\left(a\arctan\frac{y}{x}+b\cos\left(2\arctan\frac{y}{x}\right)\right)}.$$

Hence, statement (3) of Theorem 1 is proved.

The curves $H=h$ with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (3), in Cartesian coordinates are written as

$$\begin{aligned}\left(x^2+y^2\right)^2 &= 1,\\ \left(x^2+y^2\right)^2 &= \left(\frac{\exp\left(a\arctan\frac{y}{x}+b\cos\left(2\arctan\frac{y}{x}\right)\right)\left(h+f(\arctan\frac{y}{x})\right)}{-1+\exp\left(a\arctan\frac{y}{x}+b\cos\left(2\arctan\frac{y}{x}\right)\right)\left(h+f(\arctan\frac{y}{x})\right)}\right),\end{aligned}$$

where $h\in\mathbb{R}$.

Notice that system (3) has a periodic orbit if and only if equation (5) has a strictly positive 2π-periodic solution. This, moreover, is equivalent to the existence of a solution of (5) that fulfills $r(0,r_*)=r(2\pi,r_*)$ and $r(\theta,r_*)>0$ for any θ in $[0,2\pi]$.

The solution $r(\theta,r_0)$ of the differential equation (5) such that $r(0,r_0)=r_0$ is

$$r(\theta,r_0) = \left(\frac{\exp\left(a\theta+b\cos 2\theta\right)\left(\frac{r_0^4}{\left(r_0^4-1\right)\exp(b)}+f(\theta)\right)}{-1+\exp\left(a\theta+b\cos 2\theta\right)\left(\frac{r_0^4}{\left(r_0^4-1\right)\exp(b)}+f(\theta)\right)}\right)^{\frac{1}{4}},$$

where $r_0=r(0)$.

We have the particular solution $\rho(\theta)=1$ of the differential equation (5), from this solution we obtain $r^4(\theta,1)=1>0$, for all $\theta\in[0,\pi]$ is a particular solution of the differential equation (5).

This is an algebraic limit cycle for the differential systems (3), corresponding of course to an invariant algebraic curve $U(x,y)=0$.

More precisely, in Cartesian coordinates $r^2=x^2+y^2$ and $\theta=\arctan\left(\frac{y}{x}\right)$, the curve (Γ_1) defined by this limit cycle is (Γ_1): $x^4+y^4+2x^2y^2-1=0$.

Hence, statement (4) of Theorem 1 is proved.

A periodic solution of system (3) must satisfy the condition $r(2\pi,r_0)=r(0,r_0)$, which leads to a unique value $r_0=r_*$, given by

$$r_* = \sqrt[4]{\frac{e^b f(2\pi)}{1-e^{-2\pi a}+e^b f(2\pi)}},$$

r_* is the intersection of the periodic orbit with the OX_+ axis.

After the substitution of this value r_* into $r(\theta, r_0)$ we obtain

$$r(\theta, r_*) = \left(\frac{\exp(a\theta + b\cos 2\theta)\left(\frac{e^{2\pi a}}{1-e^{2\pi a}} f(2\pi) + f(\theta)\right)}{-1 + \exp(a\theta + b\cos 2\theta)\left(\frac{e^{2\pi a}}{1-e^{2\pi a}} f(2\pi) + f(\theta)\right)} \right)^{\frac{1}{4}}.$$

In what follows it is proved that $r(\theta, r_*) > 0$. Indeed

$$r(\theta, r_*) =$$

$$= \left(\frac{\exp(a\theta + b\cos 2\theta)\left(\frac{1}{1-e^{2\pi a}} f(2\pi) - \int_\theta^{2\pi} \exp(-aw - b\cos 2w)\, dw\right)}{-1 + \exp(a\theta + b\cos 2\theta)\left(\frac{1}{1-e^{2\pi a}} f(2\pi) - \int_\theta^{2\pi} \exp(-aw - b\cos 2w)\, dw\right)} \right)^{\frac{1}{4}}.$$

According to $a > 0$ and $b \in \mathbb{R} - \{0\}$, hence $\frac{1}{1-e^{2\pi a}} < 0$, $f(2\pi) > 0$ and $\int_\theta^{2\pi} \exp(-aw - b\cos 2w)\, dw \geq 0$ for all $[0, \pi]$, then we have $r(\theta, r_*) > 0$ for all $\theta \in [0, \pi]$.

