

# Asymptotically almost periodic and asymptotically almost automorphic vector-valued generalized functions

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**Abstract.** The main purpose of this paper is to introduce the notion of an asymptotically almost periodic ultradistribution and asymptotically almost automorphic ultradistribution with values in a Banach space, as well as to further analyze the classes of asymptotically almost periodic and asymptotically almost automorphic distributions with values in a Banach space. We provide some applications of the introduced concepts in the analysis of systems of ordinary differential equations.

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## 1 Introduction and preliminaries

As it is well-known, the concept of almost periodicity was introduced by Danish mathematician H. Bohr [3] around 1924–1926 and later generalized by many other authors. Let  $I = \mathbb{R}$  or  $I = [0, \infty)$ , and let  $f : I \rightarrow X$  be continuous. Given  $\epsilon > 0$ , we call  $\tau > 0$  an  $\epsilon$ -period for  $f(\cdot)$  iff  $\|f(t + \tau) - f(t)\| \leq \epsilon$ ,  $t \in I$ . The set constituted of all  $\epsilon$ -periods for  $f(\cdot)$  is denoted by  $\vartheta(f, \epsilon)$ . It is said that  $f(\cdot)$  is almost periodic, a.p. for short, iff for each  $\epsilon > 0$  the set  $\vartheta(f, \epsilon)$  is relatively dense in  $I$ , which means that there exists  $l > 0$  such that any subinterval of  $I$  of length  $l$  meets  $\vartheta(f, \epsilon)$ . The vector space consisting of all almost periodic functions is denoted by  $AP(I : X)$ .

The notion of a scalar-valued almost automorphic function was introduced by S. Bochner [2] in 1962. In a vector-valued case, definition goes as follows. A bounded continuous function  $f : \mathbb{R} \rightarrow X$  is said to be almost automorphic iff for every real sequence  $(b_n)$  there exist a subsequence  $(a_n)$  of  $(b_n)$  and a map  $g : \mathbb{R} \rightarrow X$  such that  $\lim_{n \rightarrow \infty} f(t + a_n) = g(t)$  and  $\lim_{n \rightarrow \infty} g(t - a_n) = f(t)$ , pointwise for  $t \in \mathbb{R}$ . Due to the famous Bochner's criterion, any almost periodic function has to be almost automorphic; the converse statement is not true, however [13]. By  $AA(\mathbb{R} : X)$  we denote the vector space consisting of all almost automorphic functions. The notion of an almost automorphic function on topological group was introduced and further analyzed in the landmark papers by W. A. Veech [30,31] between 1965 and 1967. For more details about almost periodic and almost automorphic functions with values in Banach spaces, we refer the reader to the monographs [13] by T. Diagana and [18] by G. M. N'Guérékata.

The notion of a scalar-valued asymptotically almost periodic distribution has been introduced by I. Cioranescu in [12], while the notion of a vector-valued asymptotically almost periodic distribution has been considered by D. N. Cheban [9] following a different approach (see also I. K. Dontvi [14] and A. Halanay, D. Wexler [15]). The notion of a scalar-valued almost automorphic distribution and a scalar-valued almost automorphic Colombeau generalized function have been introduced by C. Bouzar, Z. Tchouar [5] and C. Bouzar, M. T. Khalladi [6]. Some contributions have been also given by B. Stanković [28, 29].

In our recent joint research study with D. Velinov [25], we have analyzed the notions of an almost automorphic distribution and an almost automorphic ultradistribution in Banach space; the notion of an almost periodic ultradistribution in Banach space has been recently analyzed by M. Kostić [24] within the framework of Komatsu's theory of ultradistributions, with the corresponding sequences not satisfying the condition (M.3); see also the papers by I. Cioranescu [11] and M. C. Gómez-Collado [17] for first results in this direction. As mentioned in the abstract, the main aim of this paper is to introduce the notions of an asymptotically almost periodic ultradistribution and asymptotically almost automorphic ultradistribution in Banach space, as well as to provide some applications in the qualitative analysis of vector-valued distributional and vector-valued ultradistributional solutions to systems of ordinary differential equations (the notions of an asymptotically almost periodic ultradistribution and an asymptotically almost automorphic ultradistribution seem to be not considered elsewhere even in scalar-valued case). In such a way, we expand and contemplate the results obtained in [5, 6, 9–12, 14, 15] and [28, 29]; see also [32, 33] for some other results about the existence and uniqueness of various types of generalized almost periodic solutions of nonlinear Volterra integro-differential equations.

The organization of paper is briefly described as follows. After giving some preliminary results and definitions from the theory of vector-valued ultradistributions (Subsection 1.1), in Section 2 we analyze the notions of asymptotical almost periodicity and asymptotical almost automorphy for vector-valued distributions. Here, we recognize the importance of condition  $T * \varphi \in AAP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}$  ( $T * \varphi \in AAA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}$ ) for a bounded vector-valued distribution  $T \in \mathcal{D}'_{L^1}(X)$ , in contrast with the considerations of I. Cioranescu [12] and C. Bouzar, F. Z. Tchouar [5], where the above inclusions are required to be valid only for the test functions belonging to the space  $\mathcal{D}_0$ . Section 3 is devoted to the study of asymptotical almost periodicity and asymptotical almost automorphy for vector-valued ultradistributions. The main result of paper is Theorem 3, where we state an important structural characterization for the class of asymptotically almost periodic (automorphic) vector-valued ultradistributions. The last section of paper is reserved for certain applications to systems of ordinary differential equations in distribution and ultradistribution spaces.

We use the standard notation throughout the paper. By  $(X, \|\cdot\|)$  we denote a non-trivial complex Banach space. The abbreviations  $C_b(I : X)$  and  $C(K : X)$ , where  $K$  is a non-empty compact subset of  $\mathbb{R}$ , stand for the spaces consisting of

all bounded continuous functions  $I \mapsto X$  and all continuous functions  $K \mapsto X$ , respectively. Both spaces are Banach endowed with sup-norm. By  $C_0([0, \infty) : X)$  we denote the closed subspace of  $C_b([0, \infty) : X)$  consisting of functions vanishing at plus infinity.

We say that a continuous function  $f : \mathbb{R} \rightarrow X$  is asymptotically almost periodic (automorphic) iff there is a function  $q \in C_0([0, \infty) : X)$  and an almost periodic (automorphic) function  $g : \mathbb{R} \rightarrow X$  such that  $f(t) = g(t) + q(t)$ ,  $t \geq 0$ . Let us recall that any (asymptotically) almost periodic function  $\mathbb{R} \mapsto X$  is (asymptotically) almost automorphic. By  $AAP(\mathbb{R} : X)$ , resp.  $AAA(\mathbb{R} : X)$ , we denote the vector space consisting of all asymptotically almost periodic, resp. asymptotically almost automorphic functions. See [9, 13, 14, 18, 23, 32] and references cited therein for more details on the subject.

Let  $1 \leq p < \infty$ . Then we say that a function  $f \in L_{loc}^p(\mathbb{R} : X)$  is Stepanov  $p$ -bounded,  $S^p$ -bounded shortly, iff  $\|f\|_{S^p} := \sup_{t \in \mathbb{R}} l(\int_t^{t+1} \|f(s)\|^p ds)^{1/p} < \infty$ . The space  $L_S^p(\mathbb{R} : X)$  consisted of all  $S^p$ -bounded functions becomes a Banach space equipped with the above norm. A function  $f \in L_S^p(\mathbb{R} : X)$  is said to be Stepanov  $p$ -almost periodic,  $S^p$ -almost periodic shortly, iff the function  $\hat{f} : \mathbb{R} \rightarrow L^p([0, 1] : X)$ , defined by  $\hat{f}(t)(s) := f(t+s)$ ,  $t \in \mathbb{R}$ ,  $s \in [0, 1]$  is almost periodic. Following H. R. Henríquez [16], we say that a function  $f \in L_S^p(\mathbb{R} : X)$  is asymptotically Stepanov  $p$ -almost periodic, asymptotically  $S^p$ -almost periodic shortly, iff the function  $\hat{f} : \mathbb{R} \rightarrow L^p([0, 1] : X)$ , is asymptotically almost periodic. By  $AAPS^p(\mathbb{R} : X)$  we denote the space consisting of all asymptotically Stepanov  $p$ -almost periodic functions.

The notion of Stepanov  $p$ -almost automorphy has been introduced by G. M. N'Guérékata and A. Pankov in [19]: A function  $f \in L_{loc}^p(\mathbb{R} : X)$  is called Stepanov  $p$ -almost automorphic iff for every real sequence  $(a_n)$ , there exists a subsequence  $(a_{n_k})$  and a function  $g \in L_{loc}^p(\mathbb{R} : X)$  such that  $\lim_{k \rightarrow \infty} \int_t^{t+1} \|f(a_{n_k} + s) - g(s)\|^p ds = 0$  and  $\lim_{k \rightarrow \infty} \int_t^{t+1} \|g(s - a_{n_k}) - f(s)\|^p ds = 0$  for each  $t \in \mathbb{R}$ ; a function  $f \in L_{loc}^p(\mathbb{R} : X)$  is called asymptotically Stepanov  $p$ -almost automorphic iff there exists an  $S^p$ -almost automorphic function  $g(\cdot)$  and a function  $q \in L_S^p(\mathbb{R} : X)$  such that  $f(t) = g(t) + q(t)$ ,  $t \geq 0$  and  $\hat{q} \in C_0([0, \infty) : L^p([0, 1] : X))$ . The vector space consisting of all asymptotically  $S^p$ -almost automorphic functions will be denoted by  $AAAS^p([0, \infty) : X)$ .

By  $\mathcal{D}(X) = \mathcal{D}(\mathbb{R} : X)$  we denote the Schwartz space of test functions with values in  $X$ , by  $\mathcal{S}(X) = \mathcal{S}(\mathbb{R} : X)$  we denote the space of rapidly decreasing functions with values in  $X$ , and by  $\mathcal{E}(X) = \mathcal{E}(\mathbb{R} : X)$  we denote the space of all infinitely differentiable functions with values in  $X$ ;  $\mathcal{D} \equiv \mathcal{D}(\mathbb{C})$ ,  $\mathcal{S} \equiv \mathcal{S}(\mathbb{C})$  and  $\mathcal{E} \equiv \mathcal{E}(\mathbb{C})$ . The spaces of all linear continuous mappings from  $\mathcal{D}$ ,  $\mathcal{S}$  and  $\mathcal{E}$  into  $X$  will be denoted by  $\mathcal{D}'(X)$ ,  $\mathcal{S}'(X)$  and  $\mathcal{E}'(X)$ , respectively [27]. Set  $\mathcal{D}_0 := \{\varphi \in \mathcal{D} : \text{supp}(\varphi) \subseteq [0, \infty)\}$ .

