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The classification of a family of cubic differential systems in terms of configurations of invariant lines of the type $(3,3)$

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Abstract. In this article we consider the class of non-degenerate real planar cubic vector fields, which possess two real and two complex distinct infinite singularities and invariant straight lines, including the line at infinity, of total multiplicity 7. In addition, the systems from this class possess configurations of the type $(3,3)$. We prove that there are exactly 16 distinct configurations of invariant straight lines for this class and present corresponding examples for the realization of each one of the detected configurations.

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1 Introduction and the statement of the Main Theorem

We consider here real polynomial differential systems

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where $P,\ Q$ are polynomials in $x,\ y$ with real coefficients, i.e. $P,\,Q \in \mathbb{R}[x,y]$. We call degree of a system (1) $\max(\deg(P), \deg(Q))$. A *cubic* system (1) is of degree three.

Let

$$\mathbf{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

be the polynomial vector field corresponding to a system (1).

An algebraic curve $f(x,y)=0$ with $f(x,y) \in \mathbb{C}[x,y]$ is an invariant curve of a system of the form (1) where $P(x,y), Q(x,y) \in \mathbb{C}[x,y]$ if and only if there exists $K[x,y] \in \mathbb{C}[x,y]$ such that

$$\mathbf{X}(f) = P(x,y)\frac{\partial f}{\partial x} + Q(x,y)\frac{\partial f}{\partial y} = f(x,y)K(x,y)$$

is an identity in $\mathbb{C}[x,y]$. Since $\mathbb{R} \subset \mathbb{C}$, any system (1) over $\mathbb{R}$ generates a system of differential equations over $\mathbb{C}$.

Using the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}_2(\mathbb{C})$, $(x,y) \mapsto [x:y:1] = [X:Y:Z]$, $(x = X/Z, y = Y/Z$ and $Z \neq 0)$, we can compactify the differential equation $Q(x,y)dy - P(x,y)dx = 0$ to an associated differential equation over the complex projective plane. In fact the theory of Darboux in [13] is done for differential equations on the complex projective plane.

We compactify the space of all the polynomial differential systems (1) of degree n on $\mathbb{S}^{N-1}$ with $N = (n+1)(n+2)$ by multiplying the coefficients of each systems by $1/(\sum(a_{ij}^2 + b_{ij}^2))^{1/2}$, where a_{ij} and b_{ij} are the coefficients of the polynomials $P(x,y)$ and $Q(x,y)$, respectively.

Definition 1 (see [29])**.** (1) We say that an invariant curve $\mathcal{L} : f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ for a polynomial system (S) of degree n has *multiplicity* m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}$, $N = (n+1)(n+2)$, such that each (S_k) has m distinct invariant curves $\mathcal{L}_{1,k} : f_{1,k}(x,y) = 0, \ldots, \mathcal{L}_{m,k} : f_{m,k}(x,y) = 0$ over $\mathbb{C}$, $\deg(f) = \deg(f_{i,k}) = r$, converging to $\mathcal{L}$ as $k \to \infty$, in the topology of $P_{R-1}(\mathbb{C})$, with $R = (r+1)(r+2)/2$ and this does not occur for $m+1$.

(2) We say that the line at infinity $\mathcal{L}_\infty : Z = 0$ of a polynomial system (S) of degree n has *multiplicity* m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}$, $N = (n+1)(n+2)$, such that each (S_k) has $m-1$ distinct invariant lines $\mathcal{L}_{1,k} : f_{1,k}(x,y) = 0, \ldots, \mathcal{L}_{m-1,k} : f_{m-1,k}(x,y) = 0$ over $\mathbb{C}$, converging to the line at infinity $\mathcal{L}_\infty$ as $k \to \infty$, in the topology of $P_2(\mathbb{C})$ and this does not occur for m.

In this work we consider a particular case of invariant algebraic curves, namely the invariant straight lines of systems (1). A straight line over $\mathbb{C}$ is the locus $\{(x,y) \in \mathbb{C}^2 | f(x,y) = 0\}$ of an equation $f(x,y) = ux + vy + w = 0$ with $(u,v) \neq (0,0)$ and $(u,v,w) \in \mathbb{C}^3$. We note that by multiplying the equation by a non-zero complex number λ, the locus of the equation does not change. So that we have a bijection between the lines in $\mathbb{C}^2$ and the points in $\mathbb{P}_2(\mathbb{C}) \backslash \{[0:0:1]\}$. This bijection induces a topology on the set of lines in $\mathbb{C}^2$ from the topology of $\mathbb{P}_2(\mathbb{C})$ and hence we can talk about a sequence of lines convergent to a line in $\mathbb{C}^2$.

For an invariant line $f(x,y) = ux + vy + w = 0$ we denote $\hat{a} = (u,v,w) \in \mathbb{C}^3$ and by $[\hat{a}] = [u:v:w]$ the corresponding point in $\mathbb{P}_2(\mathbb{C})$. We say that a sequence of straight lines $f_i(x,y) = 0$ converges to a straight line $f(x,y) = 0$ if and only if the sequence of points $[\hat{a}_i]$ converges to $[\hat{a}] = [u:v:w]$ in the topology of $\mathbb{P}_2(\mathbb{C})$.

In view of the above definition of an invariant algebraic curve of a system (1), a line $f(x,y) = ux + vy + w = 0$ over $\mathbb{C}$ is an invariant line if and only if there exists $K(x,y) \in \mathbb{C}[x,y]$ which satisfies the following identity in $\mathbb{C}[x,y]$:

$$\mathbf{X}(f) = uP(x,y) + vQ(x,y) = (ux + vy + w)K(x,y).$$

We point out that if we have an invariant line $f(x,y) = 0$ over $\mathbb{C}$ it could happen that multiplying the equation by a number $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the coefficients of the new equation become real, i.e. $(u\lambda, v\lambda, w\lambda) \in \mathbb{R}^3$. In this case, along with the line

$f(x, y) = 0$ sitting in $\mathbb{C}^2$ we also have an associated real line, sitting in $\mathbb{R}^2$ defined by $\lambda f(x, y) = 0$.

Note that, since a system (1) is with real coefficients, if its associated complex system has a complex invariant straight line $ux + vy + w = 0$, then its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$ is also invariant.

A line in $\mathbb{P}_2(\mathbb{C})$ is the locus in $\mathbb{P}_2(\mathbb{C})$ of an equation $F(X, Y, Z) = uX + vY + wZ = 0$ where $(u, v, w) \in \mathbb{C}^3$ and $F(X, Y, Z) \in \mathbb{C}[X, Y, Z]$. The line $Z = 0$ in $\mathbb{P}_2(\mathbb{C})$ is called the line at infinity of the affine plane $\mathbb{C}^2$. This line is an invariant manifold of the complex differential equation on $\mathbb{P}_2(\mathbb{C})$. Clearly the lines in $\mathbb{P}_2(\mathbb{C})$ are in a one-to-one correspondence with points $[u : v : w] \in \mathbb{P}_2(\mathbb{C})$ and thus we have a topology on the set of lines in $\mathbb{P}_2(\mathbb{C})$. We can thus talk about a sequence of lines in $\mathbb{P}_2(\mathbb{C})$ convergent to a line in $\mathbb{P}_2(\mathbb{C})$.

