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Working with professor Nicolae Vulpe

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This year Professor Nicolae Vulpe turned seventy and on this occasion I start by wishing him good health, much happiness together with his large, beautiful family and a long and fruitful life ahead. I met Nicolae for the first time exactly twenty years ago when we invited him to come to Montreal so that we could learn from him about the algebraic invariant theory of polynomial differential equations, founded by C.S. Sibirschi.

For several years prior to our meeting I felt the need of using the algebraic invariant theory in my work. At the time there were numerous publications on topologically classifying special families of quadratic or cubic vector fields. These were done with respect to chosen normal forms, convenient for studying the specific classification problems. Normal forms are important because they allow us to reduce the number of parameters on which families depend. Indeed, suppose we want to study the family of quadratic differential systems. This class depends on 12 parameters, the coefficients of the systems, 6 for each one of the quadratic equations. But we have here a group action, namely the group of affine transformations and time rescaling. Due to this the family only depends on 5 parameters. A normal form can be good for studying a problem while for another problem it can be inconvenient and hence another normal form more suitable needs to be found. In the study of the whole family of quadratic differential systems a multitude of such normal forms are necessary. But how do we glue the results obtained in one normal to those obtained by using a different normal form so as to see the full picture covering the two normal forms, in case they have a nonempty intersection? This is one direction where the algebraic invariant theory of differential equations can be very helpful. So I was very keen to learn about the algebraic invariant theory and using it in my own work.

In particular in my work together with my former student Janos Pal, we studied the geometry in the neighborhood of infinity of quadratic differential systems with a weak focus. This work which was published in 2001 ([11]), was done with respect to a convenient normal form and the results were stated in terms of the coefficients of this normal form. Our work required meticulous calculations and I was thinking that if some day we would need to use these results for work where a different normal form would be necessary, our results would not be readily transferrable. I knew about the work of Nicolae with his former student Igor Nikolaev (cf.[9]), published in 1997, on the topological classification of quadratic systems at infinity. This work was done in terms of algebraic invariants and hence independent of the normal forms in which the systems may be presented. This is clearly a great advantage. On the other hand this work was purely algebraic not using any geometrical concepts. It seemed to me that if we combined our geometrical concepts with their algebraic invariants we could give geometric meaning to some of their invariant polynomials, perhaps even simplify them and we would gain much by making our work independent of the normal forms in which the systems are presented.

I thus began to think about inviting Nicolae to come to Montreal. Around the same time it came to our attention that we could apply for an AUPELF grant and so together with my colleague and former student Christiane Rousseau we made an application for an AUPELF grant and we were successful.

Nicolae came to Montreal in 1999 supported by this grant. I proposed to work with him on the project of combining the geometry Pal and I used in our work, with the algebraic invariants he used in his work with Nikolaiev. Nicolae agreed to work on this project but at the same time he proposed a second project, namely the study of quadratic differential systems possessing invariant straight lines. Work on these two projects resulted in several publications, the last one of them [**?**] on Lotka–Volterra systems, published in 2012.

In 1999 Dumitru Cozma and Iurie Calin attended the mathematical meeting in Dynamical Systems in Luminy, France, also supported by the AUPELF grant. In June 2000 I organized together with my colleague Luc Belair from the Université du Québec à Montréal the workshop "Asymptotic Series, Differential Algebra and Finiteness Problems in Nonlinear Dynamical Systems" held at the Centre de Recherches Mathématiques (CRM) of the Université de Montréal. We invited the professors Dumitru Cozma, Mihail Popa, Alexandru Şubă and Nicolae Vulpe and they came to attend this workshop.

My collaborator, professor Jaume Llibre of the Universitàt Autonoma de Barcelona, also attended this meeting where he met Nicolae. He asked me about my work with Nicolae and I told him about our work on the infinite singularities in quadratic systems and about the work on quadratic systems with invariant lines of at least five total multiplicity. He later invited Nicolae to Barcelona and worked with him on cubic systems with invariant straight lines of total mutiplicity nine [20] and they also worked together with Artés on the topological classification of finite singularities in quadratic systems [3].