## 1.1 Vector-valued ultradistributions

In this paper, we will always follow Komatsu's approach to vector-valued ultradistributions, with the sequence  $(M_p)$  of positive real numbers satisfying  $M_0 = 1$  and the following conditions: (M.1):  $M_p^2 \leq M_{p+1}M_{p-1}$ ,  $p \in \mathbb{N}$ , (M.2):  $M_p \leq$

$AH^p \sup_{0 \leq i \leq p} M_i M_{p-i}$ ,  $p \in \mathbb{N}$ , for some  $A, H > 1$ , (M.3'):  $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$ . Any use of the condition

(M.3):  $\sup_{p \in \mathbb{N}} \sum_{q=p+1}^{\infty} \frac{M_{q-1} M_{p+1}}{p M_p M_q} < \infty$ , which is slightly stronger than (M.3'), will be explicitly emphasized.

Let us recall that the Gevrey sequence  $(p!^s)$  satisfies the above conditions ( $s > 1$ ). Set  $m_p := \frac{M_p}{M_{p-1}}$ ,  $p \in \mathbb{N}$ . The space of Beurling, resp., Roumieu ultradifferentiable functions, is defined by  $\mathcal{D}^{(M_p)} := \text{indlim}_{K \in \mathbb{R}} \mathcal{D}_K^{(M_p)}$ , resp.,  $\mathcal{D}^{\{M_p\}} := \text{indlim}_{K \in \mathbb{R}} \mathcal{D}_K^{\{M_p\}}$ , where  $\mathcal{D}_K^{(M_p)} := \text{projlim}_{h \rightarrow \infty} \mathcal{D}_K^{M_p, h}$ , resp.,  $\mathcal{D}_K^{\{M_p\}} := \text{indlim}_{h \rightarrow 0} \mathcal{D}_K^{M_p, h}$ ,  $\mathcal{D}_K^{M_p, h} := \{\phi \in C^\infty(\mathbb{R}) : \text{supp} \phi \subseteq K, \|\phi\|_{M_p, h, K} < \infty\}$  and

$$\|\phi\|_{M_p, h, K} := \sup \left\{ \frac{h^p |\phi^{(p)}(t)|}{M_p} : t \in K, p \in \mathbb{N}_0 \right\}.$$

Henceforward, the asterisk  $*$  is used to denote both, the Beurling case  $(M_p)$  or the Roumieu case  $\{M_p\}$ . Set  $\mathcal{D}_0^* := \{\varphi \in \mathcal{D}^* : \text{supp}(\varphi) \subseteq [0, \infty)\}$ . The space consisted of all continuous linear functions from  $\mathcal{D}^*$  into  $X$ , denoted by  $\mathcal{D}'^*(X) := L(\mathcal{D}^* : X)$ , is said to be the space of all  $X$ -valued ultradistributions of  $*$ -class. We also need the notion of space  $\mathcal{E}^*(X)$ , defined as  $\mathcal{E}^*(X) := \text{indlim}_{K \in \mathbb{R}} \mathcal{E}_K^*(X)$ , where in Beurling case  $\mathcal{E}_K^{(M_p)}(X) := \text{projlim}_{h \rightarrow \infty} \mathcal{E}_K^{M_p, h}(X)$ , resp., in Roumieu case  $\mathcal{E}_K^{\{M_p\}}(X) := \text{indlim}_{h \rightarrow 0} \mathcal{E}_K^{M_p, h}(X)$ , and

$$\mathcal{E}_K^{M_p, h}(X) := \left\{ \phi \in C^\infty(\mathbb{R} : X) : \sup_{p \geq 0} \frac{h^p \|\phi^{(p)}\|_{C(K; X)}}{M_p} < \infty \right\}.$$

The space consisting of all linear continuous mappings  $\mathcal{E}^*(\mathbb{C}) \rightarrow X$  is denoted by  $\mathcal{E}'^*(X)$ ;  $\mathcal{E}'^* := \mathcal{E}'^*(\mathbb{C})$ . An entire function of the form  $P(\lambda) = \sum_{p=0}^{\infty} a_p \lambda^p$ ,  $\lambda \in \mathbb{C}$ , is of class  $(M_p)$ , resp., of class  $\{M_p\}$ , if there exist  $l > 0$  and  $C > 0$ , resp., for every  $l > 0$  there exists a constant  $C > 0$ , such that  $|a_p| \leq C l^p / M_p$ ,  $p \in \mathbb{N}$  [20]. The corresponding ultradifferential operator  $P(D) = \sum_{p=0}^{\infty} a_p D^p$  is said to be of class  $(M_p)$ , resp., of class  $\{M_p\}$ ; it acts as a continuous linear operator between the spaces  $\mathcal{D}^*$  and  $\mathcal{D}'^*$  ( $\mathcal{D}'^*$  and  $\mathcal{D}'^*$ ). The convolution of Banach space valued ultradistributions and scalar-valued ultradifferentiable functions of the same class will be taken in the sense of considerations given on page 685 of [22]. Let remind ourselves that, for every  $f \in \mathcal{D}'^*(X)$  and  $\varphi \in \mathcal{D}^*$ , we have  $f * \varphi \in \mathcal{E}^*(X)$  as well as that the linear mapping  $\varphi \mapsto \cdot * \varphi : \mathcal{D}'^*(X) \rightarrow \mathcal{E}^*(X)$  is continuous. The convolution of an  $X$ -valued ultradistribution  $f(\cdot)$  and a scalar-valued ultradistribution  $g \in \mathcal{E}'^*$  with compact support, defined by the identity [22, (4.9)], is an  $X$ -valued ultradistribution and the mapping  $g * \cdot : \mathcal{D}'^*(X) \rightarrow \mathcal{D}'^*(X)$  is continuous. Set  $\langle T_h, \varphi \rangle := \langle T, \varphi(\cdot - h) \rangle$ ,  $T \in \mathcal{D}'^*(X)$ ,  $\varphi \in \mathcal{D}^*$  ( $h > 0$ ). We will use a similar definition for vector-valued distributions.

Assume that the sequence  $(M_p)$  satisfies (M.1), (M.2) and (M.3). Then

$$P_l(x) = (1+x^2) \prod_{p \in \mathbb{N}} \left(1 + \frac{x^2}{l^2 m_p^2}\right), \text{ resp. } P_{r_p}(x) = (1+x^2) \prod_{p \in \mathbb{N}} \left(1 + \frac{x^2}{r_p^2 m_p^2}\right),$$

defines an ultradifferential operator of class  $(M_p)$ , resp., of class  $\{M_p\}$ ; here,  $(r_p)$  is a sequence of positive real numbers tending to infinity. The family consisting of all such sequences will be denoted by  $\mathbf{R}$  henceforth. For more details on the subject, the reader may consult [20–22].

The spaces of tempered ultradistributions of Beurling, resp., Roumieu type, are defined by S. Pilipović [26] as duals of the corresponding test spaces

$$\mathcal{S}^{(M_p)} := \text{projlim}_{h \rightarrow \infty} \mathcal{S}^{M_p, h}, \text{ resp., } \mathcal{S}^{\{M_p\}} := \text{indlim}_{h \rightarrow 0} \mathcal{S}^{M_p, h},$$

where  $\mathcal{S}^{M_p, h} := \{\phi \in C^\infty(\mathbb{R}) : \|\phi\|_{M_p, h} < \infty\}$  ( $h > 0$ ), and

$$\|\phi\|_{M_p, h} := \sup \left\{ \frac{h^{\alpha+\beta}}{M_\alpha M_\beta} (1+t^2)^{\beta/2} |\phi^{(\alpha)}(t)| : t \in \mathbb{R}, \alpha, \beta \in \mathbb{N}_0 \right\}.$$

A continuous linear mapping  $\mathcal{S}^{(M_p)} \rightarrow X$ , resp.,  $\mathcal{S}^{\{M_p\}} \rightarrow X$ , is said to be an  $X$ -valued tempered ultradistribution of Beurling, resp., Roumieu type. By  $\mathcal{S}'^{(M_p)}(X)$ , resp.  $\mathcal{S}'^{\{M_p\}}(X)$  (the common abbreviation will be  $\mathcal{S}'^*(X)$ ), we denote the space consisting of all vector-valued tempered ultradistributions of Beurling, resp., Roumieu type. It is well known that  $\mathcal{S}'^{(M_p)}(X) \subseteq \mathcal{D}'^{(M_p)}(X)$ , resp.  $\mathcal{S}'^{\{M_p\}}(X) \subseteq \mathcal{D}'^{\{M_p\}}(X)$ .

Finally, we need some preliminaries concerning the first antiderivative of a vector-valued (ultra-)distribution. Let  $\eta \in \mathcal{D}_{[1,2]}$  ( $\eta \in \mathcal{D}_{[1,2]}^*$ ) be a fixed test function satisfying  $\int_{-\infty}^{\infty} \eta(t) dt = 1$ . Then, for every fixed  $\varphi \in \mathcal{D}$  ( $\varphi \in \mathcal{D}^*$ ), we define  $I(\varphi)$  to be

$$I(\varphi)(x) := \int_{-\infty}^x \left[ \varphi(t) - \eta(t) \int_{-\infty}^{\infty} \varphi(u) du \right] dt, \quad x \in \mathbb{R}.$$

It can be simply checked that, for every  $\varphi \in \mathcal{D}$  ( $\varphi \in \mathcal{D}^*$ ) and  $n \in \mathbb{N}$ , we have  $I(\varphi) \in \mathcal{D}$  ( $I(\varphi) \in \mathcal{D}^*$ ),  $\frac{d^k}{dx^k} I(\varphi)(x) = \varphi^{(k-1)}(x) - \eta^{(k-1)}(x) \int_{-\infty}^{\infty} \varphi(u) du$ ,  $x \in \mathbb{R}$ ,  $k \in \mathbb{N}$ . Define  $G^{-1}$  by  $G^{-1}(\varphi) := -G(I(\varphi))$ ,  $\varphi \in \mathcal{D}$  ( $\varphi \in \mathcal{D}^*$ ). Then we have  $G^{-1} \in \mathcal{D}'(L(X))$  ( $G^{-1} \in \mathcal{D}'^*(L(X))$ ) and  $(G^{-1})' = G$ ; more precisely,  $-G^{-1}(\varphi') = G(I(\varphi')) = G(\varphi)$ ,  $\varphi \in \mathcal{D}$ . For the proof of Theorem 4 below, we will use the following simple fact: if  $\langle X, -\varphi' \rangle = \langle G, \varphi \rangle$ ,  $\varphi \in \mathcal{D}_0$  ( $\varphi \in \mathcal{D}_0^*$ ) for some  $X$ ,  $G \in \mathcal{D}'(X)$  ( $X, G \in \mathcal{D}'^*(X)$ ), then  $X = G^{-1} + \text{Const.}$  on  $[0, \infty)$ , i.e.,  $\langle X, \varphi \rangle = \langle G^{-1}, \varphi \rangle + \text{Const.} \cdot \int_0^{\infty} \varphi(t) dt$ ,  $\varphi \in \mathcal{D}_0$  ( $\varphi \in \mathcal{D}_0^*$ ).