To a line $f(x, y) = ux + vy + w = 0$, $(u, v) \neq (0, 0)$, $f \in \mathbb{C}[x, y]$, we associate its projective completion $F(X, Y, Z) = uX + vY + wZ = 0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}_2(\mathbb{C})$, $(x, y) \mapsto [x : y : 1] = [X, Y, Z]$ indicated above.
We first remark that in the above definition we made an abuse of language. Indeed, we talk about complex invariant lines of real systems. However we already said that to a real system one can associate a complex system and to a differential equation $Q(x, y)dy - P(x, y)dx = 0$ corresponds a differential equation in $\mathbb{P}_2(\mathbb{C})$.

We remark that the above definition is a particular case of the definition of geometric multiplicity given in [12], and namely the ”strong geometric multiplicity” with the restriction that the corresponding perturbations are cubic systems.

The set $\mathbb{CS}$ of cubic differential systems depends on 20 parameters and for this reason people began by studying particular subclasses of $\mathbb{CS}$. Some of these subclasses are on cubic systems having invariant straight lines.

We mention here some papers on polynomial differential systems possessing invariant straight lines. For cubic systems see [4, 5, 7–11, 18, 20–22, 26, 35, 36] and [27].

The existence of sufficiently many invariant straight lines of planar polynomial systems could be used for proving the integrability of such systems. During the past 15 years several articles were published on this theme (see for example [30, 31]).

According to [1], for a non-degenerate polynomial differential system of degree m, the maximum number of invariant straight lines including the line at infinity and taking into account their multiplicities is $3m$. This bound is always reached (see [12]).

In particular, the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems with a finite number of infinite singularities is 9. In [20] the authors classified all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities according to their *configurations of invariant lines*. The notion of configuration of invariant lines for a polynomial differential system was first introduced in [29].

Definition 2 (see [32])**.** Consider a real planar polynomial differential system (1). We call *configuration of invariant straight lines* of this system, the set of (complex) invariant straight lines (which may have real coefficients), including the line at infinity, of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

In [20] the authors used a weaker notion, not taking into account the multiplicities of real singularities. They detected 23 such configurations. Moreover, in [20] the necessary and sufficient conditions for the realization of each one of 23 configurations detected, are determined using invariant polynomials with respect to the action of *the group of affine transformations ($Aff(2,\mathbb{R})$) and time rescaling* (i.e. $Aff(2,\mathbb{R})\times\mathbb{R}^*$)). In [4] the author detected another class of cubic systems whose configuration of invariant lines was not detected in [20].

If two polynomial systems are equivalent under the action of the affine group and time rescaling, clearly they must have the same kinds of configurations of invariant lines. But it could happen that two distinct polynomial systems which are non-equivalent modulo the action of the affine group and time rescaling have "the same kind of configurations" of straight lines. We need to say when two configurations are considered equivalent.

Definition 3 (see [6])**.** Suppose we have two cubic systems $(S),(S')$ both with a finite number of singularities, finite and infinite, a finite set of invariant straight lines $\mathcal{L}_i : f_i(x,y)=0$, $i=1,\ldots,k$, of (S) (respectively $\mathcal{L}'_i : f'_i(x,y)=0$, $i=1,\ldots,k'$, of (S')). We say that the two configurations C, C' of invariant lines, including the line at infinity, of these systems are equivalent if there is a one-to-one correspondence ϕ between the lines of C and C' such that:

(i) ϕ sends an affine line (real or complex) to an affine line and the line at infinity to the line at infinity conserving the multiplicities of the lines and also sends an invariant line with coefficients in $\mathbb{R}$ to an invariant line with coefficients in $\mathbb{R}$;

(ii) for each line $\mathcal{L} : f(x,y)=0$ we have a one-to-one correspondence between the real singular points on $\mathcal{L}$ and the real singular points on $\phi(\mathcal{L})$ conserving their multiplicities and their order on these lines;

(iii) we have a one-to-one correspondence ϕ_∞ between the real singular points at infinity on the (real) lines at infinity of (S) and (S') such that when we list in a counter-clockwise wise sense the real singular points at infinity on (S) starting from a point p on the Poincaré disk, $p_1=p,\ldots,p_k$, ϕ_∞ preserves the multiplicities of the singular points and preserves or reverses the orientation;

(iv) consider the total curves

$$\mathcal{F} : \prod F_j(X,Y,Z)^{m_i} Z^m = 0, \mathcal{F}' : \prod F'_j(X,Y,Z)^{m'_i} Z^m = 0$$

where $F_i(X,Y,Z)=0$ (respectively $F'_i(X,Y,Z)=0$) are the projective completions of, $\mathcal{L}_i$ (respectively $\mathcal{L}'_i$) and m_i, m'_i are the multiplicities of the curves $F_i=0, F'_i=0$ and m, m' are respectively the multiplicities of $Z=0$ in the first and in the second system. Then, there is a one-to-one correspondence ψ between the real singularities of the curves $\mathcal{F}$ and $\mathcal{F}'$ conserving their multiplicities as singular points of the total curves.

Remark 1. In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [29]. Thus we denote by "(a, b)" the maximum number a (respectively b) of infinite (respectively finite) singularities which can be obtained by perturbation of a multiple infinite singular point.

The configurations of invariant straight lines which were detected for some families of systems (1), were instrumental for determining the phase portraits of those families. For example, in [30, 31] it was proved that we have a total of 57 distinct configurations of invariant lines for quadratic systems with invariant lines of total multiplicity greater than or equal to 4. These 57 configurations lead to the existence of 135 topologically distinct phase portraits. In [26, 27, 35, 36] it was proved that cubic systems with invariant lines of total parallel multiplicity six or seven (the notion of "parallel multiplicity" could be found in [36]) have 113 topologically distinct phase portraits. This was done by using the various possible configurations of invariant lines of these systems.

Definition 4. Suppose a cubic system (1) possesses 7 invariant straight lines, including the line at infinity and taking into consideration their multiplicity. We say that these lines form a *configuration of type* $(3, 3)$ if there exists two triplets of parallel lines, every set with different slope.

Notation 1. *We shall denote by* $\mathbf{CSL}^{2r2c\infty}_{(3,3)}$ *the class of cubic systems which have two real and two complex distinct infinite singularities and possess configurations of invariant straight lines of the type* $(3, 3)$.

Our main result is the following one.

Main Theorem. *The class* $\mathbf{CSL}^{2r2c\infty}_{(3,3)}$ *has a total of 16 non-equivalent configurations of invariant lines Config. 7.1a–Config. 7.16a (see Figure 1). We prove that each one of these configurations is realizable within* $\mathbf{CSL}^{2r2c\infty}_{(3,3)}$ *by constructing examples for each one of the configurations Config. 7.1a–Config. 7.16a.*

Notation 2. *We give here the directions as how to read the pictures representing the configurations. An invariant line with multiplicity* $k > 1$ *will appear in a configuration in bold face and will have next to it the number* k. *Real invariant straight lines are represented by continuous lines, whereas complex invariant straight lines are represented by dashed lines. The multiplicities of the real singular points of the system located on the invariant lines, will be indicated next to the singular points. The maximum number of parallel invariant straight lines will be shown to be three. Whenever we have three parallel line, clearly at least for one of these will be real. Due to an affine transformation we can assume this line to be* $x = 0$ *and after this transformation the system will be of the form:*

$$\dot{x} = x(a_1 + 2a_2x + a_3x^2), \quad \dot{y} = Q(\tilde{a}, x, y).$$

Here $Q(x, y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + 2a_{11}xy + a_{02}y^2 + a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3$ *and* $\tilde{a} = (a_{00}, a_{10}, ..., a_{03})$. *If two invariant lines of the triplet are*

