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Statistical Λ-Convergence in Intuitionistic Fuzzy Normed Spaces

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Abstract. The basic objective of this work is to define statistical Λ-convergence in intuitionistic fuzzy normed spaces. We have given some examples which show this method of convergence is more generalized. Further, we have defined the statistical Λ-Cauchy sequences in these spaces and given the Cauchy convergence criterion for this new notion of convergence.

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1 Introduction

The theory of fuzzy sets has shown its utility in various fields of mathematics and engineering where exact solutions of problems are not necessary. It is useful in those cases where complex physical system does not provide the precise mathematical structure or fast or initial estimate solution is needed. Zadeh [26] in 1965, introduced this new theory and later on vast number of research papers appeared in literature based on the concept of fuzzy sets/numbers (see [7, 13, 18]). Further, fuzzification of many classical theories has produced many interesting and useful applications in different areas such as population dynamics [3], computer programming [13], chaos control [26], fuzzy physics [15] etc. Park [23] presented the idea of intuitionistic fuzzy metric space and later on Saadati and Park [24] worked on the concept of intuitionistic fuzzy normed space (IFNS) using t-norm and continuous t-norm. In [11], the authors studied topological properties of intuitionistic 2-fuzzy n-normed linear spaces. The concept of intuitionistic fuzzy norm is applicable in those spaces where the exact norm of a particular vector is impossible to determine. IFNS is a highly motivated area of research due to its analytic properties and their generalizations for providing a tool for mathematical modelling of real life situations where fuzzy theory alone can't work.

In 1951, Fast [6] has introduced a new concept of convergence named as statistical convergence more generalized than the usual convergence. It has interesting applications in the various mathematical fields like Fourier Analysis [2], Measure Theory [17], Approximation Theory [12], Topology [5] etc. It has been studied by many researchers for various types of sequences in different setups like locally convex space [14], probabilistic normed space [9], etc. Karakus [10] and Mursaleen [19],

studied the notion of statistical convergence and lacunary statistical convergence in intuitionistic fuzzy normed space respectively. For more work related to statistical convergence, one may refer to [1, 4, 16, 21, 22].
Before introducing statistical Λ-convergence in intuitionistic fuzzy normed space, we first review some basic terms as follows:

Definition 1. [25] A continuous *t-norm* is a binary operation $\circledast : [0,1] \times [0,1] \to [0,1]$ with the following axioms

1. $\circledast$ is continuous, commutative and associative,
2. $a \circledast 1 = a \ \forall \ a \in [0,1]$,
3. If $a \leq c$ and $b \leq d$ then $a \circledast b \leq c \circledast d$, for each $a, b, c, d \in [0,1]$.

Definition 2. [25] A continuous *t-conorm* is a binary operation $\odot : [0,1] \times [0,1] \to [0,1]$ with the following axioms

1. $\odot$ is continuous, commutative and associative,
2. $a \odot 0 = a \ \forall \ a \in [0,1]$,
3. If $a \leq c$ and $b \leq d$ then $a \odot b \leq c \odot d$, for each $a, b, c, d \in [0,1]$.

Definition 3. [24] The 5-tuple $(Y, \varphi, \vartheta, \circledast, \odot)$ with vector space Y, continuous t-norm $\circledast$, continuous t-conorm $\odot$ and fuzzy sets φ, ϑ on $Y \times (0, \infty)$ is said to be *intuitionistic fuzzy normed space* (IFNS) if for every $y, z \in Y$ and $s, t > 0$

1. $\varphi(y,t) + \vartheta(y,t) \leq 1$,
2. $\varphi(y,t) > 0$,
3. $\varphi(y,t) = 1$ if and only if $y = 0$,
4. $\varphi(\alpha y, t) = \varphi(y, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
5. $\varphi(y,t) \circledast \varphi(z,s) \leq \varphi(y+z, t+s)$,
6. $\varphi(y, \circ) : (0, \infty) \to [0,1]$ is continuous,
7. $\lim_{t\to\infty} \varphi(y,t) = 1$ and $\lim_{t\to 0} \varphi(y,t) = 0$,
8. $\vartheta(y,t) < 1$,
9. $\vartheta(y,t) = 0$ if and only if $y = 0$,
10. $\vartheta(\alpha y, t) = \vartheta(y, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
11. $\vartheta(y,t) \odot \vartheta(z,s) \geq \vartheta(y+z, t+s)$,

12. $\vartheta(y, \circ) : (0, \infty) \to [0, 1]$ is continuous,

13. $\lim_{t\to\infty} \vartheta(y, t) = 0$ and $\lim_{t\to 0} \vartheta(y, t) = 1$.

