BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(90), 2019, Pages 99–112
ISSN 1024–7696

On the upper bound of the number of functionally independent focal quantities of the Lyapunov differential system

Mihail Popa, Victor Pricop

Abstract. Denote by $N_1 = 2\sum\limits_{i=1}^{\ell}(m_i+1)+2$ the maximal possible number of non-zero coefficients of the Lyapunov differential system $\dot{x} = y + \sum\limits_{i=1}^{\ell} P_{m_i}(x,y)$, $\dot{y} = -x + \sum\limits_{i=1}^{\ell} Q_{m_i}(x,y)$, where P_{m_i} and Q_{m_i} are homogeneous polynomials of degree m_i with respect to x and y, and $1 < m_1 < m_2 < ... < m_\ell$ $(\ell < \infty)$. Then the upper bound of functionally independent focal quantities in the center and focus problem of considered system does not exceed $N_1 - 1$.

Mathematics subject classification: 34C07, 34C14, 34C20.
Keywords and phrases: Lyapunov differential systems, the center and focus problem, focal quantities, rotation group, Lie operators, comitants and invariants.

1 Introduction

In the paper [1] we obtained the upper bound of the number of algebraically independent focal quantities that take part in solving the center and focus problem of the differential system $s(1, m_1, ..., m_\ell)$ $(\ell < \infty)$ of the form

$$\dot{x} = \sum_{i=0}^{\ell} P_{m_i}(x,y),\ \dot{y} = \sum_{i=0}^{\ell} Q_{m_i}(x,y), \tag{1}$$

where P_{m_i} and Q_{m_i} are homogeneous polynomials of degree m_i with respect to x and y, and m_i satisfy the condition $1 = m_0 < m_1 < m_2 < ... < m_\ell$ $(\ell < \infty)$.

Assuming that the origin of coordinates is a singular point of center or focus type, then in the paper [1] it is shown that the considered number is equal to $N-1$, where $N = 2\sum\limits_{i=0}^{\ell}(m_i+1)$. However, in 2017 we received an estimation of maximal number of functionally independent focal quantities that take part in solving the center and focus problem for the Lyapunov system $s\mathcal{L}(1, m_1, ..., m_\ell)$ of the form

$$\dot{x} = y + \sum_{i=1}^{\ell} P_{m_i}(x,y),\ \dot{y} = -x + \sum_{i=1}^{\ell} Q_{m_i}(x,y),\ (\ell < \infty), \tag{2}$$

which is the standard form of system for considered problem.

The last results as well as previous [1] were included in the monograph [2], which was published in Chişinău in a small edition of 50 copies in Russian. In present at the request of some foreign publishers we are working on the translation of this book in to English. Therefore, an idea was to offer the latest revised results for publication in this issue of the journal to familiarize a wide range of readers.

2 Lie operator of representation of rotation group in the space of the coefficients of system (2)

Assume that polynomials in the right side of system (2) are written as

$$\begin{aligned} P_{m_i}(x,y) &= \sum_{k=0}^{m_i} \binom{m_i}{k} \overset{i}{a}{}_k^1 x^{m_i-k} y^k, \\ Q_{m_i}(x,y) &= \sum_{k=0}^{m_i} \binom{m_i}{k} \overset{i}{a}{}_k^2 x^{m_i-k} y^k \ (i=\overline{1,\ell}), \end{aligned} \tag{3}$$

where variables and coefficients from (2) and (3) take values from the field of real numbers $\mathbb{R}$.

We study the action of the rotation group $SO(2,\mathbb{R})$ given by the formulas

$$\overline{x} = x\cos\varphi + y\sin\varphi,\ \overline{y} = -x\sin\varphi + y\cos\varphi \, (0 \le \varphi < \pi) \tag{4}$$

on system (2) – (3).

Due to transformation (4) in system (2) – (3) we obtain

$$\dot{\overline{x}} = \overline{y} + \sum_{i=1}^{\ell} P_{m_i}(\overline{x},\overline{y}),\ \dot{\overline{y}} = -\overline{x} + \sum_{i=1}^{\ell} Q_{m_i}(\overline{x},\overline{y}) \ (\ell < \infty), \tag{5}$$

where $\overline{x}$, $\overline{y}$ have the form (4), and the coefficients $\overset{i}{b}{}_k^j$ in the polynomials

$$\begin{aligned} P_{m_i}(\overline{x},\overline{y}) &= \sum_{k=0}^{m_i} \binom{m_i}{k} \overset{i}{b}{}_k^1 \overline{x}^{m_i-k} \overline{y}^k, \\ Q_{m_i}(\overline{x},\overline{y}) &= \sum_{k=0}^{m_i} \binom{m_i}{k} \overset{i}{b}{}_k^2 \overline{x}^{m_i-k} \overline{y}^k \ (i=\overline{1,\ell}) \end{aligned} \tag{6}$$

are written in the form

$$\begin{aligned} \overset{i}{b}{}_k^1 &= \overset{i}{a}{}_k^1 \cos^{m_i+1}\varphi + [(m_i-k)\overset{i}{a}{}_{k+1}^1 - k\overset{i}{a}{}_{k-1}^1 + \overset{i}{a}{}_k^2]\cos^{m_i}\varphi\sin\varphi + \\ &\quad + o(\sin\varphi), \\ \overset{i}{b}{}_k^2 &= \overset{i}{a}{}_k^2 \cos^{m_i+1}\varphi + [-\overset{i}{a}{}_k^1 + (m_i-k)\overset{i}{a}{}_{k+1}^2 - k\overset{i}{a}{}_{k-1}^2]\cos^{m_i}\varphi\sin\varphi + \\ &\quad + o(\sin\varphi). \end{aligned} \tag{7}$$

Note that $o(\sin\varphi)$ is a linear function in the coefficients of system (2) – (3) and contains in each term $\sin\varphi$ in a degree not less than two.