## 2 Asymptotical almost periodicity and asymptotical almost automorphy of vector-valued distributions

We refer the reader to [5, 10] and [25] for the basic results about vector-valued almost periodic distributions and vector-valued almost automorphic distributions.

Let  $1 \leq p \leq \infty$ . Then by  $\mathcal{D}_{L^p}(X)$  we denote the vector space consisting of all infinitely differentiable functions  $f : \mathbb{R} \rightarrow X$  satisfying that for each number  $j \in \mathbb{N}_0$  we have  $f^{(j)} \in L^p(\mathbb{R} : X)$ . The Fréchet topology on  $\mathcal{D}_{L^p}(X)$  is induced by the following system of seminorms  $\|f\|_k := \sum_{j=0}^k \|f^{(j)}\|_{L^p(\mathbb{R})}$ ,  $f \in \mathcal{D}_{L^p}(X)$  ( $k \in \mathbb{N}$ ). If  $X = \mathbb{C}$ , then the above space is simply denoted by  $\mathcal{D}_{L^p}$ . A linear continuous mapping  $f : \mathcal{D}_{L^1} \rightarrow X$  is said to be a bounded  $X$ -valued distribution; the space consisting of such vector-valued distributions will be denoted henceforth by  $\mathcal{D}'_{L^1}(X)$ . Endowed with the strong topology,  $\mathcal{D}'_{L^1}(X)$  becomes a complete locally convex space. For every  $f \in \mathcal{D}'_{L^1}(X)$ , we have that  $f|_{\mathcal{S}} : \mathcal{S} \rightarrow X$  is a tempered  $X$ -valued distribution [24].

The space of bounded vector-valued distributions tending to zero at plus infinity,  $B'_{+,0}(X)$  for short, is defined by

$$B'_{+,0}(X) := \left\{ T \in \mathcal{D}'_{L^1}(X) : \lim_{h \rightarrow +\infty} \langle T_h, \varphi \rangle = 0 \text{ for all } \varphi \in \mathcal{D} \right\}.$$

It can be simply verified that the structural characterization for the space  $B'_{+,0}(\mathbb{C})$ , proved in [12, Proposition 1], is still valid in vector-valued case as well that the space  $B'_{+,0}(\mathbb{C})$  is closed under differentiation.

Let  $T \in \mathcal{D}'_{L^1}(X)$ . Then the following assertions are equivalent [25]:

- (i)  $T * \varphi \in AP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}$ , resp.,  $T * \varphi \in AA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}$ .
- (ii) There exist an integer  $k \in \mathbb{N}$  and almost periodic, resp. almost automorphic, functions  $f_j(\cdot) : [0, \infty) \rightarrow X$  ( $1 \leq j \leq k$ ) such that  $T = \sum_{j=0}^k f_j^{(j)}$ .

We say that a distribution  $T \in \mathcal{D}'_{L^1}(X)$  is almost periodic, resp. almost automorphic, iff  $T$  satisfies any of the above two equivalent conditions. By  $B'_{AP}(X)$ ,  $B'_{AA}(X)$  we denote the space consisting of all almost periodic, resp. almost automorphic, distributions.

**Definition 1.** A distribution  $T \in \mathcal{D}'_{L^1}(X)$  is said to be asymptotically almost periodic, resp. asymptotically almost automorphic, iff there exist an almost periodic, resp. almost automorphic, distribution  $T_{ap} \in B'_{AP}(X)$ , resp.  $T_{aa} \in B'_{AA}(X)$ , and a bounded distribution tending to zero at plus infinity  $Q \in B'_{+,0}(X)$  such that  $\langle T, \varphi \rangle = \langle T_{ap}, \varphi \rangle + \langle Q, \varphi \rangle$ ,  $\varphi \in \mathcal{D}_0$ , resp.  $\langle T, \varphi \rangle = \langle T_{aa}, \varphi \rangle + \langle Q, \varphi \rangle$ ,  $\varphi \in \mathcal{D}_0$ .

By  $B'_{AAP}(X)$ , resp.  $B'_{AAA}(X)$ , we denote the vector space consisting of all asymptotically almost periodic, resp. asymptotically almost automorphic distributions.

It is well known that the representation  $T = T_{ap} + Q$  is unique in almost periodic case. This is also the case for asymptotical almost automorphy since [5, Proposition 6, 2.] continues to hold in vector-valued case (more precisely, the suppositions  $T = T_{aa}^1 + Q = T_{aa}^2 + Q_2$ , where  $T_{aa}^{1,2} \in B'_{AA}(X)$  and  $Q_{1,2} \in B'_{+,0}(X)$ , imply  $T_{aa}^1 = T_{aa}^2$  and  $Q_1 = Q_2$ ).

Further on, we would like to observe that the following structural result holds in vector-valued case:

**Theorem 1.** *Let  $T \in \mathcal{D}'_{L^1}(X)$ . Then the following assertions are equivalent:*

- (i)  $T \in B'_{AAP}(X)$ .
- (ii)  $T * \varphi \in AAP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}_0$ .
- (iii) *There exist an integer  $k \in \mathbb{N}$  and asymptotically almost periodic functions  $f_j(\cdot) : \mathbb{R} \rightarrow X$  ( $0 \leq j \leq k$ ) such that  $T = \sum_{j=0}^k f_j^{(j)}$  on  $[0, \infty)$ .*
- (iv) *There exist an integer  $k \in \mathbb{N}$  and bounded asymptotically almost periodic functions  $f_j(\cdot) : \mathbb{R} \rightarrow X$  ( $0 \leq j \leq k$ ) such that  $T = \sum_{j=0}^k f_j^{(j)}$  on  $[0, \infty)$ .*
- (v) *There exist  $S \in \mathcal{D}'_{L^1}(X)$ ,  $k \in \mathbb{N}$  and bounded asymptotically almost periodic functions  $f_j(\cdot) : \mathbb{R} \rightarrow X$  ( $0 \leq j \leq k$ ) such that  $S = \sum_{j=0}^k f_j^{(j)}$  on  $\mathbb{R}$ , and  $\langle S, \varphi \rangle = \langle T, \varphi \rangle$  for all  $\varphi \in \mathcal{D}_0$ .*
- (vi) *There exists  $S \in \mathcal{D}'_{L^1}(X)$  such that  $\langle S, \varphi \rangle = \langle T, \varphi \rangle$  for all  $\varphi \in \mathcal{D}_0$  and  $S * \varphi \in AAP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}$ .*
- (vii) *There exists a sequence  $(T_n)$  in  $\mathcal{E}(X) \cap AAP(\mathbb{R} : X)$  such that  $\lim_{n \rightarrow \infty} T_n = T$  in  $\mathcal{D}'_{L^1}(X)$ .*
- (viii)  $T * \varphi \in AAP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}$ .

*Proof.* The equivalence of (i)-(iii) can be proved as in scalar-valued case (see [12, Theorem I, Proposition 1]). It is clear that (iv) implies (iii), while the converse statement follows from the fact that there exist functions  $g_j \in AP(\mathbb{R} : X)$  and  $q_j \in C_0([0, \infty) : X)$  for  $0 \leq j \leq k$  such that

$$\begin{aligned} \langle T, \varphi \rangle &= \sum_{j=0}^k (-1)^j \int_0^\infty f_j(t) \varphi^{(j)}(t) dt \\ &= \sum_{j=0}^k (-1)^j \int_0^\infty [h_j(t) + q_{j,e}(t)] \varphi^{(j)}(t) dt, \quad \varphi \in \mathcal{D}_0, \end{aligned}$$

where  $q_{j,e}(\cdot)$  denotes the even extension of function  $q_j(\cdot)$  to the whole real axis ( $0 \leq j \leq k$ ). Therefore,  $T = \sum_{j=0}^k [h_j + q_{j,e}]^{(j)}$  on  $[0, \infty)$  and (iii) follows. The implication (iv)  $\Rightarrow$  (v) trivially follows from the fact that the expression  $S = \sum_{j=0}^k f_j^{(j)}$  defines an element from  $\mathcal{D}'_{L^1}(X)$ . Since the space  $A \equiv AAP(\mathbb{R} : X) \cap C_b(\mathbb{R} : X)$  is uniformly closed (and therefore,  $C^\infty$ -uniformly closed), closed under addition and  $A * \mathcal{D} \subseteq A$  (see [1] for the notion), an application of [1, Theorem 2.11] yields that (v) implies (vi). The implication (vi)  $\Rightarrow$  (ii) is trivial, hence we have the equivalence of assertions (i)-(vi). In order to see that (ii) imply (vii), choose arbitrarily a test function  $\rho \in \mathcal{D}$  with  $\text{supp}(\rho) \subseteq [0, 1]$ . Set  $\rho_n(\cdot) := n\rho(n\cdot)$ , for  $n \in \mathbb{N}$ . Then (ii) implies  $T * \rho_n \in \mathcal{E}(X) \cap AAP(\mathbb{R} : X)$  for all  $n \in \mathbb{N}$ . Due to the fact that  $\lim_{n \rightarrow \infty} T_n = T$  in  $\mathcal{D}'_{L^1}(X)$  (see e.g. the second part of proof of [5, Proposition 7]), we get that

(vii) holds true. The implication (vii)  $\Rightarrow$  (viii) follows from the first part of proof of [5, Proposition 7], while the implication (viii)  $\Rightarrow$  (ii) is trivial, finishing the proof of theorem.  $\square$

It can be simply proved that the following result holds true (for a proof, see [23]):