Then (φ, ϑ) is known as *intuitionistic fuzzy norm*.

Example 1. [24] Let $(Y, \| \circ \|)$ be a normed space with $a \circledast b = ab$ and $a \odot b = min\{a + b, 1\}$ for all $a, b \in [0, 1]$.
For $y \in Y$ and $t > 0$, take $\varphi(y, t) = \frac{t}{t+\|y\|}$, $\vartheta(y, t) = \frac{\|y\|}{t+\|y\|}$.
Then $(Y, \varphi, \vartheta, \circledast, \odot)$ is an IFNS.

Definition 4. [24] Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS. A sequence $y = (y_r)$ is said to be convergent to $\xi \in Y$ with respect to intuitionistic fuzzy norm (φ, ϑ) if for every $\epsilon > 0$ and $t > 0$ there exists $r_0 \in \mathbb{N}$ such that $\varphi(y_r - \xi, t) > 1 - \epsilon$ and $\vartheta(y_r - \xi, t) < \epsilon$, for all $r \geq r_0$. Symbollically, $(\varphi, \vartheta) - \lim y = \xi$.

Definition 5. [24] Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS. A sequence $y = (y_r)$ is said to be Cauchy sequence with respect to intuitionistic fuzzy norm (φ, ϑ) if for every $\epsilon > 0$ and $t > 0$ there exists $r_0 \in \mathbb{N}$ such that $\varphi(y_r - y_s, t) > 1 - \epsilon$ and $\vartheta(y_r - y_s, t) < \epsilon$, for all $r, s \geq r_0$.

Further, the concept of statistical convergence of sequences of numbers defined by Fridy [8] is given as follows. This is based on the notion of natural density and the natural density of set A, where $A \subset \mathbb{N}$, is defined as

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \mid a \leq n : a \in \mathbb{N} \mid$$

if the limits exists, where $| \, . \, |$ is the cardinality of enclosed set. A sequence $y = (y_r)$ is said to be statistically convergent to ξ if $A(\epsilon) = \{a \leq n : |y_r - \xi| > \epsilon\}$ has natural density zero.

Definition 6. [10] Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS. A sequence $y = (y_r)$ is said to be statistically convergent to $\xi \in Y$ with respect to intuitionistic fuzzy norm (φ, ϑ) if for every $\epsilon > 0$ and $t > 0$, we have

$$\delta(\{r \in \mathbb{N} : \varphi(y_r - \xi, t) \leq 1 - \epsilon \text{ or } \vartheta(y_r - \xi, t) \geq \epsilon\}) = 0,$$

or

$$\delta(\{r \in \mathbb{N} : \varphi(y_r - \xi, t) > 1 - \epsilon \text{ and } \vartheta(y_r - \xi, t) < \epsilon\}) = 1.$$

Symbollically, $S^{(\varphi,\vartheta)} - \lim y = \xi$.

2 Statistical Λ-Convergence and Statistical Λ-Cauchy Sequences

In order to explain the basic concept of statistical Λ-convergence and statistical Λ-Cauchy sequence in intuitionistic fuzzy normed space, we consider a real sequence of positive numbers $\lambda = (\lambda_j)$ such that $0 < \lambda_0 < \lambda_1 < \lambda_2 < ...... < \lambda_j < ...$ and $\lambda_j \to \infty$ as $j \to \infty$. The concept of Λ-convergence was defined by Mursaleen[20] as follows: A sequence $y = (y_r)$ is Λ-convergent to a number L if $\Lambda y_r \to L$ as $r \to \infty$ where

$$\Lambda y_r = \frac{1}{\lambda_r} \sum_{j=0}^{r} (\lambda_j - \lambda_{j-1}) y_j,$$

Here, without loss of generality we take negative subscript terms equal to zero, for example, $\lambda_{-1} = 0$ and $y_{-1} = 0$.
Next, we propose the Λ-convergence in IFNS by defining the following terms.