Denote the set of the coefficients of the polynomials (3) by A, and dimensions of its spaces by $\mathcal{N}_1$, which are written as

$$\mathcal{N}_1 = 2\sum_{i=1}^{\ell}(m_i + 1). \tag{8}$$

According to the paper [3] the Lie operator of representation of the group (4) in the space of coefficients and variables $E^{\mathcal{N}_1+2}(x, y, A)$ of system (2) – (3) is written as

$$X = \xi^1(x, y)\frac{\partial}{\partial x} + \xi^2(x, y)\frac{\partial}{\partial y} + D, \tag{9}$$

where

$$D = \sum_{j=1}^{2}\sum_{i=1}^{\ell}\sum_{k=0}^{m_i} \overset{i\,j}{\eta}_k(A)\frac{\partial}{\partial \overset{i\,j}{a}_k}. \tag{10}$$

From (4) and (7) it follows that the linear representation of the group $SO(2, \mathbb{R})$ is a one-parameter group depending on the parameter φ, whose value $\varphi = 0$ corresponds to the identity transformation

$$\overline{x} = x,\ \overline{y} = y,\ \overset{i\,j}{b}_k = \overset{i\,j}{a}_k\ (i = \overline{1, \ell};\ j = 1, 2;\ k = \overline{1, m_i}).$$

Therefore, the coordinates of the operators (9) and (10) according to the accepted condition in the Lie theory are written as

$$\xi^1(x, y) = \left.\frac{\partial \overline{x}}{\partial \varphi}\right|_{\varphi=0} = y,\ \xi^2(x, y) = \left.\frac{\partial \overline{y}}{\partial \varphi}\right|_{\varphi=0} = -x,$$

$$\overset{i\,1}{\eta}_k(A) = \left.\frac{\partial \overset{i}{b}{}^1_k}{\partial \varphi}\right|_{\varphi=0} = (m_i - k)\overset{i\,1}{a}_{k+1} - k\overset{i\,1}{a}_{k-1} + \overset{i\,2}{a}_k\ (i = \overline{1, \ell};\ k = \overline{0, m_i}),$$

$$\overset{i\,2}{\eta}_k(A) = \left.\frac{\partial \overset{i}{b}{}^2_k}{\partial \varphi}\right|_{\varphi=0} = -\overset{i\,1}{a}_k + (m_i - k)\overset{i\,2}{a}_{k+1} - k\overset{i\,2}{a}_{k-1}\ (i = \overline{1, \ell};\ k = \overline{0, m_i}).$$

Substituting these equalities into (9) and (10), we find that takes place

Theorem 1. *The Lie operator of representation of the group $SO(2, \mathbb{R})$ in the space $E^{\mathcal{N}_1+2}(x, y, A)$ of system (2) – (3) has the form*

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + D, \tag{11}$$

where

$$D=\sum_{i=1}^{\ell}\sum_{k=0}^{m_i}\left\{[(m_i-k)\overset{i}{a}{}^1_{k+1}-k\overset{i}{a}{}^1_{k-1}+\overset{i}{a}{}^2_k]\frac{\partial}{\partial\overset{i}{a}{}^1_k}+\right.$$
$$\left.+[-\overset{i}{a}{}^1_k+(m_i-k)\overset{i}{a}{}^2_{k+1}-k\overset{i}{a}{}^2_{k-1}]\frac{\partial}{\partial\overset{i}{a}{}^2_k}\right\}. \tag{12}$$

By A we denote the set of coefficients of nonlinearities in the right side of system (2) – (3).

Corollary 1. *The Lee operator of representation of the group $SO(2,\mathbb{R})$ in the coefficient space $E^{\mathcal{N}_1}(A)$ of the system* (2) – (3) *has the form* (12).

Note 1. *Using the defining equations* [3], *one can verify that the operators* (11) – (12) *are admitted by the Lyapunov system $s\mathcal{L}(1,m_1,...,m_\ell)$ from* (2) – (3).

3 Comitants of system (2) – (3) for the rotation group and the notion of functional basis

Definition 1. *The polynomial $F(x,y,A)$ in the coefficients of system* (2) – (3) *and the phase variables x, y is called an algebraic comitant of this system on the rotation group $SO(2,\mathbb{R})$ from* (4) *if for all admissible A, x, y and φ the identity*

$$F(\overline{x},\overline{y},B)=F(x,y,A), \tag{13}$$

take place, where A (B) is the set of coefficients of system (2) – (3) ((5) – (7)), *and (x,y), $(\overline{x},\overline{y})$ are phase variables of this system.*

If the comitant F of system (2) – (3) does not depend on the phase variables x, y, then, according to the paper [4], it is called an *invariant* of considered system on the rotation group.

Definition 2. *The set of algebraic comitants of the system* (2) – (3) *with respect to the rotation group*

$$\{F_\alpha(x,y,A),\ \alpha\in\mathbb{N}^+\} \tag{14}$$

is called the functional basis of comitants of considered system on the group $SO(2,\mathbb{R})$ if any comitant $F(x,y,A)$ of considered system on the group of rotations can be represented as univocal functions of comitants (14).