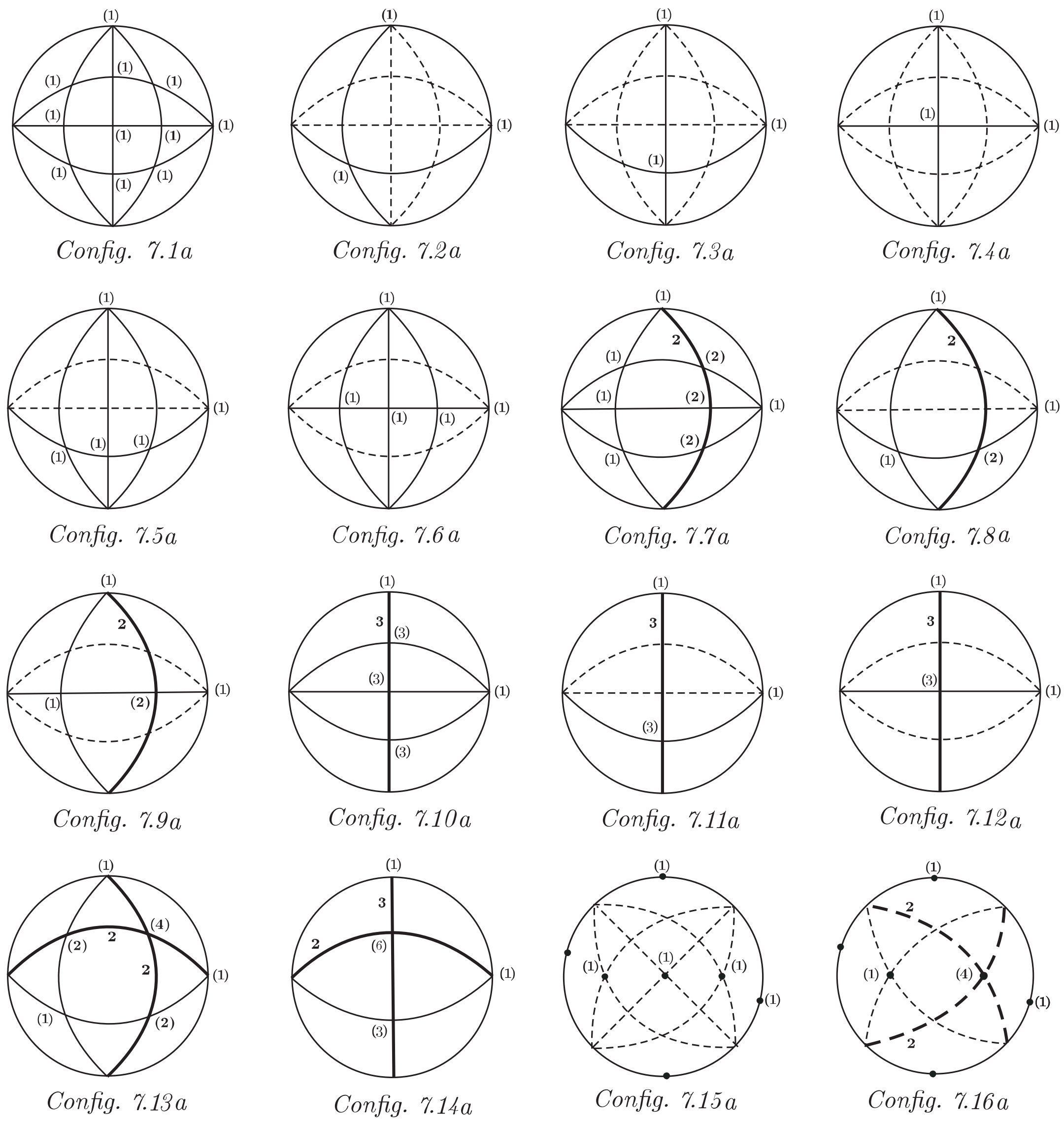

Figure 1. **Configurations of the type $(3,3)$ for cubic systems with 2 real and 2 complex infinite singularities**

complex, then the condition $b^2 - ac < 0$ must hold. This implies that $c \neq 0$ and due to time rescaling we may assume $c = 1$. Setting $b^2 - a = -u^2$ ($a = b^2 + u^2$) we obtain the system

$$\begin{aligned} \dot{x} &= x[(x+b)^2 + u^2], \\ \dot{y} &= Q(a, x, y). \end{aligned} \tag{2}$$

which has the triplet of invariant lines: $x = 0$, $x = -b + iu$, $x = -b - iu$. In case $b \neq 0$ we place both complex invariant lines on one side of the real line. If $b = 0$ we make the convention to place this line between the two complex lines.

2 Preliminaries

Consider real cubic systems, i.e. systems of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(a,x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(a,x,y) \end{aligned} \tag{3}$$

with variables x and y and real coefficients. The polynomials p_i and q_i ($i = 0,1,2,3$) are homogeneous polynomials of degree i in x and y:

$$\begin{aligned} &p_0 = a_{00}, \quad p_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3, \\ &p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ &q_0 = b_{00}, \quad q_3(x,y) = b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3, \\ &q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ be the 20-tuple of the coefficients of systems (3) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03}, x, y]$.

It is known that on the set of polynomial systems (1), in particular on the set **CS** of all cubic differential systems (3), acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane [32]. For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **CS**. We can identify the set **CS** of systems (3) with a subset of $\mathbb{R}^{20}$ via the map $\mathbf{CS} \longrightarrow \mathbb{R}^{20}$ which associates to each system (3) the 20-tuple $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03},\ b_{00}, b_{10}, b_{01}, \dots, b_{03})$ of its coefficients.

Let us consider the polynomials

$$\begin{aligned} C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 0,1,2,3, \\ D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 1,2,3. \end{aligned}$$

Let us apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $P(a,x,y)$ and $Q(a,x,y)$. We obtain $\tilde{P}(\tilde{a}(a,x_0,y_0),x',y') = P(a,x'+x_0,y'+y_0)$, $\tilde{Q}(\tilde{a}(a,x_0,y_0),x',y') = Q(a,x'+x_0,y'+y_0)$. We construct the following polynomials

$$\begin{aligned} &\Omega_i(a,x_0,y_0) \equiv \operatorname{Res}_{x'}\Big(C_i\big(\tilde{a}(a,x_0,y_0),x',y'\big), C_0\big(\tilde{a}(a,x_0,y_0),x',y'\big)\Big)/(y')^{i+1}, \\ &\qquad \Omega_i(a,x_0,y_0) \in \mathbb{R}[a,x_0,y_0],\ (i = 1,2,3) \end{aligned}$$

and we denote

$$\tilde{\mathcal{G}}_i(a,x,y) = \Omega_i(a,x_0,y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[a,x,y] \quad (i=1,2,3).$$

Notation 3. *Let $\mathcal{G}_i(a,X,Y,Z)$ $(i=1,2,3)$ be the homogenization of $\tilde{\mathcal{G}}_i(a,x,y)$, i.e.*

$$\begin{aligned}\mathcal{G}_1(a,X,Y,Z) &= Z^8\tilde{\mathcal{G}}_1(a,X/Z,Y/Z),\\ \mathcal{G}_2(a,X,Y,Z) &= Z^{10}\tilde{\mathcal{G}}_2(a,X/Z,Y/Z),\\ \mathcal{G}_3(a,X,Y,Z) &= Z^{12}\tilde{\mathcal{G}}_3(a,X/Z,Y/Z),\end{aligned}$$

and $\mathcal{H}(a,X,Y,Z) = \gcd\Big(\mathcal{G}_1(a,X,Y,Z),\ \mathcal{G}_2(a,X,Y,Z),\ \mathcal{G}_3(a,X,Y,Z)\Big)$ *in* $\mathbb{R}[a,X,Y,Z]$.