Definition 7. Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS. A sequence $y = (y_r)$ is said to be Λ-*convergent* to $\xi \in Y$ with respect to intuitionistic fuzzy norm (φ, ϑ) if for every $\epsilon > 0$ and $t > 0$ there exists $r_0 \in \mathbb{N}$ such that $\varphi(\Lambda y_r - \xi, t) > 1 - \epsilon$ and $\vartheta(\Lambda y_r - \xi, t) < \epsilon$, for all $r \geq r_0$. Symbollically, $(\varphi, \vartheta)_\Lambda - \lim y = \xi$.

Definition 8. Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS. A sequence $y = (y_r)$ is said to be Λ-*Cauchy* with respect to intuitionistic fuzzy norm (φ, ϑ) if for every $\epsilon > 0$ and $t > 0$ there exists $r_0 \in \mathbb{N}$ such that $\varphi(\Lambda y_r - \Lambda y_s, t) > 1 - \epsilon$ and $\vartheta(\Lambda y_r - \Lambda y_s, t) < \epsilon$, for all $r, s \geq r_0$.

With the help of Definition 7 and Definition 8 the concept of statistical Λ-convergence in IFNS is defined as follows:

Definition 9. Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS. A sequence $y = (y_r)$ is said to be *statistically* Λ-*convergent* to $\xi \in Y$ with respect to intuitionistic fuzzy norm (φ, ϑ) if for every $\epsilon > 0$ and $t > 0$, we have

$$\delta(\{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) \leq 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - \xi, t) \geq \epsilon\}) = 0,$$

or

$$\delta(\{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) > 1 - \epsilon \text{ and } \vartheta(\Lambda y_r - \xi, t) < \epsilon\}) = 1.$$

Symbollically, $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi$.

Definition 10. Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS. A sequence $y = (y_r)$ is said to be *statistically* Λ-*Cauchy* with respect to intuitionistic fuzzy norm (φ, ϑ) if for every $\epsilon > 0$ and $t > 0$, we have

$$\delta(\{r \in \mathbb{N} : \varphi(\Lambda y_r - \Lambda y_s, t) \leq 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - \Lambda y_s, t) \geq \epsilon\}) = 0,$$

or

$$\delta(\{r \in \mathbb{N} : \varphi(\Lambda y_r - \Lambda y_s, t) > 1 - \epsilon \text{ and } \vartheta(\Lambda y_r - \Lambda y_s, t) < \epsilon\}) = 1.$$

We may easily obtain the following lemma using Definition 9.

Lemma 1. *Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS and $\lambda = (\lambda_j)$ be a real sequence of positive non-decreasing numbers defined as above. The following statements are equivalent for the sequence $y = (y_r)$ in Y whenever $\epsilon > 0$ and $t > 0$,*

1. $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi$,
2. $\delta(\{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) \leq 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - \xi, t) \geq \epsilon\}) = 0$,
3. $\delta(\{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) > 1 - \epsilon \text{ and } \vartheta(\Lambda y_r - \xi, t) < \epsilon\}) = 1$,
4. $S_\Lambda^{(\varphi,\vartheta)} - \lim\ \varphi(\Lambda y_r - \xi, t) = 1$ *and* $S_\Lambda^{(\varphi,\vartheta)} - \lim\ \vartheta(\Lambda y_r - \xi, t) = 0$.