From the paper [3] it follows that takes place

Theorem 2. *In order a polynomial $F(x,y,A)$ to be a comitant of system* (2)–(3) *on the rotation group* (4), *i.e. to satisfy the equality* (13), *it is necessary and sufficient that it satisfies the equation*

$$X(F)=0, \tag{15}$$

where X is the Lee operators from (11) – (12).

With the help of Theorem 2 it is easy to verify that takes place

Corollary 2. *In order a polynomial* $I(A)$ *to be an invariant of system* (2) – (3) *on the rotation group* (4) *it is necessary and sufficient that it satisfies the equation*

$$D(I) = 0, \tag{16}$$

where D *is the Lee operator from* (12).

Note 2. *From* (15) – (16) *it follows that invariants and comitants of the Lyapunov system* $s\mathcal{L}(1, m_1, ..., m_\ell)$ (2) – (3) *are solutions of homogeneous partial differential equations of the first order with Lie operators* (11) – (12). *Note that the number of variables involved in the operators* (11) *and* (12) *is equal, respectively, to*

$$N_1 = 2\sum_{i=1}^{\ell}(m_i + 1) + 2. \tag{17}$$

and

$$\mathcal{N}_1 = N_1 - 2. \tag{18}$$

From the general theory of equations of the type (15) – (16) (*see, for example,* [5]) *it is known that the maximal number of functionally independent solutions is equal, respectively, to* $N_1 - 1$ *and* $N_1 - 3$.

4 General formulas interconnecting comitant's coefficients of the Lyapunov system $s\mathcal{L}(1, m_1, ..., m_\ell)$ with respect to the rotation group

Let us write the comitant K of the Lyapunov system (2) – (3) with respect to the rotations group in the form

$$\begin{gathered} K = A_0x^m + A_1x^{m-1}y + A_2x^{m-2}y^2 + A_3x^{m-3}y^3 + A_4x^{m-4}y^4 + \\ +A_5x^{m-5}y^5 + A_6x^{m-6}y^6 + ... + A_{m-7}x^7y^{m-7} + A_{m-6}x^6y^{m-6} + \\ +A_{m-5}x^5y^{m-5} + A_{m-4}x^4y^{m-4} + A_{m-3}x^3y^{m-3} + A_{m-2}x^2y^{m-2} + \\ +A_{m-1}x^1y^{m-1} + A_my^m, \end{gathered} \tag{19}$$

where A_i $(i = \overline{0, m})$ are polynomials in coefficients of system (2) – (3).

Consider equality (15) taking into account the form of the Lie operator (11). Using (19) and the Lie operator (11), we get

$$X(K) = y\frac{\partial K}{\partial x} - x\frac{\partial K}{\partial y} + D(K). \tag{20}$$

Then the terms from the right side of (20) taking into account (19) are written as

$$\begin{gathered} y\frac{\partial K}{\partial x} = mA_0x^{m-1} + (m-1)A_1x^{m-2}y + (m-2)A_2x^{m-3}y^2 + \\ +(m-3)A_3x^{m-4}y^3 + (m-4)A_4x^{m-5}y^4 + (m-5)A_5x^{m-6}y^5 + \\ +(m-6)A_6x^{m-7}y^6 + ... + 7A_{m-7}x^6y^{m-6} + 6A_{m-6}x^5y^{m-5} + \\ +5A_{m-5}x^4y^{m-4} + 4A_{m-4}x^3y^{m-3} + 3A_{m-3}x^2y^{m-2} + \\ +2A_{m-2}xy^{m-1} + A_{m-1}y^m, \end{gathered}$$

$$
\begin{gathered}
-x\frac{\partial K}{\partial y}=-A_1x^m-2A_2x^{m-1}y-3A_3x^{m-2}y^2-4A_4x^{m-3}y^3-\\
-5A_5x^{m-4}y^4-6A_6x^{m-5}y^5-...-(m-7)A_{m-7}x^8y^{m-8}-\\
-(m-6)A_{m-6}x^7y^{m-7}-(m-5)A_{m-5}x^6y^{m-6}-\\
-(m-4)A_{m-4}x^5y^{m-5}-(m-3)A_{m-3}x^4y^{m-4}-\\
-(m-2)A_{m-2}x^3y^{m-3}-(m-1)A_{m-1}x^2y^{m-2}-mA_mxy^{m-1}, \qquad (21)\\
D(K)=D(A_0)x^m+D(A_1)x^{m-1}y+D(A_2)x^{m-2}y^2+\\
+D(A_3)x^{m-3}y^3+D(A_4)x^{m-4}y^4+D(A_5)x^{m-5}y^5+\\
+D(A_6)x^{m-6}y^6+...+D(A_{m-7})x^7y^{m-7}+D(A_{m-6})x^6y^{m-6}+\\
+D(A_{m-5})x^5y^{m-5}+D(A_{m-4})x^4y^{m-4}+D(A_{m-3})x^3y^{m-3}+\\
+D(A_{m-2})x^2y^{m-2}+D(A_{m-1})x^1y^{m-1}+D(A_m)y^m.
\end{gathered}
$$

