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Existence of positive periodic solutions for fourth-order nonlinear neutral differential equations with variable coefficients

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Abstract. In this work, we study the existence of positive periodic solutions for fourth-order nonlinear neutral differential equations with variable coefficients. The results are established by using the Krasnoselskii's fixed point theorem. An example is given to illustrate this work.

Mathematics subject classification: 34K13, 34A34, 34K30, 34L30.
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1 Introduction

Delay differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, see the monograph [9, 22] and the papers [1]-[8],[10]-[28],[30]-[33] and the references therein.

Recently, the first-order nonlinear neutral functional differential equation

$$\frac{d}{dt}(x(t) - cx(t - \tau(t))) = -a(t)x(t) + f(t, x(t - \tau(t))),$$

has been investigated in [25] where $|c| < 1$. By using Krasnoselskii's fixed point theorem, the existence of positive ω-periodic solutions has been established. Ren, Siegmund and Chen [28] discussed the existence of positive ω-periodic solutions for the following neutral functional differential equation

$$\frac{d^3}{dt^3}(x(t) - cx(t - \tau(t))) = -a(t)x(t) + f(t, x(t - \tau(t))),$$

where $|c| < 1$. By employing Krasnoselskii's fixed point theorem, the authors obtained existence results for positive ω-periodic solutions.

In the present article, we study the existence of positive ω-periodic solutions for the fourth-order nonlinear neutral differential equations

$$\frac{d^4}{dt^4}(x(t) - c(t)x(t - \tau)) = a(t)x(t) - f(t, x(t - \tau)). \tag{1}$$

Throughout this paper we assume that $c \in C(\mathbb{R},\mathbb{R})$, $a \in C(\mathbb{R},(0,\infty))$, $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\tau,\omega > 0$, c and a are ω-periodic functions, f is ω-periodic with respect to first variable. To show the existence of positive ω-periodic solutions, we transform (1) into integral equation and then use Krasnoselskii's fixed point theorem. The obtained integral equations split in the sum of two mappings, one is a contraction and the other is compact.

In this paper, we have two main contributions comparing with the existing results. First, instead of constant c we take variable coefficient $c(t)$. Second, in addition to $|c(t)| < 1$, we consider the range $|c(t)| > 1$ for $c(t)$, which is new in the literature.

The organization of this paper is as follows. In section 2, we introduce some notations and lemmas, and state some preliminary results needed in later sections. Then we give the Green's function of (1) which plays an important role in this paper. Also, we present the inversion of (1), and Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [29]. In section 3, we present our main results on existence of positive ω-periodic solutions of (1). An example is also given to illustrate this work.

2 Preliminaries

For $\omega > 0$, let C_ω be the set of all continuous scalar functions x, periodic in t of period $\omega > 0$. Then $(C_\omega, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,\omega]} |x(t)| .$$

Define

$$C_\omega^- = \{x \in C_\omega,\ x < 0\} .$$

Denote

$$M = \sup\{a(t) : t \in [0,\omega]\}, \quad m = \inf\{a(t) : t \in [0,\omega]\}, \quad \rho = \sqrt[4]{M}.$$

Lemma 1 ([7]). *The equation*

$$\frac{d^4}{dt^4}u(t) - Mu(t) = h(t), \quad h \in C_\omega^-,$$

has a unique ω-periodic solution

$$u(t) = \int_t^{t+\omega} G(t,s)(-h(s))ds,$$

where

$$G(t,s) = \frac{\exp(-\rho(s-t)) + \exp(\rho(s-t-\omega))}{4\rho^3(1-\exp(-\rho\omega))} + \frac{\cos(\rho(t-s+\frac{\omega}{2}))}{4\rho^3\sin(\frac{\rho\omega}{2})}, \quad s \in [t, t+\omega].$$