Theorem 1. *Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS. A sequence $y = (y_r)$ is statistically Λ-convergent with respect to intuitionistic fuzzy norm (φ, ϑ) to a unique limit, i.e. if $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi$, then ξ is unique.*

Proof. Let if possible, $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi_1$ and $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi_2$ and $\xi_1 \neq \xi_2$.
For given $\epsilon > 0$ and $t > 0$, choose $\rho > 0$ such that $(1-\rho) \circledast (1-\rho) > 1 - \epsilon$ and $\rho \odot \rho < \epsilon$. Define

$$K_{1,\varphi}(\rho, t) = \{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi_1, t/2) \leq 1 - \rho\},$$

$$K_{2,\varphi}(\rho, t) = \{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi_2, t/2) \leq 1 - \rho\},$$

$$K_{3,\vartheta}(\rho, t) = \{r \in \mathbb{N} : \vartheta(\Lambda y_r - \xi_1, t/2) \leq 1 - \rho\},$$

$$K_{4,\vartheta}(\rho, t) = \{r \in \mathbb{N} : \vartheta(\Lambda y_r - \xi_2, t/2) \leq 1 - \rho\}.$$

Since $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi_1$, using Lemma 1 we have

$$\delta(K_{1,\varphi}(\rho, t)) = \delta(K_{1,\vartheta}(\rho, t)) = 0\ \forall\ t > 0.$$

Further $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi_2$, using Lemma 1 we have

$$\delta(K_{2,\varphi}(\rho, t)) = \delta(K_{2,\vartheta}(\rho, t)) = 0\ \forall\ t > 0.$$

Let $K_{\varphi,\vartheta}(\rho, t) = (K_{1,\varphi}(\rho, t) \bigcup K_{2,\varphi}(\rho, t)) \bigcap (K_{1,\vartheta}(\rho, t) \bigcup K_{2,\vartheta}(\rho, t))$.
Clearly,

$$\delta(K_{\varphi,\vartheta}(\rho, t)) = 0 \Leftrightarrow \delta(\mathbb{N} - K_{\varphi,\vartheta}(\rho, t)) = 1.$$

If $k \in \mathbb{N} - K_{\varphi,\vartheta}(\rho, t)$ then we have two possibilities, either $k \in \mathbb{N} - (K_{1,\varphi}(\rho, t) \bigcup K_{2,\varphi}(\rho, t))$ or $k \in \mathbb{N} - (K_{1,\vartheta}(\rho, t) \bigcup K_{2,\vartheta}(\rho, t))$.

We first consider that $k \in \mathbb{N} - (K_{1,\varphi}(\rho, t) \bigcup K_{2,\varphi}(\rho, t))$. Then

$$\varphi(\xi_1 - \xi_2, t) \geq \varphi(\Lambda y_r - L_1, t/2) \circledast \varphi(\Lambda y_r - L_2, t/2) > (1-\rho) \circledast (1-\rho) > 1 - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we get $\varphi(\xi_1 - \xi_2, t) = 1$ for all $t > 0$, then $\xi_1 = \xi_2$.

On the other hand, if $k \in \mathbb{N} - (K_{1,\vartheta}(\rho, t) \bigcup K_{2,\vartheta}(\rho, t))$, then

$$\vartheta(\xi_1 - L_2, t) \leq \vartheta(\Lambda y_r - \xi_1, t/2) \odot \vartheta(\Lambda y_r - \xi_2, t/2) < \rho \odot \rho < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we get $\varphi(\xi_1 - \xi_2, t) = 0$ for all $t > 0$, then $\xi_1 = \xi_2$. Therefore, limit is unique. □

Theorem 2. *Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS. If $(\varphi, \vartheta)_\Lambda - \lim y = \xi$, then $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi$. But converse may be not true.*

Proof. Let $(\varphi, \vartheta)_\Lambda - \lim y = \xi$. Then for given $\epsilon > 0$ and $t > 0$ there exists a number $r_0 \in \mathbb{N}$ such that

$$\varphi(\Lambda y_r - \xi, t) > 1 - \epsilon \text{ and } \vartheta(\Lambda y_r - \xi, t) < \epsilon$$

for all $r \geq r_0$. This gives that the set

$$\{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) \leq 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - \xi, t) \geq \epsilon\}$$

contains only finite number of elements. We know that natural density of a finite set is always zero. Therefore,

$$\delta\{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) \leq 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - \xi, t) \geq \epsilon\} = 0.$$

i.e.,

$$S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi.$$

The converse of the above result does not hold; this can be justified with the following example :