Taking into account the relation (15) and the last expressions from $X(K)=0$, we have the following equalities:

$$
\begin{aligned}
x^m &: D(A_0)-A_1=0,\\
x^{m-1}y &: mA_0+2A_2+D(A_1)=0,\\
x^{m-2}y^2 &: (m-1)A_1-3A_3+D(A_2)=0,\\
x^{m-3}y^3 &: (m-2)A_2-4A_4+D(A_3)=0,\\
x^{m-4}y^4 &: (m-3)A_3-5A_5+D(A_4)=0,\\
x^{m-5}y^5 &: (m-4)A_4-6A_6+D(A_5)=0,\\
x^{m-6}y^6 &: (m-5)A_5-7A_7+D(A_6)=0,\\
x^{m-7}y^7 &: (m-6)A_6-8A_8+D(A_7)=0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x^6y^{m-6} &: 7A_{m-7}-(m-5)A_{m-5}+D(A_{m-6})=0,\\
x^5y^{m-5} &: 6A_{m-6}-(m-4)A_{m-4}+D(A_{m-5})=0,\\
x^4y^{m-4} &: 5A_{m-5}-(m-3)A_{m-3}+D(A_{m-4})=0,\\
x^3y^{m-3} &: 4A_{m-4}-(m-2)A_{m-2}+D(A_{m-3})=0,\\
x^2y^{m-2} &: 3A_{m-3}-(m-1)A_{m-1}+D(A_{m-2})=0,\\
xy^{m-1} &: 2A_{m-2}-(m)A_m+D(A_{m-1})=0,\\
y^m &: A_{m-1}+D(A_m)=0.
\end{aligned}
$$

From here and Theorem 2, we obtain that

Theorem 3. *The polynomial* (19) *is a comitant of the Lyapunov system* $s\mathcal{L}(1,m_1,...,m_\ell)$ *from* (2) – (3) *with respect to the rotation group* $SO(2,\mathbb{R})$ *if and only if its coefficients satisfy the equalities*

$$
\begin{gathered}
A_1=D(A_0),\ D(A_m)=-A_{m-1},\\
A_k=\frac{1}{k}\left[D(A_{k-1})+(m-k+2)A_{k-2}\right]\ (k=\overline{2,m}),
\end{gathered} \qquad (22)
$$

where A_i $(i = \overline{0,m})$ *are from* (19)*, and* D *is from* (12).

Corollary 3. *If we know the coefficient* A_0 *for the highest degree of* x *of the comitant* K *from* (19) *of the Lyapunov system* (2) – (3) *with respect to the rotation group* $SO(2,\mathbb{R})$ *and its degree with respect to* x *and* y*, then the remaining coefficients can be constructed using formulas* (22).

By analogy with the comitants of group of center-affine transformations for system (1) we will call the coefficient A_0 of the comitant K from (19) of the Lyapunov system (2) – (3) with respect to the group $SO(2,\mathbb{R})$ *a semi-invariant.*

Remark 1. *If the comitants* $K_1, K_2, ..., K_r$ *of the Lyapunov system* (2) – (3) *with respect to the group* $SO(2,\mathbb{R})$ *are functionally independent then their semi-invariants can also be functionally independent.*

Note 3. *The number of functionally independent semi-invariants for functionally independent comitants* $K_1, K_2, ..., K_r$ *of the Lyapunov system* (2) – (3) *with respect to the rotation group* $SO(2,\mathbb{R})$ *does not exceed* r.

5 On the invariance of focal quantities in the center and focus problem with respect to the rotation group

The center and focus problem for system (2) – (3) has the following classical formulation: *for an infinite system of polynomials*

$$\left\{(x^2+y^2)^k\right\}_{k=1}^{\infty}$$

there exists a Lyapunov function

$$U(x,y) = x^2 + y^2 + \sum_{k=3}^{\infty} f_k(x,y), \tag{23}$$

where $f_k(x,y)$ *are homogeneous polynomials of degree* k *with respect to the variables* x, y*, and such constants*

$$L_1, L_2, ..., L_k, ... \tag{24}$$

that the identity

$$\frac{dU}{dt} = \sum_{k=1}^{\infty} L_k (x^2+y^2)^{k+1} \tag{25}$$

(*with respect to* x *and* y) *holds along the trajectories of the Lyapunov system* $s\mathcal{L}(1, m_1, ..., m_\ell)$ *from* (2) – (3).

The constants (24) are polynomials in coefficients of polynomials from the right side of system (2) – (3), and are called *focal quantities, Lyapunov constants or Poincaré-Lyapunov constants.*

If at least one of the quantities (24) is nonzero, then the origin of coordinates for system (2)-(3) is *a focus*, otherwise a *center.* These conditions are necessary and sufficient.

In the paper [6, p. 84] it is shown that takes place

Note 4. *Conditions for the availability of a center for the Lyapunov system $s\mathcal{L}(1, m_1, ..., m_\ell)$ from (2) – (3) are invariants of this system with respect to the rotation group $SO(2, \mathbb{R})$.*

It is known that focal quantities for the Lyapunov system $s\mathcal{L}(1, m_1, ..., m_\ell)$ from (24) are constructed ambiguously, since equations (25) with the Lyapunov function (23) contain arbitrary constants that can take different values.

We show by some examples that focal quantities in the center and focus problem for the Lyapunov system $s\mathcal{L}(1, m_1, ..., m_\ell)$ from (2) – (3) can be *invariants* and *semi-invariants* of this system with respect to the rotation group $SO(2, \mathbb{R})$.

