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A note on existence results for a nonlinear fourth-order integral boundary value problem

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Abstract. In this short note, we present some new existence results for a nonlinear fourth-order two-point boundary value problem with integral condition. The existence results are obtained by using the Leray-Schauder fixed point theorem. Our work improves the main results of Benaicha and Haddouchi [3]. In addition, examples are included to show the validity of our results.

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1 Introduction

Fourth-order ordinary differential equations are models for bending or deformation of elastic beams, and therefore have important applications in engineering and physical sciences. Recently, the two-point and multi-point boundary value problems for fourth-order nonlinear differential equations have received much attention from many authors. Many authors have studied the beam equation under various boundary conditions and by different approaches. We refer the readers to the papers [1–6, 8–13, 15–20].

In 2016, Benaicha and Haddouchi [3], by applying the Krasnoselskii's fixed point theorem in cones, established the existence of positive solutions for the following fourth-order two-point boundary value problem (BVP) with integral boundary condition:

$$u''''(t) + f(u(t)) = 0,\ t \in (0, 1), \tag{1}$$

$$u'(0) = u'(1) = u''(0) = 0,\ u(0) = \int_0^1 a(s)u(s)ds, \tag{2}$$

where

(H1) $f \in C([0, \infty), [0, \infty))$;

(H2) $a \in C([0, 1], [0, \infty))$ and $0 < \int_0^1 a(s)ds < 1$.

To obtain the existence of at least one positive solutions for this problem, they assumed that the nonlinear term f is either superlinear or sublinear. That is, defining

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u\to +\infty} \frac{f(u)}{u},$$

then, $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.

This note applies the Leray-Schauder fixed point theorem to eliminate half of the assumptions to prove the existence of a solution when using Krasnoselskii's fixed point theorem of norm type with super and sub linear hypotheses. The value of this manuscript is not in the application to this particular fourth order problem, it serves as an example of a technique that in these instances is a better technique to use to prove existence of solutions over the Krasnoselskii's fixed point theorem since it requires less hypotheses.

Motivated by the work mentioned above, the aim of this note is to improve the results in [3] by showing that the BVP (1) and (2) has at least a positive solution if $f_0 = 0$ (condition $f_\infty = \infty$ being unnecessary), as well as, for $f_\infty = 0$ (condition $f_0 = \infty$ being also unnecessary).

For our analysis we use the Leray-Schauder's fixed point theorem.

Lemma 1 ([7, 14])**.** *Let Ω be a convex subset in a Banach space X, $0 \in \Omega$ and assume that $A : \Omega \to \Omega$ is a completely continuous operator. Then, either (i) A has at least one fixed point in Ω; or (ii) the set $\{x \in \Omega / x = \lambda A x,\ 0 < \lambda < 1\}$ is unbounded.*

2 Some preliminary results

In order to prove our main results, we need some preliminary results. Consider the following two-point boundary value problem

$$u''''(t) + y(t) = 0,\ t \in (0, 1), \tag{3}$$

$$u'(0) = u'(1) = u''(0) = 0,\ u(0) = \int_0^1 a(s)u(s)ds. \tag{4}$$

For problem (3)–(4), we have the following conclusions which are derived from [3].

Lemma 2 ([3], Lemma 2.2)**.** *The problem* (3)–(4) *has a unique solution*

$$u(t) = \int_0^1 \Big(G(t, s) + \frac{1}{1-\alpha}\int_0^1 a(\tau)G(\tau, s)d\tau\Big)y(s)ds,$$

where $G(t, s) : [0, 1] \times [0, 1] \to \mathbb{R}$ is the Green's function defined by

$$G(t, s) = \frac{1}{6}\begin{cases} t^3(1-s)^2 - (t-s)^3, & 0 \le s \le t \le 1; \\ t^3(1-s)^2, & 0 \le t \le s \le 1, \end{cases} \tag{5}$$

and

$$\alpha = \int_0^1 a(t)dt.$$

Lemma 3 ([3], Lemma 2.3)**.** *Let $\theta \in (0, \frac{1}{2})$ be fixed. Then*

(i) $G(t,s) \geq 0$*, for all* $t, s \in [0,1]$*;*

(ii) $\frac{1}{6}\theta^3 s(1-s)^2 \leq G(t,s) \leq \frac{1}{6}s(1-s)^2$*, for all* $(t,s) \in [\theta, 1-\theta] \times [0,1]$*.*

Lemma 4 ([3], Lemma 2.4)**.** *Let $y(t) \in C([0,1],[0,\infty))$ and $\theta \in (0, \frac{1}{2})$. The unique solution of* (3)–(4) *is nonnegative and satisfies*

$$\min_{t \in [\theta, 1-\theta]} u(t) \geq \theta^3(1-\alpha+\beta)\|u\|,$$

where $\beta = \int_\theta^{1-\theta} a(t)dt$, $\alpha = \int_0^1 a(t)dt$, and $\|u\| = \max_{t\in[0,1]}|u(t)|$.