**Proposition 1.** *Let  $1 \leq p < \infty$ , and let  $f \in AAPS^p(\mathbb{R} : X)$ , resp.  $f \in AAAS^p(\mathbb{R} : X)$ . Then  $f \in B'_{AAP}(X)$ , resp.  $f \in B'_{AAA}(X)$ .*

The following analogue of Theorem 1 holds for asymptotical almost automorphy:

**Theorem 2.** *Let  $T \in \mathcal{D}'_{L^1}(X)$ . Then the following assertions are equivalent:*

- (i)  $T \in B'_{AAA}(X)$ .
- (ii)  $T * \varphi \in AAA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}_0$ .
- (iii) *There exist an integer  $k \in \mathbb{N}$  and asymptotically almost automorphic functions  $f_j(\cdot) : \mathbb{R} \rightarrow X$  ( $0 \leq j \leq k$ ) such that  $T = \sum_{j=0}^k f_j^{(j)}$  on  $[0, \infty)$ .*
- (iv) *There exist an integer  $k \in \mathbb{N}$  and bounded asymptotically almost automorphic functions  $f_j(\cdot) : \mathbb{R} \rightarrow X$  ( $0 \leq j \leq k$ ) such that  $T = \sum_{j=0}^k f_j^{(j)}$  on  $[0, \infty)$ .*
- (v) *There exist  $S \in \mathcal{D}'_{L^1}(X)$ ,  $k \in \mathbb{N}$  and bounded asymptotically almost automorphic functions  $f_j(\cdot) : \mathbb{R} \rightarrow X$  ( $0 \leq j \leq k$ ) such that  $S = \sum_{j=0}^k f_j^{(j)}$  on  $\mathbb{R}$ , and  $\langle S, \varphi \rangle = \langle T, \varphi \rangle$  for all  $\varphi \in \mathcal{D}_0$ .*
- (vi) *There exists  $S \in \mathcal{D}'_{L^1}(X)$  such that  $\langle S, \varphi \rangle = \langle T, \varphi \rangle$  for all  $\varphi \in \mathcal{D}_0$  and  $S * \varphi \in AAA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}$ .*
- (vii) *There exists a sequence  $(T_n)$  in  $\mathcal{E}(X) \cap AAA(\mathbb{R} : X)$  such that  $\lim_{n \rightarrow \infty} T_n = T$  in  $\mathcal{D}'_{L^1}(X)$ .*
- (viii)  $T * \varphi \in AAA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}$ .

*Proof.* The equivalence of (i)-(iii) follows similarly as in the proofs of [12, Theorem I, Proposition 1], given in almost periodic case. The equivalence of (iii) and (iv) as well as the fact that (iv) implies (v) can be proved as in the previous theorem. Since the space  $A \equiv AAA(\mathbb{R} : X) \cap C_b(\mathbb{R} : X)$  is uniformly closed ( $C^\infty$ -uniformly closed), closed under addition and  $A * \mathcal{D} \subseteq A$ , we can apply again [1, Theorem 2.11] in order to see that (v) implies (vi). The implication (vi)  $\Rightarrow$  (ii) is trivial, so that we have the equivalence of the assertions (i)-(vi). The remaining part of proof can be deduced as in almost periodic case.  $\square$

Concerning the assertions of Theorem 1 and Theorem 2, it is worth noting the following:



- Remark 1.* (i) The validity of (vi) for some  $S \in \mathcal{D}'_{L^1}(X)$  implies its validity for  $S$  replaced therein with  $S_Q = S + Q$ , where  $Q \in \mathcal{D}'_{L^1}(X)$  and  $\text{supp}(Q) \subseteq (-\infty, 0]$ . For this, it suffices to observe that  $(Q * \varphi)(x) = \langle Q, \varphi(x - \cdot) \rangle = 0$  for all  $x \geq \text{sup}(\text{supp}(\varphi))$ ,  $\varphi \in \mathcal{D}$ .
- (ii) Theorem 1 and Theorem 2 imply that the spaces  $B'_{AAP}(X)$  and  $B'_{AAA}(X)$  are closed under differentiation, as well as that  $B'_{AAP}(X)$  and  $B'_{AAA}(X)$  are closed subspaces of  $\mathcal{D}'_{L^1}(X)$  (for this, we can apply the equivalences of (i) and (ii) in the above-mentioned theorems, the fact that the spaces  $AAP(\mathbb{R} : X)$  and  $AAA(\mathbb{R} : X)$  are uniformly closed, and the fact that the translation  $\{\varphi(h - \cdot) : h \in \mathbb{R}\}$  of a given function  $\varphi \in \mathcal{D}$  forms a bounded subset of  $\mathcal{D}_{L^1}$ ).
- (iii) The implication (i)  $\Rightarrow$  (viii) of Theorem 1 (Theorem 2), can be proved directly; see Remark 2 below for ultradistributional case.

### 3 Asymptotical almost periodicity and asymptotical almost automorphy of vector-valued ultradistributions

For any  $h > 0$ , we define

$$\mathcal{D}_{L^1}((M_p), h) := \left\{ f \in \mathcal{D}_{L^1} ; \|f\|_{1,h} := \sup_{p \in \mathbb{N}_0} \frac{h^p \|f^{(p)}\|_1}{M_p} < \infty \right\}.$$

Then  $(\mathcal{D}_{L^1}((M_p), h), \|\cdot\|_{1,h})$  is a Banach space and the space of all  $X$ -valued bounded Beurling ultradistributions of class  $(M_p)$ , resp.,  $X$ -valued bounded Roumieu ultradistributions of class  $\{M_p\}$ , is defined to be the space of all linear continuous mappings from  $\mathcal{D}_{L^1}((M_p))$ , resp.,  $\mathcal{D}_{L^1}(\{M_p\})$ , into  $X$ , where  $\mathcal{D}_{L^1}l((M_p)) := \text{projlim}_{h \rightarrow +\infty} \mathcal{D}_{L^1}((M_p), h)$ , resp.,  $\mathcal{D}_{L^1}l(\{M_p\}) := \text{indlim}_{h \rightarrow 0^+} \mathcal{D}_{L^1}((M_p), h)$ . These spaces, equipped with the strong topologies, will be shortly denoted by  $\mathcal{D}'_{L^1}((M_p) : X)$ , resp.,  $\mathcal{D}'_{L^1}(\{M_p\} : X)$ . We will use the shorthand  $\mathcal{D}'_{L^1}{}^*(M_p : X)$  to denote either  $\mathcal{D}'_{L^1}((M_p) : X)$  or  $\mathcal{D}'_{L^1}(\{M_p\} : X)$ ;  $\mathcal{D}'_{L^1}{}^*(M_p) \equiv \mathcal{D}'_{L^1}{}^*(M_p : \mathbb{C})$ . As it can be easily proved,  $\mathcal{D}'_{L^1}{}^*(M_p : X)$  is a complete locally convex space.

It is well known that  $\mathcal{D}^{(M_p)}$ , resp.  $\mathcal{D}^{\{M_p\}}$ , is a dense subspace of  $\mathcal{D}_{L^1}((M_p))$ , resp.,  $\mathcal{D}_{L^1}(\{M_p\})$ , as well as that  $\mathcal{D}_{L^1}((M_p)) \subseteq \mathcal{D}_{L^1}(\{M_p\})$ . It can be simply proved that  $f|_{\mathcal{S}^{(M_p)}} : \mathcal{S}^{(M_p)} \rightarrow X$ , resp.,  $f|_{\mathcal{S}^{\{M_p\}}} : \mathcal{S}^{\{M_p\}} \rightarrow X$ , is a tempered  $X$ -valued ultradistribution of class  $(M_p)$ , resp., of class  $\{M_p\}$ . The space  $\mathcal{D}'_{L^1}{}^*(M_p : X)$  is closed under the action of ultradifferential operators of  $*$ -class.

The space of bounded vector-valued ultradistributions tending to zero at plus infinity,  $B'_{+,0}{}^*(X)$  for short, is defined by

$$B'_{+,0}{}^*(X) := \left\{ T \in \mathcal{D}'_{L^1}{}^*(M_p : X) : \lim_{h \rightarrow +\infty} \langle T_h, \varphi \rangle = 0 \text{ for all } \varphi \in \mathcal{D}^* \right\}.$$

Let  $T \in \mathcal{D}'_{L^1}{}^*(X)$ . Then we say that  $T$  is almost periodic, resp. almost automorphic, iff  $T$  satisfies:  $T * \varphi \in AP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}^*$ , resp.,  $T * \varphi \in AA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}^*$ . By  $B'_{AP}{}^*(X)$ , resp.  $B'_{AA}{}^*(X)$ , we denote the vector space consisting of all almost periodic, resp. almost automorphic, ultradistributions of  $*$ -class.

**Definition 2.** An ultradistribution  $T \in \mathcal{D}'_{L^1}(X)$  is said to be asymptotically almost periodic, resp. asymptotically almost automorphic, iff there exist an almost periodic, resp. almost automorphic, ultradistribution  $T_{ap} \in B'_{AAP}(X)$ , resp.  $T_{aa} \in B'_{AAA}(X)$ , and a bounded ultradistribution tending to zero at plus infinity  $Q \in B'_{+,0}(X)$  such that  $\langle T, \varphi \rangle = \langle T_{ap}, \varphi \rangle + \langle Q, \varphi \rangle$ ,  $\varphi \in \mathcal{D}_0^*$ , resp.  $\langle T, \varphi \rangle = \langle T_{aa}, \varphi \rangle + \langle Q, \varphi \rangle$ ,  $\varphi \in \mathcal{D}_0^*$ .

By  $B'_{AAP}(X)$ , resp.  $B'_{AAA}(X)$ , we denote the vector space consisting of all asymptotically almost periodic, resp. automorphic, ultradistributions of \*-class.

As in distribution case, decomposition of an asymptotically almost periodic (automorphic) ultradistribution of \*-class into its almost periodic (automorphic) part and bounded, tending to zero at plus infinity part, is unique. The space  $B'_{AAP}(X)$ , resp.  $B'_{AAA}(X)$ , is closed under the action of ultradifferential operators of \*-class. This follows from the fact that this is true for the space  $B'_{AP}(X)$ , resp.  $B'_{AA}(X)$  (see [24] and [25]), as well as that, for every  $Q \in B'_{+,0}(X)$  and for every ultradifferential operator  $P(D)$  of \*-class, we have  $\langle P(D)Q, \varphi(\cdot - h) \rangle = \langle Q, [P(D)\varphi](\cdot - h) \rangle$ ,  $h \in \mathbb{R}$ .