Lemma 2 ([7]). *Green's function G satisfies the following properties*

$$\int_t^{t+\omega} G(t,s)ds = \frac{1}{M},$$

and if $\max\{a(t) : t \in [0,\omega]\} < (\frac{\pi}{\omega})^4$, *then*

$$G(t,s) > 0, \quad \forall (t,s) \in [0,\omega] \times [t, t+\omega].$$

Lemma 3 ([7]). *If* $\max\{a(t) : t \in [0,\omega]\} < (\frac{\pi}{\omega})^4$ *holds, then the equation*

$$\frac{d^4}{dt^4} u(t) - a(t)u(t) = h(t), \quad h \in C_\omega^-,$$

has a unique positive ω-periodic solution

$$u(t) = (Ph)(t) = (I - TB)^{-1}(Th)(t),$$

where

$$(Th)(t) = \int_t^{t+\omega} G(t,s)(-h(s))ds, \quad (Bu)(t) = [-M + a(t)]\, u(t).$$

Lemma 4 ([7]). *If* $\max\{a(t) : t \in [0,\omega]\} < (\frac{\pi}{\omega})^4$ *holds, then P is completely continuous and*

$$0 < (Th)(t) \leq (Ph)(t) \leq \frac{M}{m} \|Th\|, \quad \forall h \in C_\omega^-.$$

The following theorem is essential for our results on existence of positive periodic solution of (1).

Theorem 1. *If $x \in C_\omega$, then x is a solution of (1) if and only if*

$$x(t) = c(t)x(t-\tau) + P(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)) \tag{2}$$

Proof. Let $x \in C_\omega$ be a solution of (1). The (1) can be rewritten as

$$\begin{aligned}
&\frac{d^4}{dt^4}(x(t) - c(t)x(t-\tau)) - M(x(t) - c(t)x(t-\tau)) \\
&= (-M + a(t))(x(t) - c(t)x(t-\tau)) - f(t, x(t-\tau)) + c(t)a(t)x(t-\tau) \\
&= B(x(t) - c(t)x(t-\tau)) - f(t, x(t-\tau)) + c(t)a(t)x(t-\tau).
\end{aligned}$$

From Lemma 1, we have

$$\begin{aligned}
&x(t) - c(t)x(t-\tau) \\
&= TB(x(t) - c(t)x(t-\tau)) + T(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)).
\end{aligned}$$

This yields

$$(I - TB)(x(t) - c(t)x(t-\tau)) = T(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)).$$

Therefore

$$\begin{aligned} x(t) - c(t)x(t-\tau) &= (I - TB)^{-1} T(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)) \\ &= P(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)). \end{aligned}$$

Obviously

$$x(t) = c(t)x(t-\tau) + P(-f(t, x(t-\tau)) + c(t)a(t)x(t-\tau)).$$

This completes the proof. □

Corollary 1. *If $x \in C_\omega$, then x is a solution of (1) if and only if*

$$x(t) = \frac{1}{c(t+\tau)} \left[x(t+\tau) + P(-c(t+\tau)a(t+\tau)x(t) + f(t+\tau, x(t))) \right]. \tag{3}$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive ω-periodic solutions to (1). For its proof we refer the reader to ([29], p. 31).

Lemma 5 (Krasnoselskii [29])**.** *Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. Suppose that A and B map $\mathbb{D}$ into $\mathcal{B}$ such that*

(i) $Ax + By \in \mathbb{D} \quad \forall x, y \in \mathbb{D}$,

(ii) A is completely continuous operator in $\mathbb{D}$,

(iii) B is a contraction operator.

Then there exists $z \in \mathbb{D}$ with $z = Az + Bz$.

3 Positive Periodic solutions

To apply Lemma 5, we need to define a Banach space $\mathcal{B}$, a closed convex subset $\mathbb{D}$ of $\mathcal{B}$ and construct two mappings, one is contraction and the other is a completely continuous. So we let $(\mathcal{B}, \|\cdot\|) = (C_\omega, \|\cdot\|)$ and $\mathbb{D} = \{\varphi \in C_\omega : M_1 \le \varphi \le M_2\}$, where M_1 is non-negative constant and M_2 is positive constant.