The geometrical meaning of these affine comitants is given by the two following lemmas (see [20]):

Lemma 1. *The straight line $\mathcal{L}(x,y) \equiv ux+vy+w=0$, $u,v,w\in\mathbb{C}$, $(u,v)\neq(0,0)$ is an invariant line for a cubic system* (3) *if and only if the polynomial $\mathcal{L}(x,y)$ is a common factor of the polynomials $\tilde{\mathcal{G}}_1(x,y)$, $\tilde{\mathcal{G}}_2(x,y)$ and $\tilde{\mathcal{G}}_3(x,y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{G}}_i(x,y) = (ux+vy+w)\widetilde{W}_i(x,y) \quad (i=1,2,3),$$

where $\widetilde{W}_i(x,y)\in\mathbb{C}[x,y]$.

Lemma 2. *Consider a cubic system* (3) *and let $\boldsymbol{a}\in\mathbb{R}^{20}$ be its 20-tuple of coefficients.*

1) If $\mathcal{L}(x,y)\equiv ux+vy+w=0$, $u,v,w\in\mathbb{C}$, $(u,v)\neq(0,0)$ is an invariant straight line of multiplicity k for a system (3) *then $[\mathcal{L}(x,y)]^k \mid \gcd(\tilde{\mathcal{G}}_1,\tilde{\mathcal{G}}_2,\tilde{\mathcal{G}}_3)$ in $\mathbb{C}[x,y]$, i.e. there exist $W_i(a,x,y)\in\mathbb{C}[x,y]$ $(i=1,2,3)$ such that*

$$\tilde{\mathcal{G}}_i(\boldsymbol{a},x,y) = (ux+vy+w)^k W_i(a,x,y), \quad i=1,2,3.$$

2) If the line $l_\infty : Z=0$ is of multiplicity $k>1$ then $Z^{k-1}\mid\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$, i.e. we have $Z^{k-1}\mid H(\boldsymbol{a},X,Y,Z)$.

Consider the differential operator $\mathcal{L} = x\cdot\mathbf{L}_2 - y\cdot\mathbf{L}_1$ constructed in [3] and acting on $\mathbb{R}[a,x,y]$, where

$$\begin{aligned}\mathbf{L}_1 =&3a_{00}\frac{\partial}{\partial a_{10}} + 2a_{10}\frac{\partial}{\partial a_{20}} + a_{01}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{02}\frac{\partial}{\partial a_{12}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{21}} + a_{20}\frac{\partial}{\partial a_{30}} +\\ &3b_{00}\frac{\partial}{\partial b_{10}} + 2b_{10}\frac{\partial}{\partial b_{20}} + b_{01}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{02}\frac{\partial}{\partial b_{12}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{21}} + b_{20}\frac{\partial}{\partial b_{30}},\\ \mathbf{L}_2 =&3a_{00}\frac{\partial}{\partial a_{01}} + 2a_{01}\frac{\partial}{\partial a_{02}} + a_{10}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{20}\frac{\partial}{\partial a_{21}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{12}} + a_{02}\frac{\partial}{\partial a_{03}} +\\ &3b_{00}\frac{\partial}{\partial b_{01}} + 2b_{01}\frac{\partial}{\partial b_{02}} + b_{10}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{20}\frac{\partial}{\partial b_{21}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{12}} + b_{02}\frac{\partial}{\partial b_{03}}.\end{aligned}$$

Using this operator and the affine invariant $\mu_0 = \text{Resultant}_x\big(p_3(a,x,y), q_3(a,x,y)\big)/y^9$ we construct the following polynomials

$$\mu_i(a,x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ \ i = 1,..,9,$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These polynomials are in fact comitants of systems (3) with respect to the group $GL(2,\mathbb{R})$ (see [3]). The polynomial $\mu_i(a,x,y)$, $i \in \{0,1,\ldots,9\}$ is homogeneous of degree 6 in the coefficients of systems (3) and homogeneous of degree i in the variables x and y. The geometrical meaning of these polynomials is revealed in the next lemma.

Lemma 3 (see [2, 3])**.** *Assume that a cubic system (S) with coefficients $\boldsymbol{a} \in \mathbb{R}^{20}$ belongs to the family* (3)*. Then:*

(i) The total multiplicity of all finite singularities of this system equals $9-k$ if and only if for every $i \in \{0,1,\ldots,k-1\}$ we have $\mu_i(\boldsymbol{a},x,y) = 0$ in the ring $\mathbb{R}[x,y]$ and $\mu_k(\boldsymbol{a},x,y) \neq 0$. In this case the factorization $\mu_k(\boldsymbol{a},x,y) = \prod_{i=1}^{k}(u_i x - v_i y) \neq 0$ over $\mathbb{C}$ indicates the coordinates $[v_i : u_i : 0]$ of singularities at infinity which in perturbations generate finite singularities of the system (S). Moreover the number of distinct factors in this factorization is less than or equal to four (the maximum number of infinite singularities of a cubic system) and the multiplicity of each one of the factors $u_i x - v_i y$ gives us the number of the finite singularities of the system (S) which have coalesced with the infinite singular point $[v_i : u_i : 0]$.

(ii) The point $M_0(0,0)$ is a singular point of multiplicity k ($1 \le k \le 9$) for the cubic system (S) if and only if for every i such that $0 \le i \le k-1$ we have $\mu_{9-i}(\boldsymbol{a},x,y) = 0$ in $\mathbb{R}[x,y]$ and $\mu_{9-k}(\boldsymbol{a},x,y) \neq 0$.

(iii) The system (S) is degenerate (i.e. $\gcd(p,q) \neq \text{const}$) if and only if $\mu_i(\boldsymbol{a},x,y) = 0$ in $\mathbb{R}[x,y]$ for every $i = 0,1,\ldots,9$.

In order to determine the degree of the common factor of the polynomials $\tilde{\mathcal{G}}_i(a,x,y)$ for $i = 1,2,3$, we shall use the notion of the k^{th} *subresultant* of two polynomials with respect to a given indeterminate (see for instance, [17],[23]).

The geometrical meaning of the subresultants is based on the following lemma.

Lemma 4 (see [17, 23])**.** *Polynomials $f(z)$ and $g(z)$ have precisely k roots in common (considering their multiplicities) if and only if the following conditions hold:*

$$R_z^{(0)}(f,g) = R_z^{(1)}(f,g) = R_z^{(2)}(f,g) = \cdots = R_z^{(k-1)}(f,g) = 0 \neq R_z^{(k)}(f,g).$$

For the polynomials in more than one variables it is easy to deduce from Lemma 4 the following result.