3 Existence results

In this section, we will state and prove our main results. Consider the Banach space $X = C([0,1],\mathbb{R})$ equipped with standard norm:

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Let $\theta \in (0, \frac{1}{2})$ be fixed, $\beta = \int_\theta^{1-\theta} a(t)dt$, $\alpha = \int_0^1 a(t)dt$, and define the cone

$$K = \left\{ u \in X : u(t) \geq 0,\ t \in [0,1],\ \min_{t\in[\theta,1-\theta]} u(t) \geq \theta^3(1-\alpha+\beta)\|u\| \right\}.$$

From Lemmas 2, 3, and 4, the function u is a positive solution of the boundary value problem (1) and (2) if and only if $u(t)$ is a fixed point of the operator

$$Au(t) := \int_0^1 \Big(G(t,s) + \frac{1}{1-\alpha}\int_0^1 a(\tau)G(\tau,s)d\tau \Big) f(u(s))ds. \tag{6}$$

Theorem 1. *Suppose that conditions* (H1) and (H2) hold. If $f_0 = 0$, then BVP (1) and (2) has at least one positive solution.

Proof. Since $f_0 = 0$, there exists $\rho_1 > 0$ such that $f(u) \leq \epsilon u$, for $0 < u \leq \rho_1$, where $\epsilon > 0$ satisfies

$$\epsilon \leq 1 - \alpha.$$

If we denote

$$\Omega = \Big\{ u \in K,\ \|u\| \leq \rho_1 \Big\},$$

then Ω is a convex subset of X.

For $u \in \Omega$, according to the proofs of Lemmas 3 and 4, we have

$$\begin{aligned} Au(t) &= \int_0^1 \Big(G(t,s) + \frac{1}{1-\alpha}\int_0^1 a(\tau)G(\tau,s)d\tau\Big) f(u(s))ds \\ &\le \int_0^1 \Big(g(s) + \frac{1}{1-\alpha}\int_0^1 g(s)a(\tau)d\tau\Big) f(u(s))ds \\ &= \frac{1}{1-\alpha}\int_0^1 g(s)f(u(s))ds, \quad t \in [0,1], \end{aligned} \tag{7}$$

where $g(s) = \frac{1}{6}s(1-s)^2$.

So,

$$\|Au\| \le \frac{1}{1-\alpha}\int_0^1 g(s)f(u(s))ds. \tag{8}$$

In view of Lemma 4 and (8), we have $Au(t) \ge 0$ and

$$\begin{aligned} Au(t) &= \int_0^1 \Big(G(t,s) + \frac{1}{1-\alpha}\int_0^1 a(\tau)G(\tau,s)d\tau\Big) f(u(s))ds \\ &\ge \theta^3 \int_0^1 \Big(g(s) + \frac{1}{1-\alpha}\int_\theta^{1-\theta} g(s)a(\tau)d\tau\Big) f(u(s))ds \\ &= \theta^3 \frac{1-\alpha+\beta}{1-\alpha}\int_0^1 g(s)f(u(s))ds \\ &\ge \theta^3(1-\alpha+\beta)\|Au\|, \quad t \in [\theta, 1-\theta]. \end{aligned} \tag{9}$$

Hence,

$$\min_{t\in[\theta,1-\theta]} Au(t) \ge \theta^3(1-\alpha+\beta)\|Au\|.$$

On the other hand,

$$\begin{aligned} Au(t) &\le \frac{1}{1-\alpha}\epsilon\|u\|\int_0^1 g(s)ds \\ &\le \|u\| \le \rho_1. \end{aligned} \tag{10}$$

Thus, $\|Au\| \le \rho_1$. Hence $A\Omega \subset \Omega$. $A : \Omega \to \Omega$ is completely continuous by an application of Arzela-Ascoli theorem.