3.1 Positive periodic solutions in the case $|c(t)| > 1$

In this subsection, we obtain the existence of positive ω-periodic solution for (1) by considering the two cases; $1 < c(t) < \infty$ and $-\infty < c(t) < -1$ for all $t \in [0, \omega]$.

Theorem 2. *Suppose that* $\max\{a(t) : t \in [0, \omega]\} < (\frac{\pi}{\omega})^4$, $1 < c_1 \le c(t) \le c_2 < \infty$ *and*

$$m \le c(t)a(t)x - f(t, x) \le c_1 M, \ \forall (t, x) \in [0, \omega] \times \left[\frac{m}{(c_2 - 1)M}, \frac{c_1 M}{(c_1 - 1)m} \right]. \tag{4}$$

Then (1) has at least one positive ω-periodic solution x in the subset $\mathbb{D}_1$ of $\mathcal{B}$ where $\mathbb{D}_1 = \left\{ \varphi \in C_\omega : \frac{m}{(c_2-1)M} \le \varphi \le \frac{c_1 M}{(c_1-1)m} \right\}$.

Proof. We express (3) as

$$\varphi(t) = (B_1\varphi)(t) + (A_1\varphi)(t) := (H_1\varphi)(t),$$

where $A_1, B_1 : \mathbb{D}_1 \to \mathcal{B}$ are defined by

$$(A_1\varphi)(t) = \frac{1}{c(t+\tau)} P(-c(t+\tau)a(t+\tau)\varphi(t) + f(t+\tau, \varphi(t))),$$

and

$$(B_1\varphi)(t) = \frac{\varphi(t+\tau)}{c(t+\tau)}.$$

It is obvious that $A_1\varphi$ and $B_1\varphi$ are continuous and ω-periodic. Now we prove that $A_1x + B_1y \in \mathbb{D}_1$, $\forall x, y \in \mathbb{D}_1$. By Corollary 1, Lemma 4 and the condition (4) we obtain

$$\begin{aligned}
&(A_1x)(t) + (B_1y)(t) \\
&= \frac{1}{c(t+\tau)} \left[P(-c(t+\tau)a(t+\tau)x(t) + f(t+\tau, x(t))) + y(t+\tau)\right] \\
&\leq \frac{1}{c_1} \left[\frac{M}{m} \|T(-c(t+\tau)a(t+\tau)x(t) + f(t+\tau, x(t)))\| + \frac{c_1 M}{(c_1-1)m}\right] \\
&= \frac{M}{mc_1} \max_{t\in[0,\omega]} \left| \int_t^{t+\omega} G(t,s)(c(s+\tau)a(s+\tau)x(s) - f(s+\tau, x(s)))ds \right| + \frac{M}{(c_1-1)m} \\
&\leq \frac{M}{mc_1} \int_t^{t+\omega} G(t,s)c_1 M ds + \frac{M}{(c_1-1)m} \\
&\leq \frac{M}{mc_1} c_1 M \frac{1}{M} + \frac{M}{(c_1-1)m} = \frac{c_1 M}{(c_1-1)m}.
\end{aligned} \tag{5}$$

On the other hand

$$\begin{aligned}
&(A_1x)(t) + (B_1y)(t) \\
&= \frac{1}{c(t+\tau)} \left[P(-c(t+\tau)a(t+\tau)x(t) + f(t+\tau, x(t))) + y(t+\tau)\right] \\
&\geq \frac{1}{c_2} \left[T(-c(t+\tau)a(t+\tau)x(t) + f(t+\tau, x(t))) + \frac{m}{(c_2-1)M}\right] \\
&\geq \frac{1}{c_2} \int_t^{t+\omega} G(t,s)(c(s+\tau)a(s+\tau)x(s) - f(s+\tau, x(s)))ds + \frac{1}{c_2}\frac{m}{(c_2-1)M} \\
&\geq \frac{1}{c_2} \int_t^{t+\omega} G(t,s) m ds + \frac{1}{c_2}\frac{m}{(c_2-1)M} \\
&\geq \frac{1}{c_2} m \frac{1}{M} + \frac{1}{c_2}\frac{m}{(c_2-1)M} = \frac{m}{(c_2-1)M}.
\end{aligned} \tag{6}$$