Example 2. Let $(\mathbb{R}, |.|)$ be the normed space of real numbers under the usual norm. Define $a \circledast b = ab$ and $a \odot b = min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For every $t > 0$ and all $y \in \mathbb{R}$, consider $\varphi(y, t) = \frac{t}{t+|y|}$, $\vartheta(y, t) = \frac{|y|}{t+|y|}$.
Then $(\mathbb{R}, \varphi, \vartheta, \circledast, \odot)$ is an IFNS.
Define the sequence

$$\Lambda y_r = \begin{cases} 1 & r = k^2, k \in Z^+, \\ 0 & \text{otherwise.} \end{cases}$$

Then for every $\epsilon > 0$ and $t > 0$, we have

$$\begin{aligned} K(\epsilon, t) &= \{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) \leq 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - \xi, t) \geq \epsilon\} \ ; \ \xi = 0 \\ &= \{r \in \mathbb{N} : \frac{t}{t + |\Lambda y_r|} \leq 1 - \epsilon \text{ or } \frac{|\Lambda yr|}{t + |\Lambda y_r|} \geq \epsilon\} \\ &= \{r \in \mathbb{N} : |\Lambda y_r| \geq \frac{\epsilon t}{1 - \epsilon} > 0\} \\ &= \{r \in \mathbb{N} : |\Lambda y_r| = 1\} \\ &= \{r \in \mathbb{N} : r = k^2, k \in Z^+\} \end{aligned}$$

Thus, for any $r_0 \in Z^+$, we have $K(\epsilon, t) \leq \{r \leq r_0 : r = k^2, k \in Z^+\}$. Now,

$$\begin{aligned} \frac{1}{r_0}|K(\epsilon, t)| &\leq \frac{1}{r_0}|\{r \leq r_0 : r = k^2, k \in Z^+\}| \\ &\leq \frac{\sqrt{r_0}}{r_0}. \end{aligned}$$

$\Rightarrow \ \lim_{r_0 \to \infty} \frac{1}{r_0}|K(\epsilon, t| = 0$.
Hence, $S_\Lambda^{(\varphi,\vartheta)} - \lim y = 0$, i.e. the sequence $y = (y_r)$ is statistically Λ-convergent in $(\mathbb{R}, \varphi, \vartheta, \circledast, \odot)$.
Also, by the above defined sequence (Λy_r), we get,

$$\varphi(\Lambda y_r, t) = \begin{cases} \frac{t}{t+1} & r = k^2, k \in Z^+, \\ 1 & \text{otherwise.} \end{cases}$$

$$\text{i.e. } \varphi(\Lambda y_r, t) \leq 1, \ \forall \ r$$

and

$$\vartheta(\Lambda y_r, t) = \begin{cases} \frac{1}{t+1} & r = k^2, k \in Z^+, \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{i.e. } \vartheta(\Lambda y_r, t) \geq 0, \ \forall \ r.$$

This shows that $(\varphi, \vartheta)_\Lambda - \lim y \neq 0$.

□

Next, we find an algebraic characterization of statistically Λ-covergent sequences in IFNS as follows:

Theorem 3. *Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS. Let $y = (y_r)$ and $z = (z_r)$ be two sequences in Y.*
(i) If $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi$, then $S_\Lambda^{(\varphi,\vartheta)} - \lim ky = k\xi \ ; \ k \in \mathbb{R}$,
(ii) If $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi_1$ and $S_\Lambda^{(\varphi,\vartheta)} - \lim z = \xi_2$, then $S_\Lambda^{(\varphi,\vartheta)} - \lim(y + z) = \xi_1 + \xi_2$.