For $u \in \mathcal{V}$ with

$$\mathcal{V} = \Big\{u \in \Omega / \ u = \lambda Au, \ 0 < \lambda < 1\Big\},$$

we have

$$u(t) = \lambda Au(t) < Au(t) \le \rho_1,$$

which implies that $\|u\| \le \rho_1$.

So, $\mathcal{V}$ is bounded. By Lemma 1, the operator A has at least one fixed point in Ω, which is a positive solution of (1) and (2). □

Theorem 2. *Suppose that conditions* (H1) and (H2) hold. If $f_\infty = 0$, then BVP (1) and (2) has at least one positive solution.

Proof. We discuss two possible cases:

Case 1. If f is bounded, then, there exists $L > 0$ such that $f(u) \leq L$.

For $u \in K$, we have $Au \in K$, and A is completely continuous. Similar to the estimates of (7), we obtain

$$Au(t) \leq \frac{L}{1-\alpha} \int_0^1 g(s)ds \quad \leq \frac{L}{6(1-\alpha)}. \tag{11}$$

Thus, $\|Au\| \leq \frac{L}{6(1-\alpha)}$. For $u \in \mathcal{V}$ with

$$\mathcal{V} = \Big\{ u \in K/ \ u = \lambda Au, \ 0 < \lambda < 1 \Big\},$$

we have

$$u(t) = \lambda Au(t) < Au(t) \leq \frac{L}{6(1-\alpha)},$$

which implies that $\|u\| \leq \frac{L}{6(1-\alpha)}$.

So, $\mathcal{V}$ is bounded. By Lemma 1, the operator A has at least one fixed point in K, which is a positive solution of (1) and (2).

Case 2. Suppose that f is unbounded, since $f_\infty = 0$, there exists $\rho_2 > 0$ such that $f(u) \leq \eta u$ for $u > \rho_2$, where $\eta > 0$ satisfies

$$\eta \leq 1 - \alpha.$$

On the other hand, from condition (H1), there is $\sigma > 0$ such that $f(u) \leq \eta\sigma$, with $0 \leq u \leq \rho_2$.

Now, set

$$\Omega = \Big\{ u \in K, \ \|u\| \leq \widehat{\rho}_2 \Big\},$$

where $\widehat{\rho}_2 = \max\{\sigma, \rho_2\}$.

If $u \in \Omega$, then we have $f(u) \leq \eta\widehat{\rho}_2$. Similar to (7), we have

$$Au(t) \leq \frac{1}{6} \cdot \frac{\eta\widehat{\rho}_2}{1-\alpha} \quad \leq \widehat{\rho}_2. \tag{12}$$

Thus, $\|Au\| \leq \widehat{\rho}_2$.

It is easy to check that $\mathcal{V} = \Big\{ u \in \Omega/u = \lambda Au, \ 0 < \lambda < 1 \Big\}$ is bounded. Therefore, by Lemma 1, the boundary value problem (1) and (2) has at least one positive solution. $\square$

4 Examples

Example 1. Consider the fourth-order boundary value problem

$$u''''(t) + u(1 - e^{-u}) = 0, \quad 0 < t < 1, \tag{13}$$

$$u'(0) = u'(1) = u''(0) = 0, \ u(0) = \int_0^1 s^2 u(s)ds, \tag{14}$$

where $f(u) = u(1 - e^{-u}) \in C([0, \infty), [0, \infty))$ and $a(t) = t^2 \geq 0$, $\int_0^1 a(s)ds = \int_0^1 s^2 ds = \frac{1}{3}$.

We have

$$f_0 = \lim_{u \to 0+} \frac{f(u)}{u} = \lim_{u \to 0+} (1 - e^{-u}) = 0.$$

Thus, it follows from Theorem 1 that the problem (13) and (14) has at least one positive solution. Notice that $f_\infty = 1$, $f_\infty \neq \infty$, so Theorem 3.1 in [3] cannot be applied to show the existence of positive solutions for the problem (13) and (14).

Example 2. As a second example we consider the fourth-order boundary value problem

$$u''''(t) + 1 - e^{-u} = 0, \quad 0 < t < 1, \tag{15}$$

$$u'(0) = u'(1) = u''(0) = 0, \ u(0) = \int_0^1 s^2 u(s)ds, \tag{16}$$

where $f(u) = 1 - e^{-u} \in C([0, \infty), [0, \infty))$ and $a(t) = t^2 \geq 0$, $\int_0^1 a(s)ds = \int_0^1 s^2 ds = \frac{1}{3}$.