Combining (5) and (6), we obtain $A_1x + B_1y \in \mathbb{D}_1$, $\forall x, y \in \mathbb{D}_1$. For $\varphi, \psi \in \mathbb{D}_1$, we have

$$|(B_1\varphi)(t) - (B_1\psi)(t)| = \left| \frac{\varphi(t+\tau)}{c(t+\tau)} - \frac{\psi(t+\tau)}{c(t+\tau)} \right|$$

$$\begin{aligned}&\leq \frac{1}{c_1}\left|\varphi(t+\tau)-\psi(t+\tau)\right|\\&\leq \frac{1}{c_1}\left\|\varphi-\psi\right\|,\end{aligned}$$

which implies that $\|B_1\varphi - B_1\psi\| \leq \frac{1}{c_1}\|\varphi-\psi\|$. Since $0 < \frac{1}{c_1} < 1$, B_1 is a contraction on $\mathbb{D}_1$. From Lemma 4, we know that P is completely continuous, so is A_1. By Lemma 5 we obtain that $A_1 + B_1$ has a fixed point $x \in \mathbb{D}_1$, i.e. (1) has a positive ω-periodic solution $x \in \mathbb{D}_1$. □

Theorem 3. *Suppose that* $\max\{a(t) : t \in [0,\omega]\} < (\frac{\pi}{\omega})^4$, $-\infty < c_3 \leq c(t) \leq c_4 < -1$ *and*

$$\frac{c_3}{c_4}M < f(t,x) - c(t)a(t)x \leq -c_4 m,\ \forall (t,x) \in [0,\omega]\times[0,1]. \tag{7}$$

Then (1) has at least one positive ω-periodic solution x in the subset $\widetilde{\mathbb{D}}_2$ of $\mathcal{B}$ where $\widetilde{\mathbb{D}}_2 = \{\varphi \in C_\omega : 0 < \varphi \leq 1\}$.

Proof. Let $\mathbb{D}_2 = \{\varphi \in C_\omega : 0 \leq \varphi \leq 1\}$. We define $A_1, B_1 : \mathbb{D}_2 \to \mathcal{B}$ as follows

$$(A_1\varphi)(t) = \frac{-1}{c(t+\tau)}P(c(t+\tau)a(t+\tau)\varphi(t) - f(t+\tau,\varphi(t))),$$

and

$$(B_1\varphi)(t) = \frac{\varphi(t+\tau)}{c(t+\tau)}.$$

Now we prove that $A_1x + B_1y \in \mathbb{D}_2$, $\forall x, y \in \mathbb{D}_2$. By Corollary 1, Lemma 4 and the condition (7) we obtain

$$\begin{aligned}&(A_1x)(t) + (B_1y)(t)\\&= \frac{-1}{c(t+\tau)}P(c(t+\tau)a(t+\tau)x(t) - f(t+\tau,x(t))) + \frac{y(t+\tau)}{c(t+\tau)}\\&\leq \frac{-1}{c_4}\frac{M}{m}\left\|T(c(t+\tau)a(t+\tau)x(t) - f(t+\tau,x(t)))\right\|\\&= \frac{-M}{mc_4}\max_{t\in[0,\omega]}\left|\int_t^{t+\omega} G(t,s)(f(s+\tau,x(s)) - c(s+\tau)a(s+\tau)x(s))ds\right|\\&\leq \frac{-M}{mc_4}\int_t^{t+\omega} G(t,s)(-c_4 m)ds\\&\leq \frac{-M}{mc_4}(-c_4 m)\frac{1}{M} = 1.\end{aligned} \tag{8}$$