My collaboration with Nicolae on the global study of singularities at infinity, which resulted in an article published in 2005 ([13]), was to be the starting point of a period of work extending over several years, culminating with the proof of the geometric classification theorem of the global configurations of finite and infinite singularities in quadratic systems. This theorem forms the content of several articles published over several years, the last results appearing for the first time in a book [5], all of these contributions co-authored by Joan Carles Artés, Jaume Llibre, Nicolae and myself. This book is going to be published by the publishing company Springer Nature Switzerland and will appear in the book series Birkhäuser.

The geometric classification of the global configurations of singularities is deeper than the topological classification including, apart from topological information, algebraic information such as for example information about the multiplicity of the singularities, finite or infinite, about the order of weak singularities, types of nodes, information about tangency of solution curves to characteristic directions. This theorem yielded 1765 distinct global geometric such configurations of singularities for this class. The geometric classification is done in the 12-parameter space of the coefficients of the systems and it gives an algorithm for deciding for any quadratic differential systems, independent of the normal form in which the system is presented, what is its global geometric configuration of singularities, finite and infinite. This algorithm was implemented in a computer program. The bifurcation diagram of all these numerous configurations was also obtained and it is expressed in terms of invariant polynomials. This work proves that the algebraic invariant theory of polynomial differential systems has a major role to play in the global studies of families of polynomial vector fields.

Our result on the geometric classification of configurations of singularities in quadratic systems together with results of Artés, Llibre on structurally stable quadratic systems and of Artés, Llibre and Alex Carducci Rezende on structurally unstable quadratic systems of codimension one, open the road for obtaining the complete topological classification of the whole quadratic class, modulo limit cycles. At the moment there is a lot of ongoing work on this topic.

Based on the algebraic invariant theory of polynomial differential systems, intrinsic classifications according to geometric properties of the systems of whole classes of polynomial vector fields were recently obtained. For example the class of all quadratic systems possessing an invariant hyperbola (cf.[10]) and the class of all quadratic systems possessing an invariant ellipse (to be submitted shortly) were recently classified in terms of the configurations of these invariant conics and invariant straight lines whenever they occur in the systems. Bifurcation diagrams of these configurations were done in terms of invariant polynomials. These intrinsic classification results also led to algorithms. Some results on limit cycles which are ellipses were also obtained.

In work by Bujac, Llibre and Vulpe other algorithms were also constructed, for example for determining whether or not a given cubic system possesses a configuration of invariant straight lines of total multiplicity greater than or equal to eight and if the answer is positive then what kind of configuration of invariant lines the system possesses. In the case of the presence of invariant lines of maximum total multiplicity nine, the systems are integrable and the integrals as well as the phase portraits are given ([6]).

The study of integrable systems is important for the study of their neighboring non-integrable systems, using methods of bifurcation theory. The cases of integrable systems are rare but as Arnold said in [1, p. 405] "...these integrable cases allow us to collect a large amount of information about the motion in more important systems...". In 1878 Darboux initiated the theory of planar polynomial differential systems in an article on integrability of such systems in terms of their particular, algebraic solutions ([8]). The Darboux' integrability theory provides a link between the number of algebraic particular solutions of the systems (or irreducible invariant algebraic curves over the complex numbers), and the integrability of the polynomial systems. The main theorem of Darboux provides a lower bound $n(n+1)/2+1$ of the number of algebraic solution curves for a differential system of degree n to have a first integral computed using the polynomials defining these curves (Darboux integrability). This is not the best lower bound for having Darboux integrability and there has been a lot work on improving Darboux' result.

A particular case of Darboux integrability is the notion of algebraic integrability, by this meaning the existence of a rational first integral of the system. The problem of Poincaré on algebraic integrability stated in 1891, which is still open, asks to recognize when a planar polynomial differential system possesses a rational first integral. Work for constructing a data base of low degree differential systems in order to get new insight into this problem or to solve the problem for specific classes of systems is in progress. Nicolae's work together with Artés and Llibre in [4],[2] could be considered to be part of this program. Work on Darboux integrability and on Poincaré's problem lie at a crossroad of ODE theory with algebraic geometry and differential algebra.