Proof. (i) Suppose $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi$. Then, for given $\epsilon > 0$ and $t > 0$, we define

$$K(\epsilon,t) = \{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) \le 1-\epsilon \text{ or } \vartheta(\Lambda y_r - \xi, t) \ge \epsilon\}.$$

Then,

$$\delta(K(\epsilon,t)) = 0 \text{ and } \delta([K(\epsilon,t)]^c) = 1.$$

Let $r \in [K(\epsilon,t)]^c$, then

$$\begin{aligned}\varphi(\Lambda(ky_r) - k\xi, t) = \varphi(k(\Lambda y_r - \xi), t) &= \varphi(\Lambda y_r - \xi, \frac{t}{|k|})\\ &\ge \varphi(\Lambda y_r - \xi, t) \circledast \varphi(0, \frac{t}{|k|} - t)\\ &= \varphi(\Lambda y_r - \xi, t) \circledast 1\\ &> 1-\epsilon\end{aligned}$$

and

$$\begin{aligned}\vartheta(\Lambda(ky_r) - k\xi, t) = \vartheta(k(\Lambda y_r - \xi), t) &= \vartheta(\Lambda y_r - \xi, \frac{t}{|k|})\\ &\le \vartheta(\Lambda y_r - \xi, t) \odot \vartheta(0, \frac{t}{|k|} - t)\\ &\le \vartheta(\Lambda y_r - \xi, t) \odot 0\\ &< \epsilon.\end{aligned}$$

Therefore, $\delta([K(\epsilon,t)]^c) = 1$, i.e.

$$\delta\{r \in \mathbb{N} : \varphi(\Lambda(ky_r) - k\xi, t) > 1-\epsilon \text{ and } \vartheta(\Lambda(ky_r) - k\xi, t) < \epsilon\} = 1.$$

Hence, $S_\Lambda^{(\varphi,\vartheta)} - \lim ky = k\xi,\ k \ne 0$.
If $k = 0$, then

$$\begin{aligned}&\varphi(0\Lambda y_r, t) = \varphi(0,t) = 1 \ > 1-\epsilon\\ &\text{and } \vartheta(0\Lambda y_r, t) = \vartheta(0,t) = 0 < \epsilon.\end{aligned}$$

Therefore, $S_\Lambda^{(\varphi,\vartheta)} - \lim ky = k\xi,\ k \in \mathbb{R}$.

(ii) Let $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi_1$ and $S_\Lambda^{(\varphi,\vartheta)} - \lim z = \xi_2$. Then, for given $\epsilon > 0$ and $t > 0$, choose $\rho > 0$ such that $(1-\rho) \circledast (1-\rho) > 1-\epsilon$ and $\rho \odot \rho < \epsilon$.
We define for the given sequences $y = (y_r)$ and $z = (z_r)$ sets

$$K_y(\rho,t) = (\{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi_1, t/2) \le 1-\rho \text{ or } \vartheta(\Lambda y_r - \xi_1, t/2) \ge \rho\},$$

and

$$K_z(\rho,t) = (\{r \in \mathbb{N} : \varphi(\Lambda z_r - \xi_2, t/2) \le 1-\rho \text{ or } \vartheta(\Lambda z_r - \xi_2, t/2) \ge \rho\}.$$

We have $\delta(K_y(\rho,t)) = \delta(K_z(\rho,t)) = 0$. Let $K(\rho,t) = K_y(\rho,t) \cap K_z(\rho,t)$, then $\delta(K(\rho,t)) = 0$, i.e. $\delta([K(\rho,t)]^c) = 1$.
For $k \in [K(\rho,t)]^c$,

$$\begin{aligned}\varphi(\Lambda(y_r + z_r) - (\xi_1 + \xi_2), t) &= \varphi(\Lambda y_r - \xi_1 + \Lambda z_r - \xi_2, t)\\ &\geq \varphi(\Lambda y_r - \xi_1, t/2) \circledast \varphi(\Lambda z_r - \xi_2, t/2)\\ &\geq (1-\rho) \circledast (1-\rho)\\ &> 1-\epsilon\end{aligned}$$

and

$$\begin{aligned}\vartheta(\Lambda(y_r + z_r) - (\xi_1 + \xi_2), t) &= \vartheta(\Lambda y_r - \xi_1 + \Lambda z_r - \xi_2, t)\\ &\leq \vartheta(\Lambda y_r - \xi_1, t/2) \odot \vartheta(\Lambda z_r - \xi_2, t/2)\\ &\leq \rho \odot \rho\\ &< \epsilon.\end{aligned}$$