On the other hand

$$\begin{aligned}&(A_1x)(t) + (B_1y)(t)\\&= \frac{-1}{c(t+\tau)}P(c(t+\tau)a(t+\tau)x(t) - f(t+\tau,x(t))) + \frac{y(t+\tau)}{c(t+\tau)}\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{-1}{c_3} T(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))) + \frac{1}{c_4} \\
&\geq \frac{-1}{c_3} \int_t^{t+\omega} G(t,s)(f(s+\tau, x(s)) - c(s+\tau)a(s+\tau)x(s))ds + \frac{1}{c_4} \\
&\geq \frac{-1}{c_3} \int_t^{t+\omega} G(t,s)(\frac{c_3}{c_4}M)ds + \frac{1}{c_4} \\
&\geq \frac{-1}{c_3}(\frac{c_3}{c_4}M)\frac{1}{M} + \frac{1}{c_4} = 0. \qquad (9)
\end{aligned}
$$

Combining (8) and (9), we obtain $A_1x + B_1y \in \mathbb{D}_2$, for all $x, y \in \mathbb{D}_2$. For $\varphi, \psi \in \mathbb{D}_2$, we have

$$
\begin{aligned}
|(B_1\varphi)(t) - (B_1\psi)(t)| &= \left| \frac{\varphi(t+\tau)}{c(t+\tau)} - \frac{\psi(t+\tau)}{c(t+\tau)} \right| \\
&\leq \frac{-1}{c_4} |\varphi(t+\tau) - \psi(t+\tau)| \\
&\leq \frac{-1}{c_4} \|\varphi - \psi\|,
\end{aligned}
$$

which implies that $\|B_1\varphi - B_1\psi\| \leq \frac{-1}{c_4} \|\varphi - \psi\|$. Since $0 < \frac{-1}{c_4} < 1$, B_1 is a contraction on $\mathbb{D}_2$. From Lemma 4, we know that P is completely continuous, so is A_1. By Lemma 5 we obtain that $A_1 + B_1$ has a fixed point $x \in \mathbb{D}_2$, i.e. (1) has a nonnegative ω-periodic solution x with $0 \leq x(t) \leq 1$. Since. $f(t,x) - c(t)a(t)x > \frac{c_3}{c_4}M$, it is easy to see that $x(t) > 0$, i.e. (1) has positive ω-periodic solution $x \in \widetilde{\mathbb{D}}_2$. ◻

3.2 Positive periodic solutions in the case $|c(t)| < 1$

In this subsection, we obtain the existence of a positive periodic solution for (1) by considering the three cases; $0 < c(t) < 1$, $-1 < c(t) \leq 0$ and $c(t) = 0$ for all $t \in [0, \omega]$.

Theorem 4. *Suppose that* $\max\{a(t) : t \in [0,\omega]\} < (\frac{\pi}{\omega})^4$, $0 < c_5 \leq c(t) \leq c_6 < 1$ *and*

$$
c_5 m \leq f(t,x) - c(t)a(t)x \leq M, \ \forall (t,x) \in [0,\omega] \times \left[\frac{c_5 m}{(1-c_5)M}, \frac{M}{(1-c_6)m} \right]. \qquad (10)
$$

Then (1) has at least one positive ω-periodic solution x in the subset $\mathbb{D}_3$ of $\mathcal{B}$ where $\mathbb{D}_3 = \left\{ \varphi \in C_\omega : \frac{c_5 m}{(1-c_5)M} \leq \varphi \leq \frac{M}{(1-c_6)m} \right\}$.

Proof. We express (2) as

$$
\varphi(t) = (B_2\varphi)(t) + (A_2\varphi)(t) := (H_2\varphi)(t),
$$

where $A_2, B_2 : \mathbb{D}_3 \to \mathcal{B}$ are defined by

$$
(A_2\varphi)(t) = P(c(t)a(t)\varphi(t-\tau) - f(t, \varphi(t-\tau))),
$$

and

$$(B_2\varphi)(t) = c(t)\varphi(t-\tau).$$

It is obvious that $A_2\varphi$ and $B_2\varphi$ are continuous and ω-periodic. Now we prove that $A_2x + B_2y \in \mathbb{D}_3$, $\forall x, y \in \mathbb{D}_3$. By Corollary 1, Lemma 4 and the condition (10) we obtain

