BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(90), 2019, Pages 137–153
ISSN 1024–7696

Invariant conditions of stability of unperturbed motion governed by critical differential systems $s(1, 2, 3)$

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Abstract. The center-affine invariant conditions of stability of unperturbed motion governed by critical differential systems $s(1, 2, 3)$ were obtained.

Mathematics subject classification: 34C14, 34C20, 34D20.
Keywords and phrases: Differential systems, stability of unperturbed motion, center-affine comitant and invariant, transvectant.

1 The Lyapunov series of a critical system $s(1, 2, 3)$

We examine the cubic nonlinear differential system $s(1, 2, 3)$ of the form

$$\begin{aligned} \dot{x} &= cx + dy + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \dot{y} &= ex + fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3, \end{aligned} \tag{1}$$

where the coefficients and the phase variables take values from the field of real numbres $\mathbb{R}$ and the center-affine transformations group $GL(2, \mathbb{R})$.

Taking into account the center-affine invariants [1–5], in [6–11] the problems of stability of unperturbed motion governed by various two-dimensional, ternary and four-dimensional polynomial differential systems were studied.

In [7] the problem of stability of unperturbed motion governed by critical systems $s(1, 2)$ and $s(1, 3)$ which has a zero root of the characteristic equation of system (1) was solved. After the work [7] appeared, the interest arose to obtain the invariant conditions of stability of unperturbed motion governed by system $s(1, 2, 3)$, as such systems have both theoretical and practical interest. It was shown in [12] and [9] that any nonlinear two-dimensional polynomial differential system, in particular the system (1) in the critical case, by center-affine transformations can be brought to the form

$$\begin{aligned} \dot{x} &= gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \dot{y} &= fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3, \end{aligned} \tag{2}$$

where $f \neq 0$.

According to [12], we analyze the noncritical equation

$$fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3 = 0.$$

Then from the last relation we express y and obtain

$$y=-\frac{l}{f}x^2-2\frac{m}{f}xy-\frac{n}{f}y^2-\frac{t}{f}x^3-3\frac{u}{f}x^2y-3\frac{v}{f}xy^2-\frac{w}{f}y^3. \tag{3}$$

We seek y as a holomorphic function of x. Then we can write

$$\begin{aligned} y=&B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+\\ &+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+\cdots \end{aligned} \tag{4}$$

Substituting (4) into (3) we have

$$\begin{gathered} B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+\\ +B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+\cdots=\\ -\frac{l}{f}x^2-2\frac{m}{f}x\Big(B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+\\ +B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+\cdots\Big)-\\ -\frac{n}{f}\Big(B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+\\ +B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+\cdots\Big)^2-\frac{t}{f}x^3-\\ -3\frac{u}{f}x^2\Big(B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+\\ +B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+\cdots\Big)-\\ -3\frac{v}{f}x\Big(B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+\\ +B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+\cdots\Big)^2-\\ -\frac{w}{f}\Big(B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+\\ +B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+\cdots\Big)^3. \end{gathered}$$

This implies that

$$\begin{gathered} B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+\\ +B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+\cdots=-\frac{l}{f}x^2-\Big[2\frac{m}{f}B_2+\frac{t}{f}\Big]x^3-\\ -\Big[\frac{n}{f}B_2^2+2\frac{m}{f}B_3+3\frac{u}{f}B_2\Big]x^4-\Big[2\frac{n}{f}B_2B_3+2\frac{m}{f}B_4+3\frac{v}{f}B_2^2+3\frac{u}{f}B_3\Big]x^5-\\ -\Big[\frac{n}{f}\Big(2B_2B_4+B_3^2\Big)+\frac{w}{f}B_2^3+6\frac{v}{f}B_2B_3+2\frac{m}{f}B_5+3\frac{u}{f}B_4\Big]x^6-\\ -\Big[2\frac{n}{f}\Big(B_2B_5+B_3B_4\Big)+3\frac{w}{f}B_2^2B_3+3\frac{v}{f}(2B_2B_4+B_3^2)+2\frac{m}{f}B_6+3\frac{u}{f}B_5\Big]x^7- \end{gathered}$$