Since

$$f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u} = \lim_{u \to +\infty} \frac{1 - e^{-u}}{u} = 0.$$

From Theorem 2, the problem (15) and (16) has at least one positive solution.

On the other hand, we have

$$f_0 = \lim_{u \to +0} \frac{f(u)}{u} = \lim_{u \to +0} \frac{1 - e^{-u}}{u} = 1.$$

Therefore, Theorem 3.2 in [3] also cannot be applied to show the existence of positive solutions for the problem (15) and (16).

References

[1] Alves E., Ma T. F., Pelicer M. L. *Monotone positive solutions for a fourth order equation with nonlinear boundary conditions.* Nonlinear Anal., 2009, **71**, 3834–3841.

[2] Anderson D. R., Avery R. I. *A fourth-order four-point right focal boundary value problem.* Rocky Mountain J. Math., 2006, **36**, 367–380.

[3] Benaicha S., Haddouchi F. *Positive solutions of a nonlinear fourth-order integral boundary value problem.* An. Univ. Vest Timis. Ser. Mat.-Inform., 2016, **54(1)**, 73-86.

[4] Graef J. R., Qian C., Yang B. *A three point boundary value problem for nonlinear fourth order differential equations*. J. Math. Anal. Appl., 2003, **287**, 217–233.

[5] Graef J. R., Yang B. *Existence and nonexistence of positive solutions of fourth order nonlinear boundary value problems*. Appl. Anal., 2000, **74**, 201–214.

[6] Graef J. R., Henderson J., Yang B. *Positive solutions to a fourth-order three point boundary value problem*. Discrete Contin. Dyn. Syst., Supplement, 2009, 269–275.

[7] Granas A., Dugundji J. *Fixed point theory*. Springer-Verlag, New York, 2003.

[8] Han X., Gao H., Xu J. *Existence of positive solutions for nonlocal fourth-order boundary value problem with variable parameter*. Fixed Point Theory Appl., 2011, Art. ID 604046, 11 pp.

[9] Henderson J., Ma D. *Uniqueness of solutions for fourth-order nonlocal boundary value problems*. Bound. Value Probl., 2006, Art. ID 23875, 12 pp.

[10] Kosmatov N. *Countably many solutions of a fourth-order boundary value problem*. Electron. J. Qual. Theory Differ. Equ., 2004, **12**, 1–15.

[11] Ma R. *Multiple positive solutions for a semipositone fourth-order boundary value problem*. Hiroshima Math. J., 2003, **33**, 217–227.

[12] Ma R., Haiyan W. *On the existence of positive solutions of fourth-order ordinary differential equations*. Appl. Anal., 1995, **59**, 225–231.

[13] Shen W. *Positive solutions for fourth-order second-point nonhomogeneous singular boundary value problems*. Adv. Fixed Point Theory., 2015, **1(5)**, 88–100.

[14] Smart D. R. *Fixed point theorems*. Cambridge University Press, London-New York, 1974.

[15] Sun Y., Zhu C. *Existence of positive solutions for singular fourth-order three-point boundary value problems*. Adv. Difference Equ., 2013, **51**, 13 pp.

[16] Webb J. R., Infante G., Franco D. *Positive solutions of nonlinear fourth-order boundary value problems with local and nonlocal boundary conditions*. Proc. Roy. Soc. Edinburgh Sect. A., 2008, **138**, 427–446.

[17] Yang B. *Positive solutions for a fourth-order boundary value problem*. Electron. J. Qual. Theory Differ. Equ., 2005, **3**, 1–17.

[18] Yao Q. *Local existence of multiple positive solutions to a singular cantilever beam equation*. J. Math. Anal. Appl., 2010, **363**, 138–154.

[19] Yong-Ping S. *Existence and multiplicity of positive solutions for an elastic beam equation*. Appl. Math. J. Chinese. Univ. Ser. B., 2011, **26(3)**, 253–264.

[20] Zhang Q., Chen S., Lü J. *Upper and lower solution method for fourth-order four-point boundary value problems*. J. Comput. Appl. Math., 2006, **196**, 387–393.

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