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Binary linear programming approach to graph convex covering problems

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Abstract. A binary linear programming (BLP) formulation of graph convex covering problems is proposed for the first time. Since the general convex covering problem of a graph is NP-complete, BLP approach will facilitate the use of convex covers and partitions of graphs in different real applications.

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1 Introduction

We denote by $G = (V; E)$ a simple undirected graph with vertex set V and edge set E. A set $S \subseteq V$ is called *nontrivial* if $3 \leq |S| \leq |V| - 1$.

Let us remind some notions defined in [1]. The *metric segment*, denoted by $\langle v, u\rangle$, is the set of all vertices lying on a shortest path between vertices $v, u \in V$. A set $S \subseteq V$ is called *convex* if $\langle v, u\rangle \subseteq S$ for any two vertices $v, u \in S$.

The family of sets $\boldsymbol{\mathcal{P}}(G)$ is called a *convex cover* of graph $G = (V; E)$ if the following properties hold:

a) every set of $\boldsymbol{\mathcal{P}}(G)$ is convex in G;

b) $V = \bigcup_{S\in\boldsymbol{\mathcal{P}}(G)} S$;

c) $S \not\subseteq \bigcup_{\substack{C\in\boldsymbol{\mathcal{P}}(G)\\ C\neq S}} C$ for every set $S \in \boldsymbol{\mathcal{P}}(G)$.

If in addition to properties a), b) and c), the family $\boldsymbol{\mathcal{P}}(G)$ satisfies the property d), set out below, then $\boldsymbol{\mathcal{P}}(G)$ is said to be a *nontrivial convex cover* of G.

d) $3 \leq |S| \leq |V| - 1$ for each set $S \in \boldsymbol{\mathcal{P}}(G)$;

e) $S \cap C = \varnothing$ for any two sets $S, C \in \boldsymbol{\mathcal{P}}(G)$, $S \neq C$.

If a convex cover $\boldsymbol{\mathcal{P}}(G)$ satisfies the property e), then this family is called a *convex partition* of G. Similarly, if $\boldsymbol{\mathcal{P}}(G)$ is a convex partition that satisfies the property d), then it is called a *nontrivial convex partition* of G. If a convex cover $\boldsymbol{\mathcal{P}}(G)$ consists of p sets, then we say that this family is a *convex p-cover* of G. By analogy, *convex p-partition*, *nontrivial convex p-partition* and *nontrivial convex p-cover* of graph G are defined. A vertex $v \in V$ is called *resident* in $\boldsymbol{\mathcal{P}}(G)$ if v belongs to only one set of $\boldsymbol{\mathcal{P}}(G)$.

Deciding whether a graph G has a convex p-cover for a fixed $p \geq 2$ is said to be convex p-cover problem. The general convex p-cover problem of a graph is NP-complete [4, 7]. It remains NP-complete even if nontrivial or pairwise disjoint convex sets are considered [5, 7]. Several classes of graphs for which there exist convex p-covers or convex p-partitions were identified in [4–6, 8, 9]. Particularly, it is NP-complete to decide whether a graph can be partitioned into an arbitrary number, greater than or equal to two, of nontrivial convex sets [9]. At the same time, there is a polynomial algorithm that determines whether a graph can be covered by an arbitrary number of nontrivial convex sets [9].

Integer linear programming is a good approach for solving NP-hard combinatorial problems. Consequently, it is of interest to propose an integer linear programming formulation for graph convex covering problems. The goal of linear programming is to optimize a linear function subject to linear constraints. In this context, we will formulate a binary linear programming (BLP) model for optimization problems related to convex covers and partitions of graphs. Several good survey on integer linear programming are available, e.g., [2, 3].

The minimum convex cover number $\varphi_c^{min}(G)$ of a graph G is the least $p \geq 2$ for which G has a convex p-cover. Likewise, the minimum convex partition number $\theta_c^{min}(G)$ of G is the least $p \geq 2$ for which G has a convex p-partition.

In the same way, we define the following numbers:

$\varphi_{cn}^{min}(G)$ is minimum nontrivial convex cover number of G;

$\theta_{cn}^{min}(G)$ is minimum nontrivial convex partition number of G;

$\varphi_{cn}^{max}(G)$ is maximum nontrivial convex cover number of G;

$\theta_{cn}^{max}(G)$ is maximum nontrivial convex partition number of G.

Additional information about these invariants can be found in papers [7–9]. It is clear that for any graph G, $\varphi_c^{min}(G) \leq \theta_c^{min}(G)$. If G can be partitioned into nontrivial convex sets, then $\theta_c^{min}(G) \leq \theta_{cn}^{min}(G)$ and:

$$\varphi_{cn}^{min}(G) \leq \theta_{cn}^{min}(G) \leq \theta_{cn}^{max}(G) \leq \varphi_{cn}^{max}(G).$$

Anyway, if G can be covered by nontrivial convex sets, then:

$$\varphi_c^{min}(G) \leq \varphi_{cn}^{min}(G) \leq \varphi_{cn}^{max}(G).$$

The problems of determining the numbers $\varphi_c^{min}(G)$, $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, $\varphi_{cn}^{max}(G)$ and $\theta_{cn}^{max}(G)$ are denoted by MinCC, MinCP, MinNCC, MinNCP, MaxNCC and MaxNCP, respectively.