The project proposed to me by Nicolae when we first met in Montreal in 1999 was to study quadratic differential systems possessing invariant straight lines. We proved that all the quadratic systems possessing invariant lines of total multiplicity 4, including the line at infinity, are integrable (cf.[15],[16]) possessing Liouvillian first integrals and those for which the lines have total multiplicity at least 5 are Darboux integrable (cf.[12],[14]). Another work was on the topological classification of all quadratic differential systems with the line at infinity filled up with singularities (cf.[17]). We also proved that all these systems are Darboux integrable.

The next step in this program was to consider the family of quadratic systems with total multiplicity at least three. This family includes the class of Lotka–Volterra systems, very important for its applications. Systems in this class are defined as having two distinct real affine invariant straight lines, intersecting at a finite point, as well as the invariant line at infinity. We worked on topologically classifying this family while in Princeton at the Institute for Advanced Studies (IAS) where Nicolae and I were invited for a period of three weeks. There were four papers all claiming to give the full topological classification of this class but these classifications turned out to be incomplete and with some erroneous phase portraits. Furthermore the work was done via several normal forms with no global tools to connect them. Thus some phase portraits appear repeated in several of these normal forms and it was very difficult to obtain a global picture. Of course no global bifurcation diagram was given. This was the situation we encountered when we started to work on this problem using the algebraic invariant theory. In [18] we gave the classification of quadratic Lotka-Volterra systems in terms of their configurations of invariant lines. In [?] we obtained the global topological classification of the Lotka-Volterra systems in invariant form and gave their bifurcation diagram in the 12-parameter space of coefficients in terms of invariant polynomials.

As already mentioned, the program later extended to include cubic differential systems. The last work in this program is on the class of non-degenerate real planar cubic vector fields, which possess four distinct real infinite singularities and invariant straight lines of total multiplicity 7, including the line at infinity ([7]).

Nicolae's contributions to all these results were absolutely essential as he is the one among us who most masterfully handles the algebraic theory of invariants founded by Sibirschi and which was further developed by Nicolae and his colleagues from the Chişinău school, among them Baltag, Bularas, Calin, Popa and the students of Popa. Our work was spread over twenty years and although all four of us, authors of the forthcoming book and joint authors of several publications, are passionate about mathematics, it took a great deal more, in particular a lot of patience, meticulousness and scrupulous checking both on the theoretical and computational sides, to complete together this work. Nicolae's determination and his extraordinary capacity for focusing were precious qualities in this work.

We always got together in the spring in Barcelona, in late summer some of us in Chişinău and in the fall in Montreal. These were periods of intense work with an occasional weekend outing on Catalonia's beautiful Costa Brava, to the wineries of Moldova or the beautiful region of Mauricie for a lunch at the Sacacomie log cabin inn in the province of Québec,... We were also together in Banff in November 2008 surrounded by the magnificent Canadian Rocky Mountains, where we attended the meeting at the Banff International Research Station (BIRS) on Classical Problems on Planar Polynomial Vector Fields, organized by Jaume Llibre and myself. I also recall the splendid colours of autumn in the parc and the woods surrounding the IAS in Princeton in October 2009 where Nicolae and I spent three weeks. So many beautiful sights and precious memories for us!

Finally I want to mention here Nicolae's beautiful work [21]. In this work he gives a complete characterization of the finite weak singularities of quadratic systems via invariant theory. This comparatively short article of 29 pages, is perhaps the best place where the reader can get to measure the mastery of invariant polynomials displayed by Nicolae. Weak singularities have a very important role to play in the production of limit cycles and hence also in Hilbert's 16th problem. They also played an important role in our classification of the global geometrical configurations of singularities in quadratic systems and of course in this we relied on Nicolae's paper [21]. His beautiful results in this paper give a very good idea of what can be done by using the algebraic invariant theory.

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