For the sequel, we need the following preparation. Let  $A \subseteq \mathcal{D}'^*(X)$ . Following B. Basit and H. Güenzler [1], whose examinations have been carried out in distributional case, we have recently introduced the following notion in [24]:

$$\mathcal{D}'_A(X) := \left\{ T \in \mathcal{D}'^*(X) : T * \varphi \in A \text{ for all } \varphi \in \mathcal{D}^* \right\}.$$

Then  $\mathcal{D}'_A(X) = \mathcal{D}'_{A \cap B}(X)$ , for any set  $B \subseteq L^1_{loc}(\mathbb{R} : X)$  that contains  $C^\infty(\mathbb{R} : X)$ , as well as the set  $\mathcal{D}'_A(X)$  is closed under the action of ultradifferential operators of \*-class. Furthermore, the following holds [24]:

- (i) Assume that there exist an ultradifferential operator  $P(D) = \sum_{p=0}^{\infty} a_p D^p$  of class  $(M_p)$ , resp., of class  $\{M_p\}$ , and  $f, g \in \mathcal{D}'_A(X)$  such that  $T = P(D)f + g$ . If  $A$  is closed under addition, then  $T \in \mathcal{D}'_A(X)$ .
- (ii) If  $A \cap C(\mathbb{R} : X)$  is closed under uniform convergence,  $T \in \mathcal{D}'_{L^1}((M_p) : X)$  and  $T * \varphi \in A$ ,  $\varphi \in \mathcal{D}^{(M_p)}$ , then there is a number  $h > 0$  such that for each compact set  $K \subseteq \mathbb{R}$  we have  $T * \varphi \in A$ ,  $\varphi \in \mathcal{D}_K^{M_p, h}$ .
- (iii) Assume that  $T \in \mathcal{D}'^{(M_p)}(X)$  and there exists  $h > 0$  such that for each compact set  $K \subseteq \mathbb{R}$  we have  $T * \varphi \in A$ ,  $\varphi \in \mathcal{D}_K^{M_p, h}$ . If  $(M_p)$  additionally satisfies (M.3), then there exist  $l > 0$  and two elements  $f, g \in A$  such that  $T = P(D)f + g$ .

Now we will focus our attention to the case that  $A = AAP(\mathbb{R} : X)$ , resp.  $A = AAA(\mathbb{R} : X)$ . Then  $A$  is closed under the uniform convergence and addition, and we have  $A \subseteq \mathcal{D}'_A(X)$  [23]. Hence, as a special case of the above assertions, we have the following lemma:

**Lemma 1.** *Let  $(M_p)$  satisfy the conditions (M.1), (M.2) and (M.3'), and let  $T \in \mathcal{D}'^{(M_p)}(X)$ , resp.,  $T \in \mathcal{D}'^{\{M_p\}}(X)$ . Then the following holds:*

- (i) Suppose that there exist an ultradifferential operator  $P(D) = \sum_{p=0}^{\infty} a_p D^p$  of class  $(M_p)$ , resp., of class  $\{M_p\}$ , and  $f, g \in \mathcal{D}'_{AAP(\mathbb{R};X)}(X)$ , resp.  $f, g \in \mathcal{D}'_{AAA(\mathbb{R};X)}(X)$ , such that  $T = P(D)f + g$ . Then  $T \in \mathcal{D}'_{AAP(\mathbb{R};X)}(X)$ , resp.  $T \in \mathcal{D}'_{AAA(\mathbb{R};X)}(X)$ .
- (ii) If  $T \in \mathcal{D}'_{L^1}((M_p) : X)$  and  $T * \varphi \in AAP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}^{(M_p)}$ , resp.  $T * \varphi \in AAA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}^{(M_p)}$ , then there is a number  $h > 0$  such that for each compact set  $K \subseteq \mathbb{R}$  we have  $T * \varphi \in AAP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}_K^{M_p, h}$ , resp.  $T * \varphi \in AAA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}_K^{M_p, h}$ .
- (iii) Assume that  $T \in \mathcal{D}'^{(M_p)}(X)$  and there is a number  $h > 0$  such that for each compact set  $K \subseteq \mathbb{R}$  we have  $T * \varphi \in AAP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}_K^{M_p, h}$ , resp.  $T * \varphi \in AAA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}_K^{M_p, h}$ . If  $(M_p)$  additionally satisfies (M.3), then there exist  $l > 0$  and bounded functions  $f, g \in AAP(\mathbb{R} : X)$ , resp.  $f, g \in AAA(\mathbb{R} : X)$ , such that  $T = P_l(D)f + g$ .

We need one more lemma, which is probably known in the existing literature.

**Lemma 2.** (Supremum formula) Let  $f \in AA(\mathbb{R} : X)$ . Then for each  $a \in \mathbb{R}$  we have  $\|f\|_{\infty} = \sup_{x \geq a} \|f(x)\|$ .

*Proof.* Essentially, we need only to prove that for each  $a, y \in \mathbb{R}$  and  $\epsilon > 0$  we have

$$\|f(y)\| \leq \sup_{x \geq a} \|f(x)\| + \epsilon. \quad (1)$$

By definition of almost automorphy, for the sequence  $(b_n := n)$  we can extract a subsequence  $(a_n)$  of it such that  $f(y) = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} f(y - a_k + a_n)$ . This, in particular, implies that we can find two integers  $n_0(\epsilon), k_0(\epsilon) \in \mathbb{N}$  such that, for every  $k \geq k_0(\epsilon)$ , we have  $\|f(y - a_k + a_n) - f(y)\| \leq \epsilon$ . This gives (1) and finishes the proof of lemma.  $\square$

Now we are able to state the following important result:

**Theorem 3.** Let  $(M_p)$  satisfy the conditions (M.1), (M.2) and (M.3'), and let  $T \in \mathcal{D}'_{L^1}(M_p : X)$ . Consider the following assertions:

- (i) There exist an element  $S \in \mathcal{D}'_{L^1}(M_p : X)$ , a number  $l > 0$  in the Beurling case/a sequence  $(r_p) \in \mathbb{R}$  in the Roumieu case, and bounded functions  $f, g \in AAP(\mathbb{R} : X)$ , resp.  $f, g \in AAA(\mathbb{R} : X)$ , such that  $S = P_l(D)f + g$ , resp.  $S = P_{r_p}(D)f + g$ , and  $S = T$  on  $[0, \infty)$ .
- (ii) There exist a number  $l > 0$ , resp. a sequence  $(r_p) \in \mathbb{R}$ , and bounded functions  $f, g \in AAP(\mathbb{R} : X)$ , resp.  $f, g \in AAA(\mathbb{R} : X)$ , such that  $T = P_l(D)f + g$ , resp.  $T = P_{r_p}(D)f + g$ , on  $[0, \infty)$ .
- (iii) There exist an ultradifferential operator  $P(D) = \sum_{p=0}^{\infty} a_p D^p$  of  $*$ -class and bounded functions  $f_1, f_2 \in AAP(\mathbb{R} : X)$ , resp.  $f_1, f_2 \in AAA(\mathbb{R} : X)$ , such that  $T = P(D)f_1 + f_2$  on  $[0, \infty)$ .

- (iv) There exist an element  $S \in \mathcal{D}'_{L^1}(M_p : X)$ , an ultradifferential operator  $P(D) = \sum_{p=0}^{\infty} a_p D^p$  of  $*$ -class and bounded functions  $f_1, f_2 \in AAP(\mathbb{R} : X)$ , resp.  $f_1, f_2 \in AAA(\mathbb{R} : X)$ , such that  $S = P(D)f_1 + f_2$  and  $S = T$  on  $[0, \infty)$ .
- (v)  $T \in B'_{AAP}(X)$ , resp.  $T \in B'_{AAA}(X)$ , there exist an element  $S \in \mathcal{D}'_{L^1}(M_p : X)$ , an ultradifferential operator  $P(D) = \sum_{p=0}^{\infty} a_p D^p$  of  $*$ -class and bounded functions  $f_1, f_2 \in AAP(\mathbb{R} : X)$ , resp.  $f_1, f_2 \in AAA(\mathbb{R} : X)$ , such that  $S = P(D)f_1 + f_2$  and  $S = T$  on  $[0, \infty)$ .
- (vi)  $T \in B'_{AAP}(X)$ , resp.  $T \in B'_{AAA}(X)$ , and there exists an element  $S \in \mathcal{D}'_{L^1}(M_p : X)$  such that  $S = T$  on  $[0, \infty)$  and  $S * \varphi \in AAP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}^*$ , resp.  $S * \varphi \in AAA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}^*$ .
- (vii)  $T \in B'_{AAP}(X)$ , resp.  $T \in B'_{AAA}(X)$ .
- (viii)  $T * \varphi \in AAP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}_0^*$ , resp.  $T * \varphi \in AAA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}_0^*$ .
- (ix) There exists a sequence  $(T_n)$  in  $\mathcal{E}^*(X) \cap AAP(\mathbb{R} : X)$ , resp.  $\mathcal{E}^*(X) \cap AAA(\mathbb{R} : X)$ , such that  $\lim_{n \rightarrow \infty} T_n = T$  in  $\mathcal{D}'_{L^1}(X)$ .
- (x)  $T * \varphi \in AAP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}^*$ , resp.  $T * \varphi \in AAA(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}^*$ .

Then we have (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v)  $\Rightarrow$  (vi)  $\Leftrightarrow$  (vii)  $\Leftrightarrow$  (viii)  $\Leftrightarrow$  (ix)  $\Leftrightarrow$  (x). Furthermore, if  $(M_p)$  additionally satisfies the condition (M.3), then the assertions (i)-(x) are equivalent for the Beurling class.