2 BLP formulation

In order to solve graph convex covering problems efficiently a BLP approach is proposed.

Consider a graph G with vertex set $V = \{v_1, v_n, ..., v_n\}$.We define n subsets S_1, S_2, ... , S_n of V, and two types of binary variables. For each vertex $v_j \in V$ there are variables $x_{ij} \in \{0,1\}$, indicating whether v_j will belong or not to S_i. Additionally, there are variables $y_{lij} \in \{0,1\}$ which will enforce the convexity constraints of S_l. We define

$$x_{ij} = \begin{cases} 1, & v_j \in S_i \\ 0, & v_j \notin S_i \end{cases}$$

and

$$y_{lij} = \begin{cases} 1, & \{v_i, v_j\} \subseteq S_l \\ 0, & \{v_i, v_j\} \not\subseteq S_l. \end{cases}$$

We denote by X the set of all variables x_{ij} and by Y the set of all variables y_{lij}. The BLP model (BLPM) that corresponds to MinCP is presented below.

$$\begin{array}{lll}
\text{minimize} & \sum_{i=1}^{n} x_{ii} & (1) \\
\text{subject to} & \sum_{i=1}^{n} x_{ii} \geq 2 & (2) \\
& \sum_{i=1}^{n} x_{ij} = 1, \qquad j = \overline{1,n} & (3) \\
& \sum_{v_k \in \langle v_i, v_j \rangle} x_{lk} - |\langle v_i, v_j \rangle| y_{lij} \geq 0, \qquad l,i,j = \overline{1,n} & (4) \\
& x_{li} + x_{lj} - y_{lij} \leq 1, \qquad l,i,j = \overline{1,n} & (5) \\
& y_{lij} - x_{li} \leq 0, \qquad l,i,j = \overline{1,n} & (6) \\
& y_{lij} - x_{lj} \leq 0, \qquad l,i,j = \overline{1,n} & (7) \\
& x_{ij} - x_{ii} \leq 0, \qquad i,j = \overline{1,n} & (8) \\
& x_{ij}, y_{lij} \in \{0,1\}, \qquad l,i,j = \overline{1,n} & (9)
\end{array}$$

Theorem 1. *Each feasible solution of BLPM with objective function (1) and restrictions (2)–(9) corresponds to a feasible solution of MinCP and vice versa.*

Proof. Let (X, Y) be a feasible solution of the BLPM. Each entry in (X, Y) is zero or one because of the integrality condition (9). Constraints (8) ensure that if there is a vertex v_j belonging to S_i, then $v_i \in S_i$. Restrictions (3) require that each vertex $v_i \in V$ belongs to exactly one set of $S_1, S_2, \ldots, S_n$. Considering these constraints, we define the family $\boldsymbol{\mathcal{P}} = \{S_i : S_i \neq \varnothing, 1 \leq i \leq n\}$ that partitions graph G. By restriction (2), $\boldsymbol{\mathcal{P}}$ consists of at least two sets. It follows that $\boldsymbol{\mathcal{P}}$ satisfies properties b), c) and e). Moreover, constraints (4)–(7) yield that if both v_i and v_j are in the same set S_l then $\langle v_i, v_j \rangle \subseteq S_l$, thus fulfilling the convexity property a) for each set of $\boldsymbol{\mathcal{P}}$. This therefore means that $\boldsymbol{\mathcal{P}}$ is an admissible convex partition of G.

Let $\boldsymbol{\mathcal{P}}(G)$ be a convex partition of G such that $|\boldsymbol{\mathcal{P}}(G)| \geq 2$, in other words, it is a feasible solution of MinCP. For each set $S \in \boldsymbol{\mathcal{P}}(G)$, we choose one resident vertex $v_i \in S$ in $\boldsymbol{\mathcal{P}}(G)$ and set x_{ij} to one if $v_j \in S$, otherwise set to zero. In the same manner, we set y_{lij} to one if $x_{li} = 1$ and $x_{lj} = 1$, otherwise set to zero. In light of all the above arguments, we can conclude that (X, Y) does not violate any of the constraints. For this reason, (X, Y) is a feasible solution for the BLP. $\square$

Corollary 1. *An optimal solution of BLPM with objective function (1) and restrictions (2)–(9) corresponds to an optimal solution of MinCP.*

Now we present the BLP model for MaxNCC.