Lemma 5. *Two polynomials $\tilde{f}(x_1, x_2, ..., x_n)$ and $\tilde{g}(x_1, x_2, ..., x_n)$ have a common factor of degree k with respect to the variable x_j if and only if the following conditions are satisfied:*

$$R^{(0)}_{x_j}(\tilde{f}, \tilde{g}) = R^{(1)}_{x_j}(\tilde{f}, \tilde{g}) = R^{(2)}_{x_j}(\tilde{f}, \tilde{g}) = \cdots = R^{(k-1)}_{x_j}(\tilde{f}, \tilde{g}) = 0 \neq R^{(k)}_{x_j}(\tilde{f}, \tilde{g}),$$

where $R^{(i)}_{x_j}(\tilde{f}, \tilde{g}) = 0$ in $\mathbb{R}[x_1, \dots x_{j-1}, x_{j+1}, \dots, x_n]$.

In articles [5, 7–10, 20] several lemmas are proved concerning the number of triplets and/or couples of parallel invariant straight lines which a cubic system could have. Taking together these lemmas produce the following theorem.

Theorem 1. *If a cubic system* (3) *possesses a given number of triplets or/and couples of invariant parallel lines real or/and complex, then the following conditions are satisfied, respectively:*

(i)	*two triplets*	$\Rightarrow$	$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$;
(ii)	*one triplet and one couple*	$\Rightarrow$	$\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$;
(iii)	*one triplet*	$\Rightarrow$	$\mathcal{V}_4 = \mathcal{U}_2 = 0$;
(iv)	*3 couples*	$\Rightarrow$	$\mathcal{V}_3 = 0$;
(v)	*2 couples*	$\Rightarrow$	$\mathcal{V}_5 = 0$.

Remark 2. The above conditions depend only on the coefficients of the cubic homogeneous parts of the systems (3).

As the mentioned above systems have invariant lines of total multiplicity 7 (where the line at infinity is considered), they could only have one of the following four possible types of configurations of invariant lines:

$$(i)\ \mathfrak{T} = (3, 3); \quad (ii)\ \mathfrak{T} = (3, 1, 1, 1); \quad (iii)\ \mathfrak{T} = (2, 2, 2); \quad (iv)\ \mathfrak{T} = (2, 2, 1, 1).$$

In this article we consider only systems possessing configurations of the type $(3, 3)$ and in this case, according to the above theorem, the condition $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ is necessary. The polynomials $\mathcal{V}_1,\ \mathcal{V}_2,\ \mathcal{U}_1$ are constructed in the following way:

$$\mathcal{V}_1(a, x, y) = S_{23} + 2D_3^2, \quad \mathcal{V}_2(a, x, y) = S_{26}, \quad \mathcal{U}_1(a) = T_{31} - 4T_{37},$$

where

$$S_{14} = (C_2, C_2)^{(2)}, \quad S_{15} = (C_2, D_2)^{(1)}, \quad S_{16} = (C_2, C_3)^{(1)}, \quad S_{17} = (C_2, C_3)^{(2)},$$

$$S_{19} = (C_2, D_3)^{(1)}, \quad S_{20} = (C_2, D_3)^{(2)}, \quad S_{21} = (D_2, C_3)^{(1)}, \quad S_{22} = (D_2, D_3)^{(1)},$$

$$S_{23} = (C_3, C_3)^{(2)}, \quad S_{26} = (C_3, D_3)^{(2)}, \quad D_3 = (S_{23}, S_{23})^{(2)} - 6C_3\,(C_3, S_{23})^{(4)},$$

$$T_3 = S_{21}/18, \quad T_4 = S_{25}/6, \quad T_{31} = (T_8, C_3)^{(2)}/24, \quad T_{37} = (T_9, C_3)^{(2)}/12,$$

$$T_8 = [5D_2(D_3^2 + 27T_3 - 18T_4) + 20D_3S_{19} + 12\,(S_{16}, D_3)^{(1)} - 8D_3S_{17}]/5/2^5/3^3,$$

$$T_9 = [9D_1(9T_3 - 18T_4 - D_3^2) + 2D_2(D_2D_3 - 3S_{17} - S_{19} - 9S_{21}) + 18\,(S_{15}, C_3)^{(1)} -$$

$$-6C_2(2S_{20}-3S_{22})+18C_1S_{26}+2D_3S_{14}]/2^4/3^3.$$

Here $S_m,\ m=1,\ldots,27$ which are constructed by C_i and D_i are the *comitants* of second degree with respect to the coefficients of the initial systems, whereas $T_n,\ n=1,\ldots,142$ are T-*comitants* of cubic systems (3) (see [29] for the definition of a T-*comitant*). We note that these invariant polynomials are the elements of the polynomial basis of $T-$*comitants* up to degree six constructed by Iu.Calin.

We rewrite the systems (3) using different coefficients:

$$\begin{aligned}\dot{x}&=a+cx+dy+gx^2+2hxy+ky^2+px^3+3qx^2y+3rxy^2+sy^3\equiv P(x,y),\\ \dot{y}&=b+ex+fy+lx^2+2mxy+ny^2+tx^3+3ux^2y+3vxy^2+wy^3\equiv Q(x,y).\end{aligned}\tag{4}$$

Let $L(x,y)=Ux+Vy+W=0$ be an invariant straight line of this family of cubic systems. Then, we have

$$UP(x,y)+VQ(x,y)=(Ux+Vy+W)(Ax^2+2Bxy+Cy^2+Dx+Ey+F),$$

and this identity provides the following 10 relations:

$$\begin{array}{ll}Eq_1=(p-A)U+tV=0, & Eq_6=(2h-E)U+(2m-D)V-2BW=0,\\ Eq_2=(3q-2B)U+(3u-A)V=0, & Eq_7=kU+(n-E)V-CW=0,\\ Eq_3=(3r-C)U+(3v-2B)V=0, & Eq_8=(c-F)U+eV-DW=0\\ Eq_4=(s-C)U+Vw=0, & Eq_9=dU+(f-F)V-EW=0,\\ Eq_5=(g-D)U+lV-AW=0, & Eq_{10}=aU+bV-FW=0.\end{array}\tag{5}$$

It is well known that the infinite singularities (real or complex) of systems (4) are determined by the linear factors of the polynomial

$$C_3=yp_3(x,y)-xq_3(x,y).$$

Remark 3. Let $C_3=\prod_{i=1}^{4}(\alpha_ix+\beta_iy),\ i=1,2,3,4.$ Since infinite singularities of systems (4) are located at the "ends" of the lines $\alpha_ix+\beta_iy=0$, the invariant affine lines must be $Ux+Vy+W=0$, where $U=\alpha_i$ and $V=\beta_i$. In this case, considering W as a fixed parameter, six equations among (5) become linear with respect to the parameters $\{A,B,C,D,E,F\}$ (with the corresponding non-zero determinant) and we can determine their values, which annihilate some of the equations (5). So in what follows we will examine only the non-zero equations containing the last parameter W.

For the proof of the Main Theorem it is useful to consider the following homogeneous cubic systems associated to systems (4):

$$x'=P_3(x,y),\quad y'=Q_3(x,y).\tag{6}$$

Clearly in the case of two real and two complex distinct infinite singularities the polynomial $C_3(x,y)$ has two real and two complex distinct linear factors. The following remark concerning the associated homogeneous cubic systems (6) is useful.

Remark 4. Assume that a cubic system (4) possesses invariant lines of total multiplicity three (respectively two) in a real direction. Then the corresponding associated homogeneous cubic system (4) has one invariant line of total multiplicity at least three (respectively two) in the same direction.