*Proof.* For the sake of brevity, we will consider only the asymptotically almost periodic case for the Beurling class (in the case of almost automorphy, it is only worth noting that, for the proof of implication (ix)  $\Rightarrow$  (vii), we need to use the supremum formula deduced in Lemma 2). The implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv) and (vi)  $\Rightarrow$  (vii) are trivial, while the implication (vii)  $\Rightarrow$  (viii) can be deduced as it has been done for scalar-valued distributions (cf. the proof of implication (i)  $\Rightarrow$  (ii) of [12, Theorem I]). In order to see that (iv) implies (v), we only need to prove that the assumptions of (iv) imply  $T \in B'^{(M_p)}(X)$ . Suppose that  $f_i(\cdot) = g_i(\cdot) + q_i(\cdot)$  on  $[0, \infty)$ , where  $f_i \in AP(\mathbb{R} : X)$  and  $g_i \in C_0([0, \infty) : X)$  ( $i = 1, 2$ ). Then  $T_1 := P(D)g_1 + g_2$  is an almost periodic ultradistribution of Beurling class due to [24, Theorem 2] (see [25, Theorem 3.1] for almost automorphic case), and all that we need to show is  $T_2 := P(D)(f_1 - g_1) + (f_2 - g_2) \in B'_{+,0}(M_p)(X)$ , i.e.,

$$\lim_{h \rightarrow +\infty} \left\langle P(D)(f_1 - g_1) + (f_2 - g_2), \varphi(\cdot - h) \right\rangle = 0, \quad \varphi \in \mathcal{D}^{(M_p)}.$$

Towards this end, assume that  $-\infty < a < b < +\infty$  and  $\text{supp}(\varphi) \subseteq [a, b]$ . Let  $\epsilon > 0$  be given. Then there exist a sufficiently large finite number  $h_0(\epsilon) > 0$  and a sufficiently large finite number  $c_\varphi > 0$ , independent of  $\epsilon$ , such that, for every  $h \geq h_0(\epsilon)$ , we have the following (cf. also the proof of [24, Theorem 1]):

$$\left\| \left\langle P(D)(f_1 - g_1) + (f_2 - g_2), \varphi(\cdot - h) \right\rangle \right\|$$

$$\begin{aligned}
&= \left\| \sum_{p=0}^{\infty} (-1)^p a_p \int_{-\infty}^{\infty} (f_1(t) - g_1(t)) \varphi^{(p)}(t-h) dt + \int_{-\infty}^{\infty} (f_2(t) - g_2(t)) \varphi(t-h) dt \right\| \\
&= \left\| \sum_{p=0}^{\infty} (-1)^p a_p \int_a^b (f_1(t+h) - g_1(t+h)) \varphi^{(p)}(t) dt + \int_a^b (f_2(t) - g_2(t)) \varphi(t) dt \right\| \\
&= \left\| \sum_{p=0}^{\infty} (-1)^p a_p \int_a^b q_1(t+h) \varphi^{(p)}(t) dt + \int_a^b q_2(t) \varphi(t) dt \right\| \\
&\leq \epsilon \left[ \sum_{p=0}^{\infty} |a_p| \|\varphi^{(p)}\|_{L^1} + \|\varphi\|_{L^1} \right] \leq \epsilon c_{\varphi}.
\end{aligned}$$

This yields (v). The implication (v)  $\Rightarrow$  (vi) follows by applying Lemma 1. In order to see that (vii) implies (vi), assume that  $T = T_{ap} + Q$  on  $[0, \infty)$ , where  $T_{ap}$  and  $Q$  satisfy the requirements of Definition 2. Put  $S := T_{ap} + Q$  on  $\mathbb{R}$ . Then we need to prove that  $S * \varphi \in AAP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}^*$ , i.e., that  $\lim_{h \rightarrow +\infty} (Q * \varphi)(h) = 0$ ,  $\varphi \in \mathcal{D}^*$ . But, this follows from the fact that  $(Q * \varphi)(h) = \langle Q, \check{\varphi}(\cdot - h) \rangle = \langle Q_h, \check{\varphi} \rangle \rightarrow 0$  as  $h \rightarrow +\infty$  by definition of space  $\mathcal{D}'_{+,0}{}^{(M_p)}(X)$ . We can show that (viii) implies (ix) as in the proof of implication of (ii)  $\Rightarrow$  (vii) in Theorem 1. The implication (ix)  $\Rightarrow$  (x) follows similarly as in the first part of the proof of [5, Proposition 7], while the implication (x)  $\Rightarrow$  (viii) is trivial. Now we will prove the implication (ix)  $\Rightarrow$  (vii). Let  $(T_n)$  be a sequence in  $\mathcal{E}^{(M_p)}(X) \cap AAP(\mathbb{R} : X)$  such that  $\lim_{n \rightarrow \infty} T_n = T$  in  $\mathcal{D}'_{L^1}{}^{(M_p)}(X)$ , and let  $T_n = G_n + Q_n$  on  $[0, \infty)$  for some  $G_n \in AP(\mathbb{R} : X)$  and  $Q_n \in C_b(\mathbb{R} : X)$  tending to zero at plus infinity ( $n \in \mathbb{N}$ ). Let  $B$  be a bounded subset of  $\mathcal{D}'_{L^1}{}^{(M_p)}(X)$ , and let  $\epsilon > 0$  be given. It can be simply shown that the set  $B' := B \cup \{\varphi(\pm \cdot \pm h) : h \in \mathbb{R}, \varphi \in B\}$  is a bounded subset of  $\mathcal{D}'_{L^1}{}^{(M_p)}(X)$ , as well. Then  $(G_n - G_m) + (Q_n - Q_m) = T_n - T_m$ ,  $m, n \in \mathbb{N}$ , so that there exists an integer  $n_0(\epsilon) \in \mathbb{N}$  such that, for every  $m, n \in \mathbb{N}$  with  $\min(m, n) \geq n_0(\epsilon)$ , we have:

$$\sup_{\varphi \in B'} \left\| \langle G_n - G_m, \varphi \rangle + \langle Q_n - Q_m, \varphi \rangle \right\| \leq \epsilon;$$

especially,

$$\sup_{\varphi \in B', h \in \mathbb{R}} \left\| \langle G_n - G_m, \varphi(h - \cdot) \rangle + \langle Q_n - Q_m, \varphi(h - \cdot) \rangle \right\| \leq \epsilon. \quad (2)$$

By the foregoing, we have  $\lim_{h \rightarrow +\infty} \|\langle Q_n - Q_m, \varphi(h - \cdot) \rangle\| = 0$ , for every  $m, n \in \mathbb{N}$  with  $\min(m, n) \geq n_0(\epsilon)$ , so that there exists a sufficiently large number  $M_{\epsilon} > 0$  such that

$$\left\| \langle G_n - G_m, \varphi(h - \cdot) \rangle \right\| \leq \epsilon, \quad m, n \geq n_0(\epsilon), h \geq M_{\epsilon}, \varphi \in B'.$$

By Lemma 2 and the almost periodicity of function on the left hand side of the above inequality, we get

$$\left\| \langle G_n - G_m, \varphi(h - \cdot) \rangle \right\| \leq \epsilon, \quad m, n \geq n_0(\epsilon), h \in \mathbb{R}, \varphi \in B'. \quad (3)$$

This clearly implies that the sequence  $(\langle G_n, \varphi \rangle)$  is Cauchy in  $X$  and therefore convergent ( $\varphi \in \mathcal{D}_{L^1}$ ). Set  $\langle G, \varphi \rangle := \lim_{m \rightarrow \infty} \langle G_m, \varphi \rangle$  for all  $\varphi \in \mathcal{D}_{L^1}$ . Letting  $m \rightarrow \infty$  in (3), we obtain

$$\left\| \langle G_n - G, \varphi(h - \cdot) \rangle \right\| \leq \epsilon, \quad n \geq n_0(\epsilon), \quad h \in \mathbb{R}, \quad \varphi \in B'. \quad (4)$$

This, in turn, implies that  $G$  is the limit of convergent sequence  $(G_n)$  in  $\mathcal{D}'_{L^1}(M_p)(X)$  and therefore  $G \in \mathcal{D}'_{L^1}(M_p)(X)$ . Furthermore, (4) shows that  $\lim_{n \rightarrow \infty} G_n * \varphi = G * \varphi$ ,  $\varphi \in \mathcal{D}^{(M_p)}$  in  $C_b(\mathbb{R} : X)$ , so that  $G * \varphi \in AP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}^{(M_p)}$ ; hence,  $G \in B'_{AAP}(X)$ . Define  $\langle Q, \varphi \rangle := \lim_{m \rightarrow \infty} \langle T - G_m, \varphi \rangle$  for all  $\varphi \in \mathcal{D}_{L^1}$ . Then  $Q \in \mathcal{D}'_{L^1}(M_p)(X)$  and all that we need to show is that  $Q$  tends to zero at plus infinity. For this, observe first that  $Q$  has to be the limit of a Cauchy sequence  $(Q_n)$  in  $\mathcal{D}'_{L^1}(M_p)(X)$  and that combining (2) and (3) yields

$$\left\| \langle Q_n - Q_m, \varphi(h - \cdot) \rangle \right\| \leq 2\epsilon, \quad m, n \geq n_0(\epsilon), \quad h \in \mathbb{R}, \quad \varphi \in B'. \quad (5)$$

Due to (5), we obtain

$$\left\| \langle Q_n - Q, \varphi(h - \cdot) \rangle \right\| \leq 2\epsilon, \quad n \geq n_0(\epsilon), \quad h \in \mathbb{R}, \quad \varphi \in B'.$$

With  $n = n_0(\epsilon)$ , the above simply yields the existence of a number  $h_0(\epsilon) > 0$  such that

$$\left\| \langle Q, \varphi(h - \cdot) \rangle \right\| \leq 3\epsilon, \quad h \geq h_0, \quad \varphi \in B'.$$

Hence, (vii) is proved. Finally, if  $(M_p)$  satisfies the condition (M.3), then the implication (x)  $\Rightarrow$  (i) follows by applying Lemma 1(ii)-(iii) with  $S = T$ , and therefore, the assertions (i)-(x) are mutually equivalent for the Beurling class.  $\square$

Using the equivalence of items (vii) and (x) in Theorem 3, as well as the continuity properties of convolution, it readily follows that  $B'_{AAP}(X)$ , resp.  $B'_{AAA}(X)$ , is a closed subspace of  $\mathcal{D}'_{L^1}(M_p : X)$ . Concerning Theorem 3, we have the following observations:

*Remark 2.* The implication (vii)  $\Rightarrow$  (x) of Theorem 3 can be proved directly, as explained below. Let  $T = T_{ap} + Q$  on  $[0, \infty)$ , with  $T_{ap}$  and  $Q$  being the same as in Definition 2. Since

$$(T * \varphi)(h) = (T_{ap} * \varphi)(h) + \left( [T - T_{ap}] * \varphi \right)(h), \quad h \in \mathbb{R}, \quad \varphi \in \mathcal{D}^*,$$

and  $T_{ap} * \varphi \in AP(\mathbb{R} : X)$ ,  $\varphi \in \mathcal{D}^*$ , we need to prove that

$$\lim_{h \rightarrow +\infty} \left( [T - T_{ap}] * \varphi \right)(h) = 0, \quad \varphi \in \mathcal{D}^*$$

i.e., that:

$$\lim_{h \rightarrow +\infty} \langle T - T_{ap}, \varphi(h - \cdot) \rangle = 0, \quad \varphi \in \mathcal{D}^*. \quad (6)$$

Since  $\text{supp}(\varphi(h - \cdot)) \subseteq [0, \infty)$  for  $h \geq \text{sup}(\text{supp}(\varphi))$ , we have that (6) is equivalent with

$$\lim_{h \rightarrow +\infty} \langle Q, \varphi(h - \cdot) \rangle = 0, \quad \varphi \in \mathcal{D}^*.$$

But, this follows from the equality  $\langle Q, \varphi(h - \cdot) \rangle = \langle Q_h, \check{\varphi} \rangle$  for all  $\varphi \in \mathcal{D}^*$ , and the fact that  $Q \in B'_{+,0}(X)$ .