$$\begin{array}{llrr}
\text{maximize} & \sum_{i=1}^{n} x_{ii} & & (10)\\
\text{subject to} & \sum_{i=1}^{n} x_{ii} \geq 2 & & (11)\\
& \sum_{j=1, j\neq i}^{n} x_{ij} - 2x_{ii} \geq 0, & i=\overline{1,n} & (12)\\
& \sum_{i=1, i\neq j}^{n} x_{ij} + Mx_{jj} \geq 1, & j=\overline{1,n} & (13)\\
& \sum_{i=1, i\neq j}^{n} x_{ij} + Mx_{jj} \leq M, & j=\overline{1,n} & (14)\\
& \sum_{v_k \in \langle v_i, v_j\rangle} x_{lk} - |\langle v_i, v_j\rangle| y_{lij} \geq 0, & l,i,j=\overline{1,n} & (15)\\
& x_{li} + x_{lj} - y_{lij} \leq 1, & l,i,j=\overline{1,n} & (16)\\
& y_{lij} - x_{li} \leq 0, & l,i,j=\overline{1,n} & (17)\\
& y_{lij} - x_{lj} \leq 0, & l,i,j=\overline{1,n} & (18)\\
& x_{ij} - x_{ii} \leq 0, & i,j=\overline{1,n} & (19)\\
& x_{ij}, y_{lij} \in \{0,1\}, & l,i,j=\overline{1,n} & (20)
\end{array}$$

Theorem 2. *Each feasible solution of BLPM with objective function (10) and restrictions (11)–(20) corresponds to a feasible solution of MaxNCC and vice versa.*

Proof. Let (X,Y) be a feasible solution of the BLPM. Restriction (20) enforce the binary nature of the variables x_{ij} and y_{lij}. By constraints (19), each nonempty set S_i contains the vertex v_i. Moreover, constraints (13) and (14) mean that for each nonempty set S_j the vertex v_j is not contained in $\bigcup_{1\leq i\leq n, i\neq j} S_i$, and there is no vertex v_i such that $v_i \notin \bigcup_{1\leq j\leq n} S_j$. Considering these constraints, we define the family $\mathcal{P} = \{S_i : S_i \neq \varnothing, 1 \leq i \leq n\}$ that covers graph G. By restriction (11), $|\mathcal{P}| \geq 2$. Constraints (12) yield that every set $S_i \in \mathcal{P}$ has at least three elements. This ensures that every set of $\mathcal{P}$ is nontrivial. It is clear that $\mathcal{P}$ satisfies properties b), c) and d). The remaining constraints (15)–(18) imply that each set of $\mathcal{P}$ is convex, thus property a) also holds. Therefore, for a feasible solution of the BLPM there is a feasible solution of MaxNCC.

As in the proof of Theorem 1, it can easily be verified that for a feasible solution of MaxNCC there exists a feasible solution of the BLPM. □

Corollary 2. *An optimal solution of BLPM with objective function (10) and restrictions (11)–(20) corresponds to an optimal solution of MaxNCC.*

Regarding other optimization problems MinCC, MinNCC, MinNCP and MaxNCP, we obtain the following corollaries:

Corollary 3. *An optimal solution of BLPM with objective function (1) and restrictions (11), (13)–(20) corresponds to an optimal solution of MinCC.*

Corollary 4. *An optimal solution of BLPM with objective function (1) and restrictions (11)–(20) corresponds to an optimal solution of MinNCC.*

Corollary 5. *An optimal solution of BLPM with objective function (1) and restrictions (2)–(9), (12) corresponds to an optimal solution of MinNCP.*

Corollary 6. *An optimal solution of BLPM with objective function (10) and restrictions (2)–(9), (12) corresponds to an optimal solution of MaxNCP.*

The above showed is a description of BLP models designed to solve optimization problems of convex cover problem of graphs. We should mention that similar BLP models can be formulated for general convex p-cover problems of graphs.

For example, by replacing objective function (1) with a constant and restriction (2) with

$$\textstyle\sum_{i=1}^{n} x_{ii} = p \tag{21}$$

we obtain the BLP formulation of convex p-partition problem with constant objective function and restrictions (3)–(9), (21).

In view of proofs of Theorems 1 and 2, the correctness of the following corollaries follows:

Corollary 7. *An optimal solution of BLPM with constant objective function and restrictions (3)–(9), (21) corresponds to a feasible solution of convex p-partition.*

Corollary 8. *An optimal solution of BLPM with constant objective function and restrictions (3)–(9), (12), (21) corresponds to a feasible solution of nontrivial convex p-partition.*

Corollary 9. *An optimal solution of BLPM with constant objective function and restrictions (13)–(20), (21) corresponds to a feasible solution of convex p-cover.*

Corollary 10. *An optimal solution of BLPM with constant objective function and restrictions (12)–(20), (21) corresponds to a feasible solution of nontrivial convex p-cover.*

3 Conclusion

We have proposed a binary linear programming formulation for convex covering problems of graphs. This yields that problems of determination of invariants $\varphi_c^{min}(G)$, $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, $\varphi_{cn}^{max}(G)$, $\theta_{cn}^{max}(G)$, and problems of existence of convex p-covers, convex p-partitions, nontrivial convex p-covers and nontrivial convex p-partitions of a given graph G can be solved by using linear programming software packages, that essentially simplifies the use of convex covers of graph in applied problem solving.

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