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Optimal control of a stochastic system related to the Kermack-McKendrick model

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Abstract. A stochastic optimal control problem for a two-dimensional system of differential equations related to the Kermack-McKendrick model for the spread of epidemics is considered. The aim is to maximize the expected value of the time during which the epidemic is under control, taking the quadratic control costs into account. An exact and explicit solution is found in a particular case.

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1 Introduction and theoretical result

Let $x(t)$ represent the percentage of individuals susceptible (and/or suspected) of being infected with a given virus in a certain population, and let $y(t)$ be the percentage of infected carriers. We consider the following controlled two-dimensional system of differential equations to model the spread of epidemics:

$$\begin{aligned} \dot{x}(t) &= -kx(t)y(t), \\ \dot{y}(t) &= kx(t)y(t) + f[x(t), y(t)] + b[x(t), y(t)]u(t) + \{v[x(t), y(t)]\}^{1/2}\, \dot{B}(t), \end{aligned}$$

where k is a positive constant, $u(t)$ is the control variable, the function $v(\cdot, \cdot)$ is positive and $B(t)$ is a standard Brownian motion.

If $u(\cdot)$ and $v(\cdot, \cdot)$ are identical to zero, then the above system is a modification of the classic two-dimensional Kermack-McKendrick model for the spread of epidemics that they proposed in their paper published in [1].

The absolute value of the variable $u(t)$ can be interpreted as the percentage increase of money spent to fight the disease, compared to the planned budget.

Let $x(0) = x$ and $y(0) = y$ be such that $0 < x + y < d$, where d is a value for which the epidemic is considered to be under control. We define the *first-passage time*

$$T(x, y) = \inf\{t > 0 : x(t) + y(t) = 0 \text{ or } d\}.$$

Our aim is to find the value u^* of the control variable that minimizes the expected value of the cost criterion

$$J(x, y) := \int_0^T \left\{ \tfrac{1}{2} q[x(t), y(t)]\, u^2(t) + \lambda \right\} \mathrm{d}t,$$

where $q(\cdot,\cdot)$ is a positive function and λ is a *negative* constant. Hence, the aim is to maximize the expected time during which the epidemic is under control, taking the quadratic control costs into account.

Depending on the application, an individual can be removed from the group of infected individuals because he/she is cured or because he/she is dead. Therefore, the optimizer might try to avoid having $x(t)+y(t)=0$.

This type of optimal control problem, for which the final time is a random variable, has been termed *LQG homing* by Whittle [6]. Such problems are generally very difficult to solve explicitly, especially in two or more dimensions. LQG homing problems have been considered, in particular, by the author and Zitouni [3, 4]. Makasu [5] solved explicitly a two-dimensional LQG homing problem. In Lefebvre [2], the author solved a related problem, namely that of ending the epidemic as soon as possible. However, he considered the three-dimensional version of the Kermack-McKendrick model.

Let

$$F(x,y) := \inf_{u(t),0\le t\le T} E[J(x,y)].$$

We will make use of dynamic programming to derive the partial differential equation (p.d.e.) that the *value function* F satisfies.

Let $u := u(0)$. We deduce from Bellman's principle of optimality that

$$\begin{aligned}
F(x,y) &= \inf_{u(t),0\le t\le T} E\Big[\int_0^{\Delta t}\Big(\frac{1}{2}qu^2(t)+\lambda\Big)\,\mathrm{d}t+\int_{\Delta t}^{T}\Big(\frac{1}{2}qu^2(t)+\lambda\Big)\,\mathrm{d}t\Big]\\
&= \inf_{u(t),0\le t\le \Delta t} E\Big[\int_0^{\Delta t}\Big(\frac{1}{2}qu^2(t)+\lambda\Big)\,\mathrm{d}t\\
&\quad +F\big(x-kxy\Delta t, y+(kxy+f+bu)\Delta t+v^{1/2}B(\Delta t)\big)+o(\Delta t)\Big]\\
&= \inf_{u(t),0\le t\le \Delta t} E\Big[\Big(\frac{1}{2}qu^2+\lambda\Big)\Delta t+F\big(x-kxy\Delta t, y+(kxy+f+bu)\Delta t\\
&\quad +v^{1/2}B(\Delta t)\big)+o(\Delta t)\Big].
\end{aligned}$$

Indeed, the optimal policy must be such that, whatever the control chosen in the interval $[0,\Delta t]$, and the resulting $x(\Delta t)$ and $y(\Delta t)$, the value of $u(t)$ must be optimal from Δt.

Assume next that F is differentiable with respect to x and twice differentiable with respect to y. Using the facts that $E[B(\Delta t)]=0$ and that $E[B^2(\Delta t)] = V[B(\Delta t)]=\Delta t$, Taylor's formula enables us to write that

$$\begin{aligned}
&E\Big[F\big(x-kxy\Delta t, y+(kxy+f+bu)\Delta t+v^{1/2}B(\Delta t)\big)\Big]\\
&\quad = F(x,y)-kxy\Delta t\frac{\partial F(x,y)}{\partial x}+(kxy+f+bu)\Delta t\frac{\partial F(x,y)}{\partial y}
\end{aligned}$$

$$+\frac{1}{2}v\Delta t\frac{\partial^2 F(x,y)}{\partial y^2}+o(\Delta t).$$

Hence, we have

$$\begin{aligned}0 \quad = \quad & \inf_{u(t),0\leq t\leq \Delta t}\left\{\left(\frac{1}{2}qu^2+\lambda\right)\Delta t - kxy\Delta t F_x + (kxy+f+bu)\Delta t F_y \right. \\ & \left. +\frac{1}{2}v\Delta t F_{yy} + o(\Delta t)\right\}.\end{aligned}$$

Finally, if we divide both sides of the above equation by Δt, and if we let Δt decrease to 0, we obtain the following *dynamic programming equation*:

$$0=\inf_u\left\{\frac{1}{2}qu^2+\lambda-kxyF_x+(kxy+f+bu)F_y+\frac{1}{2}vF_{yy}\right\}. \tag{1}$$

We deduce from Eq. (1) that the optimal control u^* can be expressed in terms of the function F as follows:

$$u^*=-\frac{b}{q}F_y.$$

Substituting this expression into Eq. (1), we obtain the following proposition.