$$\begin{aligned}
&(A_2x)(t) + (B_2y)(t) \\
&= P(c(t)a(t)x(t-\tau) - f(t, x(t-\tau))) + c(t)y(t-\tau) \\
&\le \frac{M}{m} \|T(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t)))\| + c_6\frac{M}{(1-c_6)m} \\
&= \frac{M}{m} \max_{t\in[0,\omega]} \left| \int_t^{t+\omega} G(t,s)(f(s+\tau, x(s)) - c(s+\tau)a(s+\tau)x(s))ds \right| + c_6\frac{M}{(1-c_6)m} \\
&\le \frac{M}{m} \int_t^{t+\omega} G(t,s)Mds + c_6\frac{M}{(1-c_6)m} \\
&\le \frac{M}{m}M\frac{1}{M} + c_6\frac{M}{(1-c_6)m} = \frac{M}{(1-c_6)m}.
\end{aligned} \tag{11}$$

On the other hand

$$\begin{aligned}
&(A_2x)(t) + (B_2y)(t) \\
&= P(c(t)a(t)x(t-\tau) - f(t, x(t-\tau))) + c(t)y(t-\tau) \\
&\ge T(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))) + c_5\frac{c_5m}{(1-c_5)M} \\
&\ge \int_t^{t+\omega} G(t,s)(f(s+\tau, x(s)) - c(s+\tau)a(s+\tau)x(s))ds + c_5\frac{c_5m}{(1-c_5)M} \\
&\ge \int_t^{t+\omega} G(t,s)c_5mds + c_5\frac{c_5m}{(1-c_5)M} \\
&\ge c_5m\frac{1}{M} + c_5\frac{c_5m}{(1-c_5)M} = \frac{c_5m}{(1-c_5)M}.
\end{aligned} \tag{12}$$

Combining (11) and (12), we obtain $A_2x + B_2y \in \mathbb{D}_3$,$\forall x, y \in \mathbb{D}_3$. For $\varphi, \psi \in \mathbb{D}_3$, we have

$$\begin{aligned}
|(B_2\varphi)(t) - (B_2\psi)(t)| &= |c(t)\varphi(t-\tau) - c(t)\psi(t-\tau)| \\
&\le c_6 |\varphi(t-\tau) - \psi(t-\tau)| \\
&\le c_6 \|\varphi - \psi\|,
\end{aligned}$$

which implies that $\|B_2\varphi - B_2\psi\| \le c_6 \|\varphi - \psi\|$. Since $0 < c_6 < 1$, B_2 is a contraction on $\mathbb{D}_3$. From Lemma 4, we know that P is completely continuous, so is A_2. By Lemma 5 we obtain that $A_2 + B_2$ has a fixed point $x \in \mathbb{D}_3$, i.e. (1) has a positive ω-periodic solution $x \in \mathbb{D}_3$. □

Theorem 5. *Suppose that* $\max\{a(t) : t \in [0,\omega]\} < (\frac{\pi}{\omega})^4$, $-1 < c_7 \le c(t) \le c_8 < 0$ *and*

$$-c_7 M < f(t,x) - c(t)a(t)x \le m, \ \forall (t,x) \in [0,\omega] \times [0,1]. \tag{13}$$

Then (1) has at least one positive ω*-periodic solution* x *in the subset* $\widetilde{\mathbb{D}}_4$ *of* $\mathcal{B}$ *where* $\widetilde{\mathbb{D}}_4 = \{\varphi \in C_\omega : 0 < \varphi \le 1\}$.

Proof. Let $\mathbb{D}_4 = \{\varphi \in C_\omega : 0 \le \varphi \le 1\}$. Now we prove that $A_2x + B_2y \in \mathbb{D}_4$, $\forall x, y \in \mathbb{D}_4$. By Corollary 1, Lemma 4 and the condition (13) we obtain

$$\begin{aligned}
&(A_2x)(t) + (B_2y)(t) \\
&= P(c(t)a(t)x(t-\tau) - f(t, x(t-\tau))) + c(t)y(t-\tau) \\
&\le \frac{M}{m} \|T(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t)))\| \\
&= \frac{M}{m} \max_{t\in[0,\omega]} \left| \int_t^{t+\omega} G(t,s)(f(s+\tau, x(s)) - c(s+\tau)a(s+\tau)x(s))ds \right| \\
&\le \frac{M}{m} \int_t^{t+\omega} G(t,s) m ds \\
&\le \frac{M}{m} m \frac{1}{M} = 1.
\end{aligned} \tag{14}$$