Consider the Lyapunov system $s\mathcal{L}(1, 2)$ written in the form

$$\begin{aligned} \dot{x} &= y + gx^2 + 2hxy + ky^2, \\ \dot{y} &= -x + lx^2 + 2mxy + ny^2, \end{aligned} \tag{26}$$

for which the Lie operators (11) – (12) have the form

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + D, \tag{27}$$

where

$$\begin{aligned} D = (2h + l)\frac{\partial}{\partial g} + (-g + k + m)\frac{\partial}{\partial h} + (2h + n)\frac{\partial}{\partial k} + \\ +(-g + 2m)\frac{\partial}{\partial l} + (-h - l + n)\frac{\partial}{\partial m} - (k + 2m)\frac{\partial}{\partial n}. \end{aligned} \tag{28}$$

In the paper of A. P. Sadovsky [7, p. 110] 3 focal values are given that solve the center and focus problem for the system $s\mathcal{L}(1, 2)$ from (26), having the form

$$\begin{gathered} L_1 S = \frac{1}{2}(-gh - hk + gl + lm - kn + mn), \\ L_2 S = \frac{1}{24}(62g^3h - 2gh^3 + 119g^2hk - 2h^3k + 62ghk^2 + 5hk^3 - \\ -62g^3l + 27gh^2l - 63g^2kl + 29h^2kl - 15gk^2l - 8ghl^2 + 15hkl^2 - \\ -5gl^3 + 101g^2hm + 138ghkm + 37hk^2m - 175g^2lm - 6h^2lm - \\ -116gklm - 15k^2lm - 13hl^2m - 5l^3m + 54ghm^2 + 54hkm^2 - \\ -159glm^2 - 53klm^2 - 46lm^3 + 6g^3n + 37gh^2n + 72g^2kn + 39h^2kn + \\ +57gk^2n + 5k^3n + 14ghln + 68hkln - 33gl^2n + 15kl^2n - 72g^2mn - \\ -6h^2mn + 34gkmn + 32k^2mn - 42hlmn - 38l^2mn - 109gm^2n - \\ -3km^2n - 46m^3n + 48ghn^2 + 79hkn^2 - 48gln^2 + 39kln^2 - \\ -29hmn^2 - 71lmn^2 - 6gn^3 + 38kn^3 - 38mn^3), \end{gathered}$$

$$
\begin{gathered}
L_3S = \frac{1}{2304}(-44725g^5h + 11142g^3h^3 - 88gh^5 - 124537g^4hk+ \\
+30842g^2h^3k - 88h^5k - 121728g^3hk^2 + 27186gh^3k^2 - 45492g^2hk^3+ \\
+7486h^3k^3 - 2651ghk^4 + 925hk^5 + 44725g^5l - 51066g^3h^2l- \\
-1216gh^4l + 84372g^4kl - 95044g^2h^2kl - 1704h^4kl + 53320g^3k^2l- \\
-44602gh^2k^2l + 10096g^2k^3l - 2880h^2k^3l - 465gk^4l + 28368g^3hl^2- \\
-3362gh^3l^2 + 7708g^2hkl^2 - 5802h^3kl^2 - 13582ghk^2l^2 - 3650hk^3l^2+ \\
+2436g^3l^3 + 1858gh^2l^3 + 9332g^2kl^3 - 1700h^2kl^3 + 3650gk^2l^3- \\
-875ghl^4 + 465hkl^4 - 925gl^5 - 157320g^4hm + 7528g^2h^3m- \\
-356424g^3hkm + 19904gh^3km - 264096g^2hk^2m + 12376h^3k^2m- \\
-67288ghk^3m - 2296hk^4m + 211165g^4lm - 72038g^2h^2lm+ \\
+888h^4lm + 310816g^3klm - 109100gh^2klm + 144112g^2k^2lm- \\
-26630h^2k^2lm + 17812gk^3lm - 465k^4lm + 67344g^2hl^2m+ \\
+2776h^3l^2m + 12024ghkl^2m - 7552hk^2l^2m + 14808g^2l^3m+ \\
+978h^2l^3m + 17844gkl^3m + 3650k^2l^3m - 1800hl^4m - 925l^5m- \\
-214106g^3hm^2 - 4560gh^3m^2 - 214106g^2hkm^2 - 4560h^3km^2- \\
-29106ghk^2m^2 - 29106hk^3m^2 + 389610g^3lm^2 - 16864gh^2lm^2+ \\
+249820g^2klm^2 - 32896h^2klm^2 + 131634gk^2lm^2 + 7716k^3lm^2+ \\
+60798ghl^2m^2 + 9126hkl^2m^2 + 15478gl^3m^2 + 8512kl^3m^2- \\
-131528g^2hm^3 - 188448ghkm^3 - 56920hk^2m^3 + 350410g^2lm^3+ \\
+7632h^2lm^3 + 83832gklm^3 + 40842k^2lm^3 + 15912hl^2m^3+ \\
+3106l^3m^3 - 29432ghm^4 - 29432hkm^4 + 152800glm^4+ \\
+60456klm^4 + 25560lm^5 - 4560g^5n - 49528g^3h^2n- \\
-2168gh^4n - 56129g^4kn - 71382g^2h^2kn - 2656h^4kn- \\
-86332g^3k^2n - 8356gh^2k^2n - 42376g^2k^3n + 11242h^2k^3n- \\
-3576gk^4n + 925k^5n - 8288g^3hln - 21396gh^3ln- \\
-107608g^2hkln - 28148h^3kln - 78028ghk^2ln - 6580hk^3ln+ \\
+41832g^3l^2n - 3496gh^2l^2n + 29628g^2kl^2n - 28690h^2kl^2n- \\
-7552gk^2l^2n - 3650k^3l^2n + 7132ghl^3n - 5780hkl^3n- \\
-520gl^4n + 465kl^4n + 44753g^4mn - 87182g^2h^2mn+ \\
+888h^4mn - 78864g^3kmn - 108972gh^2kmn - 155168g^2k^2mn- \\
-11358h^2k^2mn - 60956gk^3mn - 3221k^4mn + 51096g^2hlmn+ \\
+9632h^3lmn - 124544ghklmn - 50104hk^2lmn + 144264g^2l^2mn+ \\
+16334h^2l^2mn + 63884gkl^2mn - 1522k^2l^2mn + 5272hl^3mn- \\
-1445l^4mn + 173116g^3m^2n - 43264gh^2m^2n + 17830g^2km^2n- \\
-59296h^2km^2n + 82532gk^2m^2n - 25890k^3m^2n + 121900ghlm^2n-
\end{gathered}
$$