Indeed, if a system (4) possesses a triplet of parallel invariant lines (distinct or coinciding) in a real direction then via an affine transformation this system could be brought to the form

$$\dot{x} = x[(x+b)^2 + u], \quad \dot{y} = Q(a, x, y).$$

It is clear that if $u < 0$ (respectively $u > 0$) then we have three real (respectively one real and two complex) all distinct invariant lines. In the case $u = 0$ we either have one simple and one double invariant lines if $b \neq 0$, or one triple invariant line if $b = 0$. It remains to observe that in all four cases the corresponding associated homogeneous cubic systems possess the invariant line $x = 0$ of total multiplicity at least three. The case of a couple of parallel invariant lines can be examined similarly.

According to [20] (see also [25]) we have the following result.

Lemma 6. *Assume that a cubic system* (4) *has 2 real and 2 complex all distinct infinite singularities. Then its associated homogeneous cubic systems* (6) *could be brought via a linear transformation to the canonical form*

$$(S_I) \quad \begin{cases} x' = (u+1)x^3 + (s+v)x^2y + rxy^2, & C_3 = x(sx+y)(x^2+y^2), \\ y' = -sx^3 + ux^2y + vxy^2 + (r-1)y^3. \end{cases} \tag{7}$$

3 Systems with the configuration of the type $(3,3)$. The proof of the Main Theorem

Considering Lemma 6 systems possessing two real and two complex infinite singularities via a linear transformation could be brought to the family of systems

$$\begin{aligned} \dot{x} =& a + cx + dy + gx^2 + 2hxy + ky^2 + (u+1)x^3 + (s+v)x^2y + rxy^2, \\ \dot{y} =& b + ex + fy + lx^2 + 2mxy + ny^2 - sx^3 + ux^2y + vxy^2 + (r-1)y^3 \end{aligned} \tag{8}$$

with $C_3 = x(sx+y)(x^2+y^2)$.

Since we have two triplets of parallel invariant lines, according to Theorem 1 the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ are necessary for systems (8). In [20, Section 6.1] it was proved that in this case via a linear transformation and time rescaling the homogeneous cubic systems associated to systems (8) could be brought either to the system

$$\dot{x} = x^3, \quad \dot{y} = -y^3, \tag{9}$$

or to the system

$$\dot{x} = x^3 - 3xy^2, \quad \dot{y} = 3x^2y - y^3. \tag{10}$$

So we examine each one of these two possibilities.

3.1 Systems with the associated homogeneous cubic system (9)

Then applying a translation we may assume $g = n = 0$ in the quadratic parts of systems (8) with the cubic homogeneities $(x^3,\ -y^3)$. In such a way we get the family of systems

$$\begin{aligned} \dot{x} &= a + cx + dy + 2hxy + ky^2 + x^3, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy - y^3, \end{aligned} \tag{11}$$

for which we have $C_3(x, y) = xy(x^2 + y^2)$.

In order to find out the directions of two triplets, according to Remark 4, we determine the multiplicity of the invariant lines of system (9). For this system we calculate (see the definition of the polynomial $H(X, Y, Z)$ on page 86, Notation 3):

$$H(X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 3X^3Y^3(X^2 + Y^2). \tag{12}$$

So system (9) possesses two triple invariant lines $x = 0$ and $y = 0$ and by Remark 4, systems (11) could have triplets of parallel invariant lines only in these two directions.

(i) The direction $x = 0$. Considering (5) and Remark 3 we obtain

$$Eq_7 = k, \quad Eq_9 = d - 2hW, \quad Eq_{10} = a - cW - W^3$$

and obviously we can have a triplet of parallel invariant lines (which could coincide) in the direction $x = 0$ if and only if $k = d = h = 0$.

(ii) The direction $y = 0$. In this case we have

$$Eq_5 = l, \quad Eq_8 = e - 2mW, \quad Eq_{10} = b - fW - W^3$$

and again we conclude that for the existence of a triplet of parallel invariant lines for systems (11) the conditions $e = l = m = 0$ have to be satisfied.

Thus we arrive at the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + fy - y^3 \tag{13}$$

which possess the invariant lines defined by the equations

$$x^3 + cx + a = 0, \quad -y^3 + fy + b = 0.$$

We observe that the number of distinct invariant lines and their types (real and/or complex) depend on the discriminants of the cubic polynomials $x^3 + cx + a$ and $-y^3 + fy + b$, i.e.

$$\xi_1 = -(27a^2 + 4c^3), \quad \xi_2 = -(27b^2 - 4f^3),$$

respectively. Moreover, we observe that the polynomial $x^3 + cx + a$ (respectively $-y^3 + fy + b$) has a triple root if and only if $\nu_1 = a^2 + c^2 = 0$ (respectively $\nu_2 = b^2 + f^2 = 0$).

Remark 5. We remark that for systems (13) we could not have simultaneously $\nu_1 = \nu_2 = 0$, otherwise we get the homogeneous cubic system $\dot{x} = x^3$, $\dot{y} = -y^3$ which possesses invariant lines of total multiplicity nine (see the value (12) of $H(X,Y,Z)$ for homogeneous system (9)).

In what follows we examine the possibilities provided by the discriminants ξ_1 and ξ_2.

1) *The case* $\xi_1\xi_2 > 0$ *and* $\xi_1 + \xi_2 > 0$. Then each one of the mentioned cubic polynomials factorizes in three distinct real factors, i.e. we get the systems

$$\dot{x} = (x-\alpha_1)(x-\beta_1)(x-\delta_1), \quad \dot{y} = (y-\alpha_2)(y-\beta_2)(\delta_2-y), \tag{14}$$

where $\alpha_i, \beta_i, \delta_i \in \mathbb{R}$, $i = 1, 2$. As all the lines are distinct then via the transformation

$$(x,y,t) \mapsto \big(\alpha_1 - (\alpha_1-\beta_1)x,\ \alpha_2 - (\alpha_1-\beta_1)y,\ t/(\alpha_1-\beta_1)^2\big) \tag{15}$$

we arrive at the following 3-parameter family of systems

$$\dot{x} = x(x-1)(x-a), \quad \dot{y} = y(y-b)(c-y), \quad a(a+1)bc(b-c) \neq 0, \tag{16}$$

where $a = \dfrac{\alpha_1-\delta_1}{\alpha_1-\beta_1}$, $b = \dfrac{\alpha_2-\beta_2}{\alpha_1-\beta_1}$ and $c = \dfrac{\alpha_2-\delta_2}{\alpha_1-\beta_1}$. These systems possess 9 finite real singularities which are located at the intersections of these two triplets of invariant straight lines. As a result we get *Config. 7.1a*.

2) *The case* $\xi_1\xi_2 > 0$ *and* $\xi_1 + \xi_2 < 0$. Then in each one of the directions $x = 0$ and $y = 0$ systems (13) possess one real and two complex invariant lines. After the translation of the origin of coordinates at the intersections of the real invariant lines we arrive at the systems

$$\dot{x} = x(x^2 + 2\beta_1 x + \delta_1), \quad \dot{y} = -y(y^2 + 2\beta_2 y + \delta_2),$$

where $\beta_1^2 - \delta_1 < 0$ and $\beta_2^2 - \delta_2 < 0$. So we can set $\beta_1^2 - \delta_1 = -u^2 \neq 0$ and $\beta_2^2 - \delta_2 = -v^2 \neq 0$ respectively, the above systems become

$$\dot{x} = x\big[u^2 + (x+\beta_1)^2\big], \quad \dot{y} = -y\big[v^2 + (y+\beta_2)^2\big].$$

Since $u \neq 0$ then we may assume $u = 1$ due to the rescaling $(x,y,t) \mapsto (ux,\ uy,\ t/u^2)$ and we get the following 3-parameter family of systems

$$\dot{x} = x\big[(x+a)^2 + 1\big], \quad \dot{y} = -y\big[(y+b)^2 + c^2\big], \quad c \neq 0. \tag{17}$$

These systems possess 1 real and 8 complex finite singularities which are located at the intersections of these two triplets of invariant lines (real and complex).