$$\Rightarrow S_\Lambda^{(\varphi,\vartheta)} - \lim(y+z) = \xi_1 + \xi_2.$$

□

Theorem 4. *Let $(Y, \varphi, \vartheta, \circledast, \odot)$ be an IFNS. Then for a sequence $y = (y_r)$ in Y, $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi$ if and only if there exists a set $J = \{j_1 < j_2 < j_3.....\} \subseteq \mathbb{N}$ such that $\delta(J) = 1$ and $S_\Lambda^{(\varphi,\vartheta)} - \lim y_{j_n} = \xi$.*

Proof. Necessity:

Assume $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi$. Then for $t > 0$ and $\rho \in \mathbb{N}$, we take

$$M(\rho,t) = \{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) > 1 - 1/\rho \text{ and } \vartheta(\Lambda y_r - \xi, t) < 1/\rho\},$$

and

$$K(\rho,t) = \{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) \leq 1 - 1/\rho \text{ or } \vartheta(\Lambda y_r - \xi, t) \geq 1/\rho\}.$$

Since $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi$, then $\delta(K(\rho,t)) = 0$. Also $M(\rho,t) \supset M(\rho+1,t)$, and

$$\delta(M(\rho,t)) = 1 \text{ for } t > 0 \text{ and } \rho \in \mathbb{N} \tag{1}$$

For $k \in M(\rho,t)$, we prove $S_\Lambda^{(\varphi,\vartheta)} - \lim_{r\to\infty} y_r = \xi$.
We prove this by contradiction. Suppose for some $k \in M(\rho,t)$ the sequence $y = (y_r)$ is not statistically Λ-convergent to ξ. Then, there exists $\alpha > 0$ and positive integer r_0 such that

$$\begin{aligned}\varphi(\Lambda y_r - \xi, t) \leq 1 - \alpha \text{ or } \vartheta(\Lambda y_r - \xi, t) \geq \alpha \text{ , for all } r \geq r_0\\ \Rightarrow \varphi(\Lambda y_r - \xi, t) > 1 - \alpha \text{ and } \vartheta(\Lambda y_r - \xi, t) < \alpha \text{ , for all } r < r_0\end{aligned}$$

Therefore, $\delta\{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) > 1 - \alpha \text{ and } \vartheta(\Lambda y_r - \xi, t) < \alpha\} = 0$.

Since $\alpha > \frac{1}{\rho}$, we have $\delta(M(\rho, t)) = 0$, which is a contradiction to (1). This shows that there exists a set $M(\rho, t)$ for which $\delta(M(\rho, t)) = 1$ and the sequence $y = (y_r)$ is statistically Λ-convergent to ξ.

Sufficiency:

Suppose there exists a subset $J = \{j_1 < j_2 < j_3.....\} \subseteq \mathbb{N}$ such that $\delta(J) = 1$ and $S_\Lambda^{(\varphi,\vartheta)} - \lim y_{j_n} = \xi$, i.e. $\exists\ N_0 \in \mathbb{N}$ such that for every $\alpha > 0$ and $t > 0$

$$\varphi(\Lambda y_r - \xi, t) > 1 - \alpha \text{ and } \vartheta(\Lambda y_r - \xi, t) < \alpha\ ;\ r \geq N_0.$$

Now, let

$$K(\alpha, t) = \{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t) \leq 1 - \alpha \text{ or } \vartheta(\Lambda y_r - \xi, t) \geq \alpha\}.$$

Then,

$$K(\alpha, t) \subseteq \mathbb{N} - \{j_{N_0+1}, j_{N_0+2}, ...\}.$$

As $\delta(J) = 1 \Rightarrow\ \delta(K(\alpha, t)) \leq 0$. Hence, $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi$. □

But, in the next result we establish the Cauchy criterion for statistically Λ-convergent sequences in IFNS.