On the other hand

$$\begin{aligned}
&(A_2x)(t) + (B_2y)(t) \\
&= P(c(t)a(t)x(t-\tau) - f(t, x(t-\tau))) + c(t)y(t-\tau) \\
&\ge T(c(t+\tau)a(t+\tau)x(t) - f(t+\tau, x(t))) + c_7 \\
&\ge \int_t^{t+\omega} G(t,s)(f(s+\tau, x(s)) - c(s+\tau)a(s+\tau)x(s))ds + c_7 \\
&\ge \int_t^{t+\omega} G(t,s)(-c_7 M)ds + c_7 \\
&\ge (-c_7 M)\frac{1}{M} + c_7 = 0.
\end{aligned} \tag{15}$$

Combining (14) and (15), we obtain $A_2x + B_2y \in \mathbb{D}_4$, $\forall x, y \in \mathbb{D}_4$. From Lemma 4, we know that P is completely continuous, so is A_2. For $\varphi, \psi \in \mathbb{D}_4$, we have

$$\begin{aligned}
|(B_2\varphi)(t) - (B_2\psi)(t)| &= |c(t)\varphi(t-\tau) - c(t)\psi(t-\tau)| \\
&\le -c_7 |\varphi(t-\tau) - \psi(t-\tau)| \\
&\le -c_7 \|\varphi - \psi\|,
\end{aligned}$$

which implies that $\|B_2\varphi - B_2\psi\| \le -c_7 \|\varphi - \psi\|$. Since $0 < -c_7 < 1$, B_2 is a contraction on $\mathbb{D}_4$. By Lemma 5 we obtain that $A_2 + B_2$ has a fixed point $x \in \mathbb{D}_4$, i.e. (1) has a nonnegative ω-periodic solution x with $0 \le x(t) \le 1$. Since. $f(t,x) - c(t)a(t)x > -c_7 M$, it is easy to see that $x(t) > 0$, i.e. (1) has positive ω-periodic solution $x \in \mathbb{D}_4$. ∎

Theorem 6 ([7]). *If* $\max\{a(t) : t \in [0, \omega]\} < (\frac{\pi}{\omega})^4$ *holds,* $c(t) = 0$ *and*

$$0 < f(t, x) \leq M, \ \forall (t, x) \in [0, \omega] \times \left[0, \frac{M}{m}\right].$$

Then (1) has at least one positive ω-periodic solution x with $0 < x(t) \leq \frac{M}{m}$.

Example 1. Consider the forth-order nonlinear neutral differential equation

$$\begin{aligned} &\frac{d^4}{dt^4}\left(x(t) - \left(2 + \frac{1}{0.9 - 0.1\cos^2 t}\right) x(t - 4\pi)\right) \\ &= \frac{1}{10^3}\left(1 - \frac{1}{10^2}\cos^2 t\right) x(t) - \frac{1}{10^4}(2 + \cos t) - \frac{1}{10^3}\exp(\sin(x(t - 4\pi))). \end{aligned} \tag{16}$$

Note that (16) of the form (1) with $\omega = 2\pi$, $c(t) = 2 + \frac{1}{0.9 - 0.1\cos^2 t}$, $a(t) = \frac{1}{10^3}(1 - \frac{1}{10^2}\cos^2 t)$, $f(t, x(t - 4\pi)) = \frac{1}{10^4}(2 + \cos t) + \frac{1}{10^3}\exp(\sin(x(t - 4\pi)))$ and $\tau = 4\pi$. It is easy to verify that the conditions of Theorem 2 are satisfied with $m = \frac{99}{10^5}$ and $M = \frac{1}{10^3}$. Thus (16) has at least one positive ω-periodic solution.

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