*Remark 3.* Assume that  $(M_p)$  additionally satisfies (M.3). We feel duty bound to say that, in this case, it seems very plausible that the assertion of Theorem 3 can be slightly generalized for  $\omega$ -ultradistributions (see [17] for more details). The interested reader may also try to prove analogues of [9, Theorem 4.3.1, p. 123; Theorem 4.3.2, p. 124; Theorem 4.5.1, p. 128] for asymptotically almost periodic (automorphic) ultradistributions of  $*$ -class.

The assertion of [12, Proposition 2] can be formulated for vector-valued almost periodic distributions and vector-valued almost periodic ultradistributions of Beurling class, with the corresponding sequence  $(M_p)$  satisfying (M.1), (M.2) and (M.3). Now we would like to state the following automorphic version of [12, Proposition 2] (vector-valued case):

**Proposition 2.** (i) Consider the equation  $T' = f \cdot T + U$  in  $\mathcal{D}'(X)$ , where  $f \in \mathcal{E} \cap AA(\mathbb{R} : \mathbb{C})$  and  $U \in B'_{AA}(X)$ . If  $T \in B'_{AAA}(X)$  is a solution of this equation on  $[0, \infty)$ , then its almost automorphic part is a solution of this equation on  $\mathbb{R}$ .

(ii) Let  $(M_p)$  satisfy (M.1), (M.2) and (M.3). Consider the equation  $T' = f \cdot T + U$  in  $\mathcal{D}'^{(M_p)}(X)$ , where  $f \in \mathcal{E}^{(M_p)} \cap AA(\mathbb{R} : \mathbb{C})$  and  $U \in B'^{(M_p)}_{AA}(X)$ . If  $T \in B'^{(M_p)}_{AAA}(X)$  is a solution of this equation on  $[0, \infty)$ , then its almost automorphic part is a solution of this equation on  $\mathbb{R}$ .

*Proof.* Since the pointwise product of an almost automorphic  $X$ -valued function with a scalar-valued almost automorphic function is again an almost automorphic  $X$ -valued function, we can use the structural characterization of  $B'_{AA}(X)$ , resp.  $B'^{(M_p)}_{AA}(X)$ , in order to see that these spaces are closed under pointwise multiplication with a scalar-valued function  $f \in \mathcal{E} \cap AA(\mathbb{R} : \mathbb{C})$ , resp.  $f \in \mathcal{E}^{(M_p)} \cap AA(\mathbb{R} : \mathbb{C})$ ; furthermore, the almost automorphy of a bounded distribution, resp. bounded ultradistribution of Beurling class, is equivalent to say that, for every real sequence  $(b_n)$ , there exist a subsequence  $(a_n)$  of  $(b_n)$  and a vector-valued distribution  $S \in \mathcal{D}'(X)$ , resp. ultradistribution  $S \in \mathcal{D}'^{(M_p)}(X)$ , such that  $\lim_{n \rightarrow \infty} \langle T_{a_n}, \varphi \rangle = \langle S, \varphi \rangle$  for all  $\varphi \in \mathcal{D}$ , resp.  $\varphi \in \mathcal{D}^{(M_p)}$ , and  $\lim_{n \rightarrow \infty} \langle S_{-a_n}, \varphi \rangle = \langle T, \varphi \rangle$  for all  $\varphi \in \mathcal{D}$ , resp.  $\varphi \in \mathcal{D}^{(M_p)}$ ; see [25]. Using these facts, we can repeat almost literally the proof

of [12, Proposition 2] to deduce the validity of (i) and (ii); the only thing worth noting is that we do not need Bochner's criterion for the proof because the limits appearing on lines 10 and -4 on p. 258 of [12] follows, actually, from the almost automorphy of corresponding distributions.  $\square$

#### 4 An application to systems of ordinary differential equations in distribution and ultradistribution spaces

Let  $n \in \mathbb{N}$ , and let  $A = [a_{ij}]_{1 \leq i, j \leq n}$  be a given complex matrix such that  $\sigma(A) \subseteq \{z \in \mathbb{C} : \Re z < 0\}$ . In this section, we analyze the existence of asymptotically almost periodic (automorphic) solutions of equation

$$T' = AT + F, \quad T \in \mathcal{D}'(X^n) \quad \text{on } [0, \infty) \quad (7)$$

and equation

$$T' = AT + F, \quad T \in \mathcal{D}'^*(X^n) \quad \text{on } [0, \infty), \quad (8)$$

where  $F$  is an asymptotically almost periodic (automorphic)  $X^n$ -valued distribution in (7) and  $F$  is an asymptotically almost periodic (automorphic)  $X^n$ -valued ultradistribution of  $*$ -class in (8). By a solution of (7), resp. (8), we mean any distribution  $T \in \mathcal{D}'(X^n)$ , resp.  $\mathcal{D}'^*(X^n)$ , such that (7), resp. (8), holds in distributional, resp. ultradistributional, sense on  $[0, \infty)$ .

The main result of this section reads as follows:

**Theorem 4.** (i) Let  $F = [F_1 \ F_2 \ \cdots \ F_n]^T \in B'_{AAP}(X^n)$ , resp.  $B'_{AAA}(X^n)$ . Then there exists a solution  $T = [T_1 \ T_2 \ \cdots \ T_n]^T \in B'_{AAP}(X^n)$ , resp.  $B'_{AAA}(X^n)$  of (7). Furthermore, any distributional solution  $T$  of (7) belongs to the space  $B'_{AAP}(X^n)$ , resp.  $B'_{AAA}(X^n)$ .

(ii) Let  $(M_p)$  satisfy the conditions (M.1), (M.2) and (M.3'), and let  $F = [F_1 \ F_2 \ \cdots \ F_n]^T \in \mathcal{D}'^*_{L^1}(M_p : X^n)$  be such that, for every  $i \in \mathbb{N}_n$ , there exist an ultradifferential operator  $P_i(D) = \sum_{p=0}^{\infty} a_{i,p} D^p$  of  $*$ -class and bounded functions  $f_{1,i}, f_{2,i} \in AAP(\mathbb{R} : X)$ , resp.  $AAA(\mathbb{R} : X)$ , for  $1 \leq i \leq n$ , such that  $F_i = P_i(D)f_{1,i} + f_{2,i}$  on  $[0, \infty)$ . Then there exist ultradifferential operators  $P_{ij}(D)$  of  $*$ -class, bounded functions  $h_{ij} \in AAP(\mathbb{R} : X)$ , resp.  $AAA(\mathbb{R} : X)$  ( $1 \leq i, j \leq n$ ), and bounded functions  $h_{2,i} \in AAP(\mathbb{R} : X)$ , resp.  $AAA(\mathbb{R} : X)$  ( $1 \leq i \leq n$ ), such that  $T = [T_1 \ T_2 \ \cdots \ T_n]^T \in B'^*_{AAP}(X^n)$ , resp.  $B'^*_{AAA}(X^n)$ , is a solution of (8), where  $T_i = \sum_{j=1}^n P_{ij}(D)h_{ij} + h_{2,i}$  for  $i \in \mathbb{N}_n$ . Furthermore, any ultradistributional solution  $T$  of  $*$ -class to the equation (8) has such a form.

*Proof.* We will prove only (ii). Since  $\sigma(A) \subseteq \{z \in \mathbb{C} : \Re z < 0\}$ , [7, Lemma 1] and a careful inspection of the proof of [7, Theorem 3, pp. 117-118] yield that it is sufficient to prove the required assertion in one-dimensional case. So, let  $\Re \lambda < 0$  and let us consider the equation

$$T' = \lambda T + P(D)f + g, \quad T \in \mathcal{D}'^*(X) \quad \text{on } [0, \infty), \quad (9)$$



with given bounded functions  $f, g \in AAP(\mathbb{R} : X)$ , resp.  $f, g \in AAA(\mathbb{R} : X)$ , and  $P(D) = \sum_{p=0}^{\infty} a_p D^p$ . Then  $(e^{-\lambda \cdot} T)' = e^{-\lambda \cdot} [P(D)f + g]$  on  $[0, \infty)$ , which implies  $\langle e^{-\lambda \cdot} T, \varphi \rangle = -\langle e^{-\lambda \cdot} [P(D)f + g], I(\varphi) \rangle + \langle \text{Const.}, \varphi \rangle$ ,  $\varphi \in \mathcal{D}_0^*$ . Therefore, there exists a constant  $c \in \mathbb{C}$  such that

$$\begin{aligned} \langle T, \varphi \rangle &= \left\langle P(D)f + g, e^{-\lambda \cdot} \int_{-\infty}^{\cdot} \left[ e^{\lambda t} \varphi(t) - \nu(t) \int_{-\infty}^{\infty} e^{\lambda u} \varphi(u) du \right] dt \right\rangle \\ &\quad + c \int_{-\infty}^{\infty} e^{\lambda u} \varphi(u) du, \quad \varphi \in \mathcal{D}_0^*. \end{aligned}$$