Theorem 5. *A sequence $y = (y_r)$ in IFNS $(Y, \varphi, \vartheta, \circledast, \odot)$ is statistically Λ-convergent with respect to (φ, ϑ) if and only if it is statistically Λ-Cauchy with respect to (φ, ϑ).*

Proof. Let $S_\Lambda^{(\varphi,\vartheta)} - \lim y = \xi$.
Then, for $\epsilon > 0$ and $t > 0$, choose $\rho > 0$ such that $(1 - \rho) \circledast (1 - \rho) > 1 - \epsilon$ and $\rho \odot \rho < \epsilon$. Let $K(\rho, t) = \{r \in \mathbb{N} : \varphi(\Lambda y_r - \xi, t/2) \leq 1 - \rho \text{ or } \vartheta(\Lambda y_r - \xi, t/2) \geq \rho\}$
$\therefore \delta(K(\rho, t)) = 0$ and $\delta([K(\rho, t)]^c) = 1$.

Let $M(\epsilon, t) = \{r \in \mathbb{N} : \varphi(\Lambda y_r - \Lambda y_s, t) \leq 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - \Lambda y_s, t) \geq \epsilon\}$.
Here, it is enough to prove that $M(\epsilon, t) \subset K(\rho, t)$.
As $r \in M(\epsilon, t) - K(\rho, t) \Rightarrow \varphi(\Lambda y_r - \xi, t/2) \leq 1 - \rho$ or $\vartheta(\Lambda y_r - \xi, t/2) \geq \rho$.

$$\begin{aligned} 1 - \epsilon \geq \varphi(\Lambda y_r - \Lambda y_s, t) &\geq \varphi(\Lambda y_r - \xi, t/2) \circledast \varphi(\Lambda y_s - \xi, t/2) \\ &> (1 - \rho) \circledast (1 - \rho) \\ &> 1 - \epsilon \end{aligned}$$

and

$$\begin{aligned} \epsilon \leq \vartheta(\Lambda y_r - \Lambda y_s, t) &\leq \vartheta(\Lambda y_r - \xi, t/2) \odot \vartheta(\Lambda y_s - \xi, t/2) \\ &< \rho \odot \rho \\ &< \epsilon, \end{aligned}$$

which is not possible. Therefore $M(\epsilon,t) \subset K(\rho,t)$ and $\delta(M(\epsilon,t)) = 0$, i.e. $y = (y_r)$ is statistically Λ-Cauchy with respect to (φ,ϑ).

Conversely, let $y = (y_r)$ be statistically Λ-Cauchy with respect to intuitionistic fuzzy norm (φ,ϑ) but not statistically Λ-convergent with respect to (φ,ϑ). Thus for $\epsilon > 0$ and $t > 0$, $\delta(K(\epsilon,t)) = 0$ where

$$K(\epsilon,t) = \{r \in \mathbb{N} : \varphi(\Lambda y_r - \Lambda y_{r_0}, t) \leq 1 - \epsilon \text{ or } \vartheta(\Lambda y_r - \Lambda y_{r_0}, t) \geq \epsilon\}.$$

Choose $\rho > 0$ such that $(1-\rho) \circledast (1-\rho) > 1-\epsilon$ and $\rho \odot \rho < \epsilon$
Now

$$\begin{aligned} \varphi(\Lambda y_r - \Lambda y_{r_0}, t) &\geq \varphi(\Lambda y_r - \xi, t/2) \circledast \varphi(\Lambda y_{r_0} - \xi, t/2) \\ &> (1-\rho) \circledast (1-\rho) \\ &> 1 - \epsilon \end{aligned}$$

and

$$\begin{aligned} \vartheta(\Lambda y_r - \Lambda y_{r_0}, t) &\leq \vartheta(\Lambda y_r - \xi, t/2) \odot \vartheta(\Lambda y_{r_0} - \xi, t/2) \\ &< \rho \odot \rho \\ &< \epsilon. \end{aligned}$$

Since $y = (y_r)$ is not statistically Λ-convergent. Therefore, $\delta([K(\epsilon,t)]^c) = 0$, i.e. $\delta(K(\epsilon,t)) = 1$, which is a contradiction as $y = (y_r)$ is not statistically Λ-Cauchy. Hence, $y = (y_r)$ is statistically Λ-convergent with respect to (φ,ϑ). □

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