$$\begin{gathered}
-26836hklm^2n+146916gl^2m^2n+39066kl^2m^2n+222962g^2m^3n+\\
+7632h^2m^3n-103272gkm^3n-18814k^2m^3n+53184hlm^3n+\\
+38574l^2m^3n+125304gm^4n+32960km^4n+25560m^5n-\\
-56942g^3hn^2-23378gh^3n^2-141270g^2hkn^2-27690h^3kn^2-\\
-60328ghk^2n^2+6856hk^3n^2+48734g^3ln^2-38598gh^2ln^2-\\
-11808g^2kln^2-81368h^2kln^2-38236gk^2ln^2-3700k^3ln^2+\\
+22272ghl^2n^2-43344hkl^2n^2+12584gl^3n^2-4080kl^3n^2-\\
-57664g^2hmn^2+6856h^3mn^2-184664ghkmn^2-\\
-49232hk^2mn^2+202966g^2lmn^2+36790h^2lmn^2+\\
+13848gklmn^2-28284k^2lmn^2+43488hl^2mn^2+\\
+11604l^3mn^2+28654ghm^2n^2-68410hkm^2n^2+\\
+243030glm^2n^2+34596klm^2n^2+37272hm^3n^2+75942lm^3n^2-\\
-288g^3n^3-43772gh^2n^3-48542g^2kn^3-64906h^2kn^3-\\
-30696gk^2n^3+3100k^3n^3+4176ghln^3-85408hkln^3+\\
+31676gl^2n^3-20456kl^2n^3+55598g^2mn^3+21434h^2mn^3-\\
-58400gkmn^3-31408k^2mn^3+69384hlmn^3+39200l^2mn^3+\\
+100760gm^2n^3-6790km^2n^3+40474m^3n^3-18481ghn^4-\\
-56701hkn^4+25249gln^4-30484kln^4+32968hmn^4+\\
+43993lmn^4+3408gn^5-16917kn^5+16917mn^5).
\end{gathered} \tag{29}$$

In L_iS, the letter S emphasizes that L_iS are focal quantities of A. P. Sadovsky.
Note 5. *Note that for the focal quantities* (29) *using the Lie operator* (28) *we find*

$$D(L_1S)=0,\ D(L_2S)\neq 0,\ D(L_3S)\neq 0,$$

whence it follows that L_1S is an invariant of the system $s\mathcal{L}(1,2)$ with respect to the rotation group $SO(2,\mathbb{R})$, but L_2S and L_3S are not.

Taking into account Theorems 2 and 3 and operators (27) – (28) we find that for the focal quantities L_2S and L_3S from (29) the following comitants of the rotation group correspond:

$$K(L_2S)=L_2Sx^4+A_1x^3y+A_2x^2y^2+A_3xy^3+A_4y^4, \tag{30}$$

where

$$A_1=D(L_2S),\ A_2=\frac{1}{2}[D(A_1)+4L_2S], A_3=\frac{1}{3}[D(A_3)+2A_2],\ A_4=\frac{1}{4}[D(A_3)+2A_2], \tag{31}$$

and

$$\begin{gathered}
K(L_3S)=L_3Sx^{12}+A_1x^{11}y+A_2x^{10}y^2+A_3x^9y^3+A_4x^8y^4+\\
+A_5x^7y^5+A_6x^6y^6+A_7x^5y^7+A_8x^4y^8+A_9x^3y^9+A_{10}x^2y^{10}+\\
+A_{11}xy^{11}+A_{12}y^{12},
\end{gathered} \tag{32}$$

for

$$\begin{gathered}
A_1 = D(L_3S),\ A_2 = \frac{1}{2}[D(A_1) + 12L_3S],\ A_3 = \frac{1}{3}[D(A_2) + 11A_1],\\
A_4 = \frac{1}{4}[D(A_3) + 10A_2],\ A_5 = \frac{1}{5}[D(A_4) + 9A_3],\\
A_6 = \frac{1}{6}[D(A_5) + 8A_4],\ A_7 = \frac{1}{7}[D(A_6) + 7A_5],\\
A_8 = \frac{1}{8}[D(A_7) + 6A_6],\ A_9 = \frac{1}{9}[D(A_8) + 5A_7],\\
A_{10} = \frac{1}{10}[D(A_9) + 4A_8],\ A_{11} = \frac{1}{11}[D(A_{10}) + 3A_9],\\
A_{12} = \frac{1}{12}[D(A_{11}) + 2A_{10}].
\end{gathered} \tag{33}$$

Using the Lee operator (27) – (28) from (30) – (31) and (32) – (33) we find that

$$X[K(L_2S)] = X[K(L_3S)] = 0. \tag{34}$$

Consequently, from Remark 5 and equalities (34) we obtain that

Lemma 1. *The focal quantity L_1S is an invariant, but L_2S and L_3S are semi-invariants of system $s\mathcal{L}(1,2)$ from (26) with respect to the rotation group $SO(2,\mathbb{R})$.*