Applying product rule for the term containing the function  $f(\cdot)$ , as well as the Fubini theorem and partial integration for the term containing the function  $g(\cdot)$ , we get:

$$\begin{aligned} \langle T, \varphi \rangle &= \sum_{p=0}^{\infty} (-1)^{p+1} a_p \int_{-\infty}^{\infty} f(t) \left[ e^{-\lambda \cdot} I(e^{\lambda \cdot} \varphi)(\cdot) \right]^{(p)}(t) dt + c \int_{-\infty}^{\infty} e^{\lambda u} \varphi(u) du \\ &\quad + \int_{-\infty}^{\infty} g(t) e^{-\lambda t} \int_{-\infty}^t \left[ e^{\lambda u} \varphi(u) - \nu(u) \int_{-\infty}^{\infty} e^{\lambda v} \varphi(v) dv \right] du dt \\ &= \sum_{p=0}^{\infty} a_p \int_{-\infty}^{\infty} f(t) \sum_{k=1}^p (-1)^{k+1} \binom{p}{k} \lambda^{p-k} e^{-\lambda t} \left[ \sum_{j=0}^{k-1} \binom{k-1}{j} \lambda^{k-j} e^{\lambda t} \varphi^{(j)}(t) \right. \\ &\quad \left. - \nu^{(k-1)}(t) \int_{-\infty}^{\infty} e^{\lambda u} \varphi(u) du \right] \\ &\quad - \sum_{p=0}^{\infty} a_p \int_{-\infty}^{\infty} f(t) \lambda^p e^{-\lambda t} \int_{-\infty}^t \left[ e^{\lambda u} \varphi(u) - \nu(t) \int_{-\infty}^{\infty} e^{\lambda v} \varphi(v) dv \right] du dt \\ &\quad + c \int_0^{\infty} e^{\lambda u} \varphi(u) du + \int_0^{\infty} \left[ \int_0^t e^{\lambda(t-s)} g(s) ds \right] \varphi(t) dt \\ &\quad - \int_0^{\infty} e^{\lambda v} \left[ \int_{-\infty}^{\infty} \nu(t) \int_0^t e^{-\lambda s} g(s) ds dt \right] \varphi(v) dv, \quad \varphi \in \mathcal{D}_0^*. \end{aligned}$$

Applying again the Fubini theorem for the term containing the part  $\nu^{(k-1)}(\cdot)$  as well as the Fubini theorem and partial integration for the term containing the part  $f(t) \lambda^p e^{-\lambda t}$ , we get

$$\begin{aligned} \langle T, \varphi \rangle &= \int_0^{\infty} f(t) \left[ \sum_{p'=p+1}^{\infty} a_{p'} \sum_{k=p}^{p'} (-1)^{k+1} \binom{p'}{k} \binom{k-1}{p} \lambda^{p'-p} \right] \varphi^{(p)}(t) dt \\ &\quad - \int_0^{\infty} e^{\lambda v} \left[ \sum_{p=0}^{\infty} a_p \int_1^2 f(t) \sum_{k=1}^p (-1)^{k+1} \binom{p}{k} \lambda^{p-k} e^{-\lambda t} \nu^{(k-1)}(t) dt \right] \varphi(v) dv \\ &\quad - \int_0^{\infty} \left[ \sum_{p=0}^{\infty} a_p \lambda^p \int_0^t e^{\lambda(t-s)} f(s) ds \right] \varphi(t) dt \end{aligned}$$

$$\begin{aligned}
& + \int_0^\infty e^{\lambda v} \left[ \sum_{p=0}^\infty a_p \lambda^p \int_{-\infty}^\infty \nu(t) \int_0^t e^{-\lambda s} f(s) ds dt \right] \varphi(v) dv + c \int_0^\infty e^{\lambda u} \varphi(u) du \\
& + \int_0^\infty \left[ \int_0^t e^{\lambda(t-s)} g(s) ds \right] \varphi(t) dt \\
& - \int_0^\infty e^{\lambda v} \left[ \int_{-\infty}^\infty \nu(t) \int_0^t e^{-\lambda s} g(s) ds dt \right] \varphi(v) dv, \quad \varphi \in \mathcal{D}_0^*.
\end{aligned}$$

The series in the first and fourth addend converge, whereas the functions  $\int_0^\cdot e^{\lambda(\cdot-s)} g(s) ds$ ,  $ce^{\lambda \cdot}$  are asymptotically almost periodic, resp. asymptotically almost automorphic, due to our assumption  $\Re \lambda < 0$  (see [23] for more details). The series  $\sum_{p=0}^\infty a_p \lambda^p \int_{-\infty}^\infty \nu(t) \int_0^t e^{-\lambda s} f(s) ds dt$  converges uniformly for  $t \geq 0$  and, to complete the whole proof, it suffices to show that the series

$$\sum_{p=0}^\infty a_p \int_1^2 f(t) \sum_{k=1}^p (-1)^{k+1} \binom{p}{k} \lambda^{p-k} e^{-\lambda t} \nu^{(k-1)}(t) dt$$

is absolutely convergent. Towards this end, observe that (M.3') yields the existence of a finite constant  $c' > 0$  such that the absolute value of

$$a_p \int_1^2 f(t) \sum_{k=1}^p (-1)^{k+1} \binom{p}{k} \lambda^{p-k} e^{-\lambda t} \nu^{(k-1)}(t) dt$$

does not exceed  $\|f\|_\infty \text{Const.} |a_p| \sum_{k=0}^p \frac{M_k}{h^k} \leq \|f\|_\infty \text{Const.} |a_p| (p+1) m^p \frac{M_p}{h^p}$ , where  $m := \sum_{p=1}^\infty 1/m_p < \infty$ . This completes the proof in a routine manner.  $\square$

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## References

- [1] BASIT, B., GÜENZLER, H. *Generalized vector valued almost periodic and ergodic distributions*, J. Math. Anal. Appl. **314** (2006), 363–381.
- [2] BOCHNER, S. *A new approach to almost periodicity*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 2039–2043.
- [3] BOHR, H. *Zur theorie der fastperiodischen Funktionen I; II; III*, Acta Math. **45** (1924), 29–127; **H6** (1925), 101–214; **HT** (1926), 237–281.
- [4] BOUZAR, C., KHALLADI, M. T., TCHOUAR, F. Z. *Almost automorphic generalized functions*, Novi Sad J. Math. **45** (2015), 207–214.
- [5] BOUZAR, C., TCHOUAR, F. Z. *Almost automorphic distributions*, Mediterr. J. Math. **14:151** (2017), <https://doi.org/10.1007/s00009-017-0953-3>.

- [6] BOUZAR, C., KHALLADI, M. T. *On asymptotically almost periodic generalized solutions of differential equations*, Operator Theory: Advances and Applications, Vol. **245**, eds. S. Pilipović, J. Toft, (2015), 35–43.
- [7] BOUZAR, C., KHALLADI, M. T. *Linear differential equations in the algebra of almost periodic generalized functions*, Rend. Sem. Mat. Univ. Politec. Torino **70** (2012), 111–120.
- [8] CARMICHEAL, R., KAMIŃSKI, A., PILIPOVIĆ, S. *Notes on Boundary Values in Ultradistribution Spaces*, Lect. Notes Ser., Seoul **49**, 1999.
- [9] CHEBAN, D. N. *Asymptotically Almost Periodic Solutions of Differential Equations*, Hindawi Publishing Corporation, 2009.
- [10] CIORANESCU, I. *On the abstract Cauchy problem in spaces of almost periodic distributions*, J. Math. Anal. Appl. **148** (1990), 440–462.
- [11] CIORANESCU, I. *The characterization of the almost periodic ultradistributions of Beurling type*, Proc. Amer. Math. Soc. **116** (1992), 127–134.
- [12] CIORANESCU, I. *Asymptotically almost periodic distributions*, Appl. Anal. **34** (1989), 251–259.
- [13] DIAGANA, T. *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*, Springer-Verlag, New York, 2013.
- [14] DONTVI, I. K. *Bounded, almost periodic and asymptotically almost periodic solutions of linear differential equations with generalized perturbations*, PhD. thesis, Odessa, Odessa, Ukraine, 1988.
- [15] HALANAY, A., WEXLER, D. *Teoria Calitativa a Sistemelor cu Impulsuri*, Editura Academiei Republicii Socialiste Romania, Bucuresti, Romania, 1968.
- [16] HENRÍQUEZ, H. R. *On Stepanov-almost periodic semigroups and cosine functions of operators*, J. Math. Anal. Appl. **146** (1990), 420–433.
- [17] GÓMEZ-COLLADO, M. C. *Almost periodic ultradistributions of Beurling and of Roumieu type*, Proc. Amer. Math. Soc. **129** (2000), 2319–2329.
- [18] GUÉRÉKATA, G.M.N'. *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Acad. Publ., Dordrecht, 2001.
- [19] GUÉRÉKATA, G.M.N'. , PANKOV, A. *Stepanov-like almost automorphic functions and monotone evolution equations*, Nonlinear Anal. **68** (2008), 2658–2667.
- [20] KOMATSU, H. *Ultradistributions, I. Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **20** (1973), 25–105.
- [21] KOMATSU, H. *Ultradistributions, II. The kernel theorem and ultradistributions with support in a manifold*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), 607–628.
- [22] KOMATSU, H. *Ultradistributions, III. Vector valued ultradistributions. The theory of kernels*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1982), 653–718.
- [23] KOSTIĆ, M. *Almost Periodic and Almost Automorphic Solutions of Integro-Differential Equations*, W. de Gruyter, Berlin, 2019.
- [24] KOSTIĆ, M. *Vector-valued almost periodic ultradistributions and their generalizations*, Mat. Bilten **42** (2018), 5–20.
- [25] KOSTIĆ, M., VELINOV, D. *Vector-valued almost automorphic distributions and vector-valued almost ultradistributions*, Novi Sad J. Math. **48** (2018), 111–121.
- [26] PILIPOVIĆ, S. *Tempered ultradistributions*, Boll. Un. Mat. Ital. **7 (2-B)** (1998), 235–251.
- [27] SCHWARTZ, L. *Theorie des Distributions, 2 vols.*, Hermann, Paris, 1950-1951.
- [28] STANKOVIĆ, B. *Asymptotic almost distributions, structural theorem*, Serdica **19** (1993), 198–203.

- [29] STANKOVIĆ, B. *Asymptotic almost periodic distributions, application to partial differential equations*, Trudy Mat. Inst. Steklov. **203** (1994), 449–460.
- [30] VEECH, W. A. *Almost automorphic functions on groups*, Amer. J. Math. **87** (1965), 719–751.
- [31] VEECH, W. A. *On a theorem of Bochner*, Ann. of Math. **86** (1967), 117–137.
- [32] XIA, Z. *Piecewise asymptotically almost periodic solution of neutral Volterra integrodifferential equations with impulsive effects*, Turkish J. Math. **41** (2017), 1656–1672.
- [33] XIA, Z., LI, Z., CHAI, J. *Pseudo almost periodic dynamics of impulsive Nicholson's blowflies model with nonlinear density-dependent mortality term*, Jpn. J. Ind. Appl. Math. **35** (2018), 311–333.

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