Proposition 1.1. *The value function satisfies the following second-order non-linear p.d.e.:*

$$-\frac{b^2}{2q}F_y^2+\lambda-kxyF_x+(kxy+f)F_y+\tfrac{1}{2}vF_{yy}=0, \tag{2}$$

where all the functions are evaluated at $t = 0$. The boundary conditions are

$$F(x,y)=0 \quad \text{if } x+y=0 \text{ or } d.$$

We will try to solve Eq. (2) by making use of the *method of similarity solutions.* More precisely, based on the boundary conditions, we look for a solution of the form

$$F(x,y)=H(w),$$

where $w := x + y$ is the *similarity variable.* We find that Eq. (2) is transformed into the non-linear *ordinary* differential equation

$$-\frac{b^2}{2q}[H'(w)]^2+\lambda+fH'(w)+\tfrac{1}{2}vH''(w)=0. \tag{3}$$

The new boundary conditions are

$$H(w)=0 \quad \text{if } w=0 \text{ or } d. \tag{4}$$

We can now state the following result.

Proposition 1.2. *If the ratio b^2/q and the functions f and v can be expressed in terms of the similarity variable w, then the optimal control can be deduced from the solution of* (3), (4).

2 A particular case

Let us choose the following particular values for the various terms that appear in the problem set up in the previous section:

$$\lambda = -1,\ b \equiv 1,\ q \equiv 1,\ v \equiv 1 \text{ and } f(x,y) = -(x+y).$$

Thus, we assume that the functions b, q and v are all identical to 1. Moreover, we take $d = 1$. Notice that Proposition 1.2 then applies, because the ratio b^2/q and v are both constant functions, and $f(x,y) = -w$.

Hence, the epidemic is deemed under control as long as the sum $x(t) + y(t)$ is smaller than 1 percent. In practice, it is probably very difficult to have the sum $x(t) + y(t)$ equal to zero, especially in the case of a large population. Therefore, the objective is to keep this sum below 1% of the population for as long as possible, taking the quadratic control costs into account.

Next, the differential equation that we must solve becomes

$$-\tfrac{1}{2}[H'(w)]^2 - 1 - wH'(w) + \tfrac{1}{2}H''(w) = 0,$$

which is a Riccati equation for $H'(w)$. Making use of the mathematical software *Maple*, we find that the solution of the above equation that satisfies the boundary conditions $H(0) = H(1) = 0$ is the following:

$$H(w) = -\frac{1}{2}\ln\left(\frac{\left(w\sqrt{\pi}(1-e) + iw\pi\left(\operatorname{erf}(iw) - \operatorname{erf}(i)\right) + \sqrt{\pi}e^{w^2}\right)^2}{\pi}\right),$$

where erf is the error function. From $H(w)$, we obtain the value function $F(x,y)$, and hence the optimal control $u^* = -F_y$ explicitly. It can be expressed as a function of $w = x + y$:

$$u^* = \frac{\sqrt{\pi}(e-1) + i\pi[\operatorname{erf}(i) - \operatorname{erf}(iw)]}{-\sqrt{\pi}e^{w^2} + \sqrt{\pi}(e-1)w + i\pi[\operatorname{erf}(i) - \operatorname{erf}(iw)]w} \quad \text{for } 0 \le w \le 1.$$

See Figure 1. This optimal solution is very different from the one found for the problem considered in Lefebvre [2]. Indeed, in this paper, the optimal control was actually a constant.

3 Concluding remarks

In this note, we were able to obtain an explicit solution to a two-dimensional LQG homing problem by making use of the method of similarity solutions. It would be interesting to obtain at least approximate solutions to problems for which we cannot make use of this method. We could also try to solve the appropriate partial differential equation numerically.

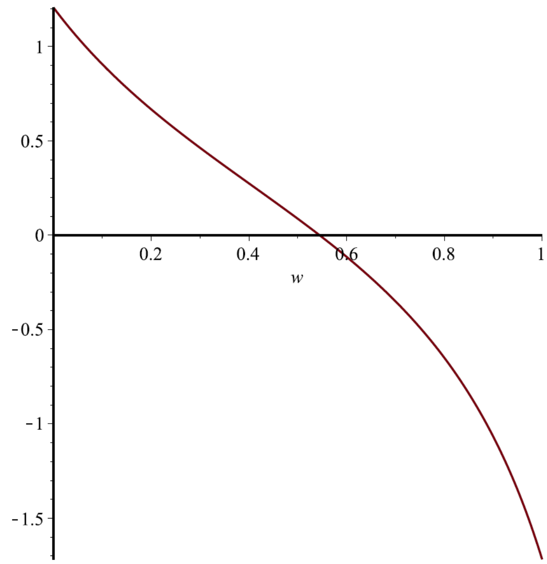

Figure 1. Optimal control u^*, as a function of $w = x + y$.

Finally, we could try to find suboptimal controls, either by making some approximations, or by choosing the form of the control variable (for instance, a linear control).

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