Note 6. *The other three focal quantities constructed by another method that solve the center and focus problem for the system $s\mathcal{L}(1,2)$ from (26), were presented to us by Yu. F. Kalin. They have the form*

$$\begin{gathered}
L_1S = L_2C = \frac{1}{2}(-gh - hk + gl + lm - kn + mn),\\
L_2C = \frac{1}{24}(53g^3h + 16gh^3 + 95g^2hk + 16h^3k + 47ghk^2 + 5hk^3-\\
-53g^3l + 18gh^2l - 48g^2kl + 38h^2kl - 15gk^2l - 17ghl^2 + 15hkl^2-\\
-5gl^3 + 62g^2hm + 84ghkm + 22hk^2m - 127g^2lm - 24h^2lm - 86gklm-\\
-15k^2lm - 22hl^2m - 5l^3m + 24ghm^2 + 24hkm^2 - 90glm^2 - 38klm^2-\\
-16lm^3 + 6g^3n + 70gh^2n + 63g^2kn + 90h^2kn + 42gk^2n + 5k^3n-\\
-10ghln + 86hkln - 42gl^2n + 15kl^2n - 63g^2mn - 24h^2mn + 10gkmn+\\
+17k^2mn - 84hlmn - 47l^2mn - 70gm^2n - 18km^2n - 16m^3n + 63ghn^2+\\
+127hkn^2 - 63gln^2 + 48kln^2 - 62hmn^2 - 95lmn^2 - 6gn^3+\\
+53kn^3 - 53mn^3),\\
L_3C = \frac{1}{2304}(-31393g^5h - 17022g^3h^3 - 1144gh^5 - 77985g^4hk-\\
-27330g^2h^3k - 1144h^5k - 67264g^3hk^2 - 8666gh^3k^2 - 21292g^2hk^3+\\
+1642h^3k^3 + 305ghk^4 + 925hk^5 + 31393g^5l - 27138g^3h^2l - 10960gh^4l+\\
+50720g^4kl - 70216g^2h^2kl - 13080h^4kl + 30636g^3k^2l - 41898gh^2k^2l+
\end{gathered}$$

$$+5940g^2k^3l-5844h^2k^3l-465gk^4l+35076g^3hl^2-1706gh^3l^2+$$
$$+16672g^2hkl^2-14946h^3kl^2-6030ghk^2l^2-3650hk^3l^2+828g^3l^3+$$
$$+7270gh^2l^3+6664g^2kl^3-3560h^2kl^3+3650gk^2l^3+265ghl^4+465hkl^4-$$
$$-925gl^5-79756g^4hm-30792g^2h^3m-164120g^3hkm-37792gh^3km-$$
$$-106536g^2hk^2m-7000h^3k^2m-21512ghk^3m+660hk^4m+119405g^4lm+$$
$$+4582g^2h^2lm+3096h^4lm+157960g^3klm-48148gh^2klm+$$
$$+72848g^2k^2lm-19738h^2k^2lm+9500gk^3lm-465k^4lm+74424g^2hl^2m+$$
$$+12424h^3l^2m+31600ghkl^2m+6844g^2l^3m+7530h^2l^3m+12508gkl^3m+$$
$$+3650k^2l^3m-660hl^4m-925l^5m-74114g^3hm^2-15376gh^3m^2-$$
$$-127550g^2hkm^2-15376h^3km^2-60966ghk^2m^2-7530hk^3m^2+$$
$$+168022g^3lm^2+41568gh^2lm^2+177140g^2klm^2+57158gk^2lm^2+$$
$$+3560k^3lm^2+52258ghl^2m^2+19738hkl^2m^2+4374gl^3m^2+5844kl^3m^2-$$
$$=29432g^2hm^3-41856ghkm^3-12424hk^2m^3+104770g^2lm^3+$$
$$+15376h^2lm^3+82980gklm^3+14946k^2lm^3+7000hl^2m^3-1642l^3m^3-$$
$$-3096ghm^4-3096hkm^4+25904glm^4+13080klm^4+1144lm^5-$$
$$-4128g^5n-89564g^3h^2n-23784gh^4n-41357g^4kn-183390g^2h^2kn-$$
$$-25904h^4kn-51912g^3k^2n-91176gh^2k^2n-21132g^2k^3n-4374h^2k^3n-$$
$$-620gk^4n+925k^5n+30248g^3hln-43620gh^3ln-64504g^2hkln-$$
$$-82980h^3kln-67900ghk^2ln-12508hk^3ln+43332g^3l^2n+17324gh^2l^2n+$$
$$+30812g^2kl^2n-57158h^2kl^2n-3650k^3l^2n+21092ghl^3n-9500hkl^3n+$$
$$+620gl^4n+465kl^4n+32573g^4mn-89970g^2h^2mn+3096h^4mn-$$
$$=32600g^3kmn-175220gh^2kmn-61672g^2k^2mn-52258h^2k^2mn-$$
$$-21092gk^3mn-265k^4mn+162352g^2hlmn+41856h^3lmn-$$
$$-31600hk^2lmn+119932g^2l^2mn+60966h^2l^2mn+67900gkl^2mn+$$
$$+6030k^2l^2mn+21512hl^3mn-305l^4mn+93296g^3m^2n+38762g^2km^2n-$$
$$-41568h^2km^2n-17324gk^2m^2n-7270k^3m^2n+175220ghlm^2n+$$
$$+48148hklm^2n+91176gl^2m^2n+41898kl^2m^2n+78650g^2m^3n+$$
$$+15376h^2m^3n+43620gkm^3n+1706k^2m^3n+37792hlm^3n+$$
$$+8666l^2m^3n+23784gm^4n+10960km^4n+1144m^5n-$$
$$-68550g^3hn^2-78650gh^3n^2-204974g^2hkn^2-104770h^3kn^2-$$
$$-119932ghk^2n^2-6844hk^3n^2+59766g^3ln^2-38762gh^2ln^2-$$
$$-177140h^2kln^2-30812gk^2ln^2-6664k^3ln^2+61672ghl^2n^2-$$
$$-72848hkl^2n^2+21132gl^3n^2-5940kl^3n^2+29432h^3mn^2-$$
$$-162352ghkmn^2-74424hk^2mn^2+204974g^2lmn^2+$$
$$+127550h^2lmn^2+64504gklmn^2-16672k^2lmn^2+$$