As a result, considering Notation 2 we get *Config. 7.2a* if $ab \neq 0$, *Config. 7.3a* if $ab = 0$ and $a + b \neq 0$ and *Config. 7.4a* if $a = 0 = b$.

3) *The case* $\xi_1\xi_2 < 0$. Without loss of generality we may assume $\xi_1 > 0$ and $\xi_2 < 0$ due to the change $(x, y, t, a, b, c, f) \mapsto (-y, -x, -t, b, a, -f, -c)$ which conserves systems (13). Then we have the following factorization of the right hand sides of these systems

$$\dot{x} = (x - \alpha_1)(x - \beta_1)(x - \delta_1), \quad \dot{y} = (\alpha_2 - y)(y^2 + 2\beta_2 y + \delta_2), \tag{18}$$

where $\alpha_i, \beta_i, \delta_i \in \mathbb{R}$, $i = 1, 2$ and $\beta_2^2 - \delta_2 < 0$. So we can set $\beta_2^2 - \delta_2 = -u^2 \neq 0$ and then applying the transformation (15) we get the following 3-parameter family of systems

$$\dot{x} = x(x - 1)(x - a), \quad \dot{y} = -y\big[(y + b)^2 + c^2\big], \quad a(a - 1)c \neq 0. \tag{19}$$

These systems possess 3 real and 6 complex finite singularities which are located at the intersections of these two triplets of invariant lines (real and complex).

So considering Notation 2 we get *Config. 7.5a* if $b \neq 0$ and *Config. 7.6a* if $b = 0$.

4) *The case* $\xi_1\xi_2 = 0$, $\xi_1 + \xi_2 > 0$, $\nu_1\nu_2 \neq 0$. As it was mentioned earlier due to the change $(x, y, t, a, b, c, f) \mapsto (-y, -x, -t, b, a, -f, -c)$ (which conserves systems (13)) we may assume $\xi_1 = 0$ and $\xi_2 > 0$ (in this case the condition $\nu_1 \neq 0$ holds). Following the same arguments as before, systems (13) can be written in the form (14) with $\beta_1 = \alpha_1 \neq \delta_1$. Then applying the transformation (15) in which we substitute β_1 by $\alpha_1 \neq \delta_1$ we arrive at the following 2-parameter family of systems

$$\dot{x} = x^2(x - 1), \quad \dot{y} = y(y - b)(y - c), \quad bc(b - c) \neq 0. \tag{20}$$

These systems possess three double real singularities (located at the intersections of the double invariant line $x = 0$ with three simple ones) and 3 simple real singularities, located at the intersections of the simple invariant line $x = 1$ with the triplet in the direction $y = 0$. As a result we obtain *Config. 7.7a*.

5) *The case* $\xi_1\xi_2 = 0$, $\xi_1 + \xi_2 < 0$, $\nu_1\nu_2 \neq 0$. We may assume again $\xi_1 = 0$, $\xi_2 < 0$ and $\nu_1 \neq 0$. In this case we consider systems (18) with $\beta_1 = \alpha_1 \neq \delta_1$ and following the same steps and applying the corresponding similar transformation we get the following 2-parameter family of systems

$$\dot{x} = x^2(x - 1), \quad \dot{y} = y\big[(y + b)^2 + c^2\big], \quad c \neq 0. \tag{21}$$

Clearly these systems possess three double singularities (one real and two complex) on the double line $x = 0$ and three simple singularities (one real and two complex) located on the simple invariant line $x = 1$.

So considering Notation 2 we get the configuration of invariant lines given by *Config. 7.8a* if $b \neq 0$ and *Config. 7.9a* if $b = 0$.

6) *The case* $\xi_1\xi_2 = 0$, $\xi_1 + \xi_2 > 0$, $\nu_1\nu_2 = 0$. As it was mentioned above we may consider $\xi_1 = 0$ which implies $\xi_2 > 0$. Then $\nu_2 \neq 0$ and hence we have $\nu_1 = 0$. In this case we have a triple line in the direction $x = 0$ and after a translation we get the systems

$$\dot{x} = x^3, \quad \dot{y} = y(y-b)(y-c)$$

with $bc \neq 0$. Then applying the rescaling $(x, y, t) \mapsto (cx,\, cy,\, t/c^2)$ we force $c = 1$ and we arrive at the following 1-parameter family of systems

$$\dot{x} = x^3, \quad \dot{y} = y(y-1)(y-b), \quad b(b-1) \neq 0. \tag{22}$$

It is easy to determine that these systems possess three real singularities, each one of multiplicity 3, located on the triple invariant line $x = 0$. This leads to the configuration *Config. 7.10*.

7) *The case* $\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 < 0,\ \nu_1\nu_2 = 0$. So similarly as before, we may consider $\xi_1 = \nu_1 = 0$ and $\xi_2 < 0$. In this case we have a triple line in the direction $x = 0$ and after a translation setting some new parameters (see the second equation of systems (21)) we get the systems

$$\dot{x} = x^3, \quad \dot{y} = y\big[c^2 + (y+b)^2\big]$$

with $c \neq 0$. Then applying the rescaling $(x, y, t) \mapsto (cx,\, cy,\, t/c^2)$ we arrive at the following 1-parameter family of systems

$$\dot{x} = x^3, \quad \dot{y} = y\big[1 + (y+b)^2\big]. \tag{23}$$

These systems possess three triple singularities (one real and two complex) located on the triple invariant line $x = 0$. Considering Notation 2 this leads to the configuration *Config. 7.11a* if $b \neq 0$ and *Config. 7.12a* if $b = 0$.

8) *The case* $\xi_1 = \xi_2 = 0,\ \nu_1\nu_2 \neq 0$. Then we have two double real invariant lines (one in the direction $x = 0$ and the second in the direction $y = 0$). Due to $\nu_1\nu_2 \neq 0$ none of them could be triple. So after a translation which moves the origin of coordinates at the intersection of the double lines we arrive at the systems

$$\dot{x} = x^2(x-a), \quad \dot{y} = y^2(y-b),$$

where $ab \neq 0$. Then applying the rescaling $(x, y, t) \mapsto (ax,\, ay,\, t/a^2)$ we get the following 1-parameter family of systems

$$\dot{x} = x^2(x-1), \quad \dot{y} = y^2(y-b), \quad b \neq 0. \tag{24}$$

It is not difficult to determine that these systems possess four distinct real finite singularities: one of multiplicity four (located at the intersection of the double lines) two double and one simple. As a result we obtain the configuration *Config. 7.13a*.

9) *The case* $\xi_1 = \xi_2 = 0,\ \nu_1\nu_2 = 0$. According to Remark 5 the condition $\nu_1^2 + \nu_2^2 \neq 0$ is necessary and by the same reasons as above we may assume $\nu_1 = 0$ and $\nu_2 \neq 0$. Therefore we have a triple invariant line in the direction $x = 0$ and a double one in the direction $y = 0$. As a result via a translation we get the systems

$$\dot{x} = x^3, \quad \dot{y} = y^2(y-a),$$

with $a \neq 0$. Then we may assume $a = 1$ due to the rescaling $(x, y, t) \mapsto (ax,\ ay,\ t/a^2)$ and we arrive at the system

$$\dot{x} = x^3, \quad \dot{y} = y^2(y-1). \tag{25}$$

We observe that this system has only two distinct finite singularities: one of the multiplicity six and one triple both located on the invariant line $x = 0$. So we get *Config. 7.14a*.