This is a limit cycle for the differential system (3). This limit cycle is not algebraic, due to the expression

$$\exp\left(a \arctan \frac{y}{x} + b\cos\left(2\arctan\frac{y}{x}\right)\right).$$

More precisely, in Cartesian coordinates $r^2 = x^2 + y^2$ and $\theta = \arctan\left(\frac{y}{x}\right)$, the curve (Γ_2) defined by this limit cycle is (Γ_2): $F(x, y) = 0$ where

$$F(x, y) = \left(x^2 + y^2\right)^2 -$$

$$- \frac{\exp\left(a\arctan\frac{y}{x} + b\cos\left(2\arctan\frac{y}{x}\right)\right)\left(\frac{r_*^4}{(r_*^4 - 1)\exp(b)} + f(\arctan\frac{y}{x})\right)}{-1 + \exp\left(a\arctan\frac{y}{x} + b\cos\left(2\arctan\frac{y}{x}\right)\right)\left(\frac{r_*^4}{(r_*^4 - 1)\exp(b)} + f(\arctan\frac{y}{x})\right)}.$$

If the limit cycle is algebraic this curve must be given by a polynomial, but a polynomial $F(x, y)$ in the variables x and y satisfies that there is a positive integer n such that $\frac{\partial^{(n)} F(x, y)}{(\partial x)^n} = 0$, and this is not the case, therefore the curve (Γ_2): $F(x, y) = 0$ is non-algebraic and the limit cycle will also be non-algebraic.

According to $a > 0$ and $b \in \mathbb{R} - \{0\}$, hence $\frac{1}{1-e^{2\pi a}} < 0$, $f(2\pi) > 0$ and $\int_\theta^{2\pi} \exp(-aw - b\cos 2w)\, dw \geq 0$ for all $[0, \pi]$, we get

$$\exp(a\theta + b\cos 2\theta)\left(\frac{1}{1-e^{2\pi a}} f(2\pi) - \int_\theta^{2\pi} \exp(-aw - b\cos 2w)\, dw\right) < 0,$$

for all $[0,\pi]$, then we have $r(\theta,r_*)<1$ for all $\theta\in[0,\pi]$.

We conclude that system (3) has two limit cycles, the non-algebraic (Γ_2) lies inside the algebraic one (Γ_1).

This completes the proof of statement (5) of Theorem 1. □

4 Example

The following examples are given to illustrate our result.

Example 1 If we take $a=3$ and $b=1$, then system (3) reads

$$\begin{cases} x' = x-5x^5+8x^4y-10x^3y^2+12x^2y^3-5xy^4+4y^5+4x^9-4x^8y+ \\ 16x^7y^2-12x^6y^3+24x^5y^4-12x^4y^5+16x^3y^6-4x^2y^7+4xy^8, \\ \\ y' = y-4x^5-5x^4y-4x^3y^2-10x^2y^3-5y^5+4x^8y-4x^7y^2+16x^6y^3- \\ 12x^5y^4+24x^4y^5-12x^3y^6+16x^2y^7-4xy^8+4y^9. \end{cases} \tag{9}$$

The system (9) has the first integral

$$H(x,y)=$$

$$=\frac{\left(x^2+y^2\right)^2+\left(1-\left(x^2+y^2\right)^2\right)\exp\left(3\arctan\frac{y}{x}+\cos\left(2\arctan\frac{y}{x}\right)\right)f(\arctan\frac{y}{x})}{\left(\left(x^2+y^2\right)^2-1\right)\exp\left(3\arctan\frac{y}{x}+\cos\left(2\arctan\frac{y}{x}\right)\right)},$$

where $f(\arctan\frac{y}{x})=\int_0^{\arctan\frac{y}{x}}\exp(-3s-\cos 2s)ds$.