$$
\begin{gathered}
+106536hl^2mn^2+21292l^3mn^2+89970ghm^2n^2- \\
-4582hkm^2n^2+183390glm^2n^2+70216klm^2n^2+ \\
+30792hm^3n^2+27330lm^3n^2-93296gh^2n^3- \\
+59766g^2kn^3-168022h^2kn^3-43332gk^2n^3-828k^3n^3+ \\
+32600ghln^3-157960hkln^3+51912gl^2n^3-30636kl^2n^3+ \\
+68550g^2mn^3+74114h^2mn^3-30248gkmn^3-35076k^2mn^3+ \\
+164120hlmn^3+67264l^2mn^3+89564gm^2n^3+27138km^2n^3+ \\
+17022m^3n^3-32573ghn^4-119405hkn^4+41357gln^4- \\
-50720kln^4+79756hmn^4+77985lmn^4+4128gn^5- \\
-31393kn^5+31393mn^5).
\end{gathered} \tag{35}
$$

In L_iC, the letter C emphasizes that L_iC are focal quantities of Yu. F. Kalin. Using the operator D from (28) and (35) we find that

$$D(L_1C)=D(L_2C)=D(L_3C)=0.$$

Hence, taking into account Corollary 2, we have that the focal quantities (35) are invariants of the system $s\mathcal{L}(1,2)$ from (26) with respect to the rotation group $SO(2,\mathbb{R})$.

In a similar way that was used for the focal quantities of the system $s\mathcal{L}(1,2)$ one can find a sequences of focal quantities for the systems $s\mathcal{L}(1,3)$, $s\mathcal{L}(1,4)$, $s\mathcal{L}(1,2,3)$ and other, which are invariants and semi-invariants of these systems with respect to the rotation group $SO(2,\mathbb{R})$.

6 On the upper bound of number of functionally independent focal quantities that take part in solving the center and focus problem for the Lyapunov systems $s\mathcal{L}(1,m_1,...,m_\ell)$

Consider the set of comitants $\{F(x,y,A)\}$ of the Lyapunov system $s\mathcal{L}(1,m_1,...,m_\ell)$ of the form (2) – (3) with respect to the rotation group $SO(2,\mathbb{R})$. From Theorem 2 it follows that the equality (15) is true, where X is a Lie operator of linear representation of the rotation group $SO(2,\mathbb{R})$ in the space $E_1^N(x,y,A)$ of the system $s\mathcal{L}(1,m_1,...,m_\ell)$. Then, according to Note 2, we have that the maximal number of functionally independent solutions of the equation (15) is equal to

$$N_1-1, \tag{36}$$

where N_1 is from (17). Since from examples considered in Section 5 we have that focal quantities in the center and focus problem for the Lyapunov systems $s\mathcal{L}(1,m_1,...,m_\ell)$ from (2) – (3), generally speaking, are semi-invariants of this system with respect to the rotation group $SO(2,\mathbb{R})$, then according to Remark 3, the number of functionally independent focal quantities for this system does not exceed the number (36).

Thus we obtain

Theorem 4. *The number of functionally independent focal quantities in the center and focus problem for the Lyapunov system* $s\mathcal{L}(1, m_1, ..., m_\ell)$ *does not exceed the number* (36).

Acknowledgements. This research was supported by grants 15.817.02.03F.

References

[1] Popa M. N., Pricop V. V. *Applications of algebraic methods in solving the center-focus problem.* Bul. Acad. Ştiinţe Repub. Moldova, Matematica, 2013, No 1 (71), 45–71. http://arxiv.org/abs/1310.4343.pdf.

[2] Popa M. N., Pricop V. V. *The center and focus problem: algebraic solutions and hypotheses.* Academy of Sciences of Moldova, Chişinău, 2018 (in Russian).

[3] Popa M. N. *Algebraic methods for differential systems.* Piteshty University, The Flower Power Edit, 2004 (in Romanian).

[4] Sibirsky K. S. *Introduction to the algebraic theory of invariants of differential equations.* Translated from Russian. Nonlinear Science: Theory and Applications. Manchester University Press, Manchester, 1988.

[5] Pontryagin L. S. *Ordinary differential equations.* VI + 298 S. London 1962. Addison Wesley Publishing Company.

[6] Sibirsky K. S. *Algebraic Invariants of Differential Equations and Matrices.* Shtiintsa, Kishinev, 1976 (in Russian).

[7] Sadovsky A. P. *Polynomial ideals and varieties.* Minsk, BSU, 2008 (in Russian).

Mihail Popa, Victor Pricop
Vladimir Andrunachievici Institute of Mathematics
and Computer Science
5, Academiei street, Chişinău
Republic of Moldova, MD 2028
E-mail: *mihailpomd@gmail.com; pricopvv@gmail.com*

Received August 12, 2019