Thus we have proved the following lemma.

Lemma 7. *Systems* (13) *possess one of the configurations Config. 7.1a – 7.14a if and only if the corresponding conditions are satisfied. Moreover in each one of the cases these systems could be brought via affine transformations and time rescaling to the indicated canonical forms, respectively:*

Config. 7.1a $\Leftrightarrow$	$\xi_1\xi_2 > 0,\ \xi_1+\xi_2 > 0$	$\Rightarrow$ (16);
Config. 7.2a $\Leftrightarrow$	$\xi_1\xi_2 > 0,\ \xi_1+\xi_2 < 0,\ ab \neq 0$	$\Rightarrow$ (17) *with* $ab \neq 0$;
Config. 7.3a $\Leftrightarrow$	$\xi_1\xi_2 > 0,\ \xi_1+\xi_2 < 0,\ ab = 0,\ a+b \neq 0$	$\Rightarrow$ (17) *with* $ab = 0$ *and* $a+b\neq 0$;
Config. 7.4a $\Leftrightarrow$	$\xi_1\xi_2 > 0,\ \xi_1+\xi_2 < 0,\ a = b = 0$	$\Rightarrow$ (17) *with* $a=b=0$;
Config. 7.5a $\Leftrightarrow$	$\xi_1\xi_2 < 0,\ b \neq 0$	$\Rightarrow$ (19) *with* $b \neq 0$;
Config. 7.6a $\Leftrightarrow$	$\xi_1\xi_2 < 0,\ b = 0$	$\Rightarrow$ (19) *with* $b = 0$;
Config. 7.7a $\Leftrightarrow$	$\xi_1\xi_2 = 0,\ \xi_1+\xi_2 > 0,\ \nu_1\nu_2 \neq 0$	$\Rightarrow$ (20);
Config. 7.8a $\Leftrightarrow$	$\xi_1\xi_2 = 0,\ \xi_1+\xi_2 < 0,\ \nu_1\nu_2 \neq 0,\ b \neq 0$	$\Rightarrow$ (21) *with* $b \neq 0$;
Config. 7.9a $\Leftrightarrow$	$\xi_1\xi_2 = 0,\ \xi_1+\xi_2 < 0,\ \nu_1\nu_2 \neq 0,\ b = 0$	$\Rightarrow$ (21) *with* $b = 0$;
Config. 7.10a $\Leftrightarrow$	$\xi_1\xi_2 = 0,\ \xi_1+\xi_2 > 0,\ \nu_1\nu_2 = 0$	$\Rightarrow$ (22);
Config. 7.11a $\Leftrightarrow$	$\xi_1\xi_2 = 0,\ \xi_1+\xi_2 < 0,\ \nu_1\nu_2 = 0,\ b \neq 0$	$\Rightarrow$ (23) *with* $b \neq 0$;
Config. 7.12a $\Leftrightarrow$	$\xi_1\xi_2 = 0,\ \xi_1+\xi_2 < 0,\ \nu_1\nu_2 = 0,\ b = 0$	$\Rightarrow$ (23) *with* $b = 0$;
Config. 7.13a $\Leftrightarrow$	$\xi_1\xi_2 = 0,\ \xi_1+\xi_2 = 0,\ \nu_1\nu_2 \neq 0$	$\Rightarrow$ (24);
Config. 7.14a $\Leftrightarrow$	$\xi_1\xi_2 = 0,\ \xi_1+\xi_2 = 0,\ \nu_1\nu_2 = 0$	$\Rightarrow$ (25).

3.1.1 Systems with the associated homogeneous cubic system (10)

Then applying a translation we may assume $g = n = 0$ in the quadratic parts of systems (8) with the cubic homogeneities $x^3 - 3xy^2,\ 3x^2y - y^3$. In such a way we get the family of systems

$$\begin{aligned}\dot{x} &= a + cx + dy + 2hxy + ky^2 + x^3 - 3xy^2,\\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + 3x^2y - y^3,\end{aligned} \tag{26}$$

for which we have $C_3(x,y) = -2xy(x^2+y^2)$.

In order to find out the directions of two triplets, according to Remark 4, we determine the multiplicity of the invariant lines of system (10). For this system we calculate (see the definition of the polynomial $H(X,Y,Z)$, Notation 3):

$$H(X,Y,Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 6XY(X^2+Y^2)^3. \tag{27}$$

So system (10) possesses two triple invariant lines $y = ix$ and $y = -ix$ and by Remark 4, systems (26) could have triplets of parallel invariant lines only in these two complex directions. Therefore we have to examine only these directions and since the systems are real it is sufficient to consider only one direction.

The direction $x + iy = 0$. Then $U = 1$, $V = i$ and considering (5) and Remark 3 we obtain

$$Eq_7 = k + 2m - i(l + 2h), \quad Eq_9 = d + e + i(f - c) - 2(l + h + im)W,$$
$$Eq_{10} = a + ib - (c + ie)W + ilW^2 - W^3.$$

So in order to have a triplet of parallel invariant lines in this direction we must force the equations $Eq_7 = 0$ and $Eq_9 = 0$ to vanish identically. Since all the parameters are real we obtain: $k = -2m,\ l = -2h,\ d = -e,\ f = c,\ l = -h,\ m = 0$ which implies $k = l = h = m = 0$, $d = -e$ and $f = c$. In this case we arrive at the family of systems

$$\begin{aligned} \dot{x} &= a + cx - ey + x^3 - 3xy^2, \\ \dot{y} &= b + ex + cy + 3x^2y - y^3, \end{aligned} \tag{28}$$

which possess the following two triplets of complex parallel invariant lines:

$$(x + iy)^3 + (c + ie)(x + iy) + a + ib = 0, \quad (x - iy)^3 + (c - ie)(x - iy) + a - ib = 0.$$

So setting $z = x + iy$ (then $\bar{z} = x - iy$) we get two equations

$$z^3 + (c + ie)z + a + ib = 0, \quad \bar{z}^3 + (c - ie)\bar{z} + a - ib = 0$$

with the corresponding discriminants

$$27b^2 - 27a^2 - 4c^3 + 12ce^2 \pm i(54ab + 12c^2e - 4e^3) \equiv \phi \pm i\psi.$$

It is clear that we could have a multiple complex invariant line if and only if $\phi = \psi = 0$. Moreover this line could be of multiplicity 3 if and only if $a = b = c = e = 0$ however in this case we get homogeneous system (10) which possesses invariant lines of total multiplicity 9 (see (27)).

On the other hand the equations $\phi = \psi = 0$ have real solutions as shows us the example:

$$a = 0,\ b = -\frac{4}{3}\sqrt{\frac{2}{3}},\ c = -1, e = -\sqrt{3}.$$

As a result we arrive at the *Config. 7.15a* if $\phi^2 + \psi^2 \neq 0$ and at the *Config. 7.16a* if $\phi = \psi = 0$ (see the above example). It remains to note that in the canonical systems (28) we may assume $b \in \{0, 1\}$ due to a rescaling in the case $b \neq 0$.

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