The system (9) has an algebraic limit cycle (Γ_1) whose expression is (Γ_1): $x^4+y^4+2x^2y^2-1=0$.

This system (9) has a non-algebraic limit cycle (Γ_2) whose expression in polar coordinates (r,θ) is

$$r(\theta,r_*)=\left(\frac{\exp(3\theta+\cos 2\theta)\left(\dfrac{e^{6\pi}}{1-e^{6\pi}}f(2\pi)+f(\theta)\right)}{-1+\exp(3\theta+\cos 2\theta)\left(\dfrac{e^{6\pi}}{1-e^{6\pi}}f(2\pi)+f(\theta)\right)}\right)^{\frac{1}{4}},$$

where $\theta\in\mathbb{R}$, with $f(\theta)=\int_0^{\theta}\exp(-3s-\cos 2s)ds$, and the intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_*=\sqrt[4]{\frac{e\int_0^{2\pi}\exp(-3s-\cos 2s)ds}{1-e^{-6\pi}+e\int_0^{2\pi}\exp(-3s-\cos 2s)ds}}=0.764\,60$$

We conclude that system (9) has two limit cycles. Since $r_*=0.764\,60<1$, the non-algebraic one lies inside the algebraic one.

Example 2 If we take $a = 5$ and $b = -2$, then system (3) reads

$$\begin{cases} x' = x - 7x^5 - 4x^4y - 14x^3y^2 - 7xy^4 + 4y^5 + 6x^9 + 8x^8y + 24x^7y^2 + 24x^6y^3 + \\ 36x^5y^4 + 24x^4y^5 + 24x^3y^6 + 8x^2y^7 + 6xy^8, \\ \\ y' = y - 4x^5 - 7x^4y - 16x^3y^2 - 14x^2y^3 - 12xy^4 - 7y^5 + 6x^8y + 8x^7y^2 + \\ 24x^6y^3 + 24x^5y^4 + 36x^4y^5 + 24x^3y^6 + 24x^2y^7 + 8xy^8 + 6y^9, \end{cases} \tag{10}$$

The system (10) has the first integral

$$H(x,y) =$$

$$= \frac{\left(x^2+y^2\right)^2 + \left(1 - \left(x^2+y^2\right)^2\right) \exp\left(5\arctan\frac{y}{x} - 2\cos\left(2\arctan\frac{y}{x}\right)\right) f(\arctan\frac{y}{x})}{\left(\left(x^2+y^2\right)^2 - 1\right) \exp\left(5\arctan\frac{y}{x} - 2\cos\left(2\arctan\frac{y}{x}\right)\right)},$$

where $f(\arctan\frac{y}{x}) = \int_0^{\arctan\frac{y}{x}} \exp(-5s + 2\cos 2s)ds$.

The system (10) has an algebraic limit cycle (Γ_1) whose expression is (Γ_1): $x^4 + y^4 + 2x^2y^2 - 1 = 0$.

This system (10) has a non-algebraic limit cycle (Γ_2) whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = \left(\frac{\exp(5\theta - 2\cos 2\theta)\left(\dfrac{e^{10\pi}}{1-e^{10\pi}} f(2\pi) + f(\theta)\right)}{-1 + \exp(5\theta - 2\cos 2\theta)\left(\dfrac{e^{10\pi}}{1-e^{10\pi}} f(2\pi) + f(\theta)\right)} \right)^{\frac{1}{4}},$$

where $\theta \in \mathbb{R}$, with $f(\theta) = \int_0^\theta \exp(-5s + 2\cos 2s)ds$, and the intersection of the limit cycle with the OX_+ axis is the point having r_*

$$r_* = \sqrt[4]{\frac{e^{-2}\int_0^{2\pi} \exp(-5s + 2\cos 2s)ds}{1 - e^{-10\pi} + e^{-2}\int_0^{2\pi} \exp(-s + 2\cos 2s)ds}} = 0.584\,60$$

We conclude that system (10) has two limit cycles. Since $r_* = 0.584\,60 < 1$, the non-algebraic limit cycle lies inside the algebraic limit cycle.

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