$$-\Big[\frac{n}{f}\Big(2B_2B_6+2B_3B_5+B_4^2\Big)+3\frac{w}{f}\Big(B_2^2B_4+B_2B_3^2\Big)+6\frac{v}{f}\Big(B_2B_5+B_3B_4\Big)+2\frac{m}{f}B_7+$$
$$+3\frac{u}{f}B_6\Big]x^8-\Big[2\frac{n}{f}\Big(B_2B_7+B_3B_6+B_4B_5\Big)+\frac{w}{f}\Big(3B_2^2B_5+6B_2B_3B_4+B_3^3\Big)+$$
$$+3\frac{v}{f}\Big(2B_2B_6+2B_3B_5+B_4^2\Big)+2\frac{m}{f}B_8+3\frac{u}{f}B_7\Big]x^9-\Big[\frac{n}{f}\Big(2B_2B_8+2B_3B_7+$$
$$+2B_4B_6+B_5^2\Big)+3\frac{w}{f}\Big(B_2^2B_6+2B_2B_3B_5+B_2B_4^2+B_3^2B_4\Big)+6\frac{v}{f}\Big(B_2B_7+B_3B_6+$$
$$+B_4B_5\Big)+2\frac{m}{f}B_9+3\frac{u}{f}B_8\Big]x^{10}-\Big[2\frac{n}{f}\Big(B_2B_9+B_3B_8+B_4B_7+B_5B_6\Big)+$$
$$+3\frac{w}{f}\Big(B_2^2B_7+2B_2B_3B_6+2B_2B_4B_5+B_3^2B_5+B_3B_4^2\Big)+3\frac{v}{f}\Big(2B_2B_8+2B_3B_7+$$
$$+2B_4B_6+B_5^2\Big)+2\frac{m}{f}B_{10}+3\frac{u}{f}B_9\Big]x^{11}-\Big[\frac{n}{f}\Big(2B_2B_{10}+2B_3B_9+2B_4B_8+2B_5B_7+B_6^2\Big)+$$
$$+\frac{w}{f}\Big(3B_2^2B_8+6B_2B_3B_7+6B_2B_4B_6+3B_2B_5^2+3B_3^2B_6+6B_3B_4B_5+B_4^3\Big)+$$
$$+6\frac{v}{f}\Big(B_2B_9+B_3B_8+B_4B_7+B_5B_6\Big)+2\frac{m}{f}B_{11}+3\frac{u}{f}B_{10}\Big]x^{12}-$$
$$-\Big[2\frac{n}{f}\Big(B_2B_{11}+B_3B_{10}+B_4B_9+B_5B_8+B_6B_7\Big)+3\frac{w}{f}\Big(B_2^2B_9+2B_2B_3B_8+$$
$$+2B_2B_4B_7+2B_2B_5B_6+B_3^2B_7+2B_3B_4B_6+B_3B_5^2+B_4^2B_5\Big)+$$
$$+3\frac{v}{f}\Big(2B_2B_{10}+2B_3B_9+2B_4B_8+2B_5B_7+B_6^2\Big)+2\frac{m}{f}B_{12}+3\frac{u}{f}B_{11}\Big]x^{13}-$$
$$-\Big[\frac{n}{f}\Big(2B_2B_{12}+2B_3B_{11}+2B_4B_{10}+2B_5B_9+2B_6B_8+B_7^2\Big)+3\frac{w}{f}\Big(B_2^2B_{10}+2B_2B_3B_9+$$
$$+2B_2B_4B_8+2B_2B_5B_7+B_2B_6^2+B_3^2B_8+2B_3B_4B_7+2B_3B_5B_6+B_4^2B_6+B_4B_5^2\Big)+$$
$$+6\frac{v}{f}\Big(B_2B_{11}+B_3B_{10}+B_4B_9+B_5B_8+B_6B_7\Big)+2\frac{m}{f}B_{13}+3\frac{u}{f}B_{12}\Big]x^{14}-$$
$$-\Big[2\frac{n}{f}\Big(B_2B_{13}+B_3B_{12}+B_4B_{11}+B_5B_{10}+B_6B_9+B_7B_8\Big)+\frac{w}{f}\Big(3B_2^2B_{11}+6B_2B_3B_{10}+$$
$$+6B_2B_4B_9+6B_2B_5B_8+6B_2B_6B_7+3B_3^2B_9+6B_3B_4B_8+6B_3B_5B_7+3B_3B_6^2+$$
$$+3B_4^2B_7+6B_4B_5B_6+B_5^3\Big)+3\frac{v}{f}\Big(2B_2B_{12}+2B_3B_{11}+2B_4B_{10}+2B_5B_9+$$
$$+2B_6B_8+B_7^2\Big)+2\frac{m}{f}B_{14}+3\frac{u}{f}B_{13}\Big]x^{15}+\cdots$$

From this identity we have

$$B_2=-\frac{l}{f},\ \ B_3=-\Big(2\frac{m}{f}B_2+\frac{t}{f}\Big),\ \ B_4=-\Big(\frac{n}{f}B_2^2+2\frac{m}{f}B_3+3\frac{u}{f}B_2\Big),$$
$$B_5=-\Big(2\frac{n}{f}B_2B_3+2\frac{m}{f}B_4+3\frac{v}{f}B_2^2+3\frac{u}{f}B_3\Big),$$
$$B_6=-\Big[\frac{n}{f}\Big(2B_2B_4+B_3^2\Big)+\frac{w}{f}B_2^3+6\frac{v}{f}B_2B_3+2\frac{m}{f}B_5+3\frac{u}{f}B_4\Big],$$

$$B_7=-\Big[2\frac{n}{f}\Big(B_2B_5+B_3B_4\Big)+3\frac{w}{f}B_2^2B_3+3\frac{v}{f}\Big(2B_2B_4+B_3^2\Big)+2\frac{m}{f}B_6+3\frac{u}{f}B_5\Big],$$

$$B_8=-\Big[\frac{n}{f}\Big(2B_2B_6+2B_3B_5+B_4^2\Big)+3\frac{w}{f}\Big(B_2^2B_4+B_2B_3^2\Big)+6\frac{v}{f}\Big(B_2B_5+B_3B_4\Big)+$$
$$+2\frac{m}{f}B_7+3\frac{u}{f}B_6\Big],$$

$$B_9=-\Big[2\frac{n}{f}\Big(B_2B_7+B_3B_6+B_4B_5\Big)+\frac{w}{f}\Big(3B_2^2B_5+6B_2B_3B_4+B_3^3\Big)+$$
$$+3\frac{v}{f}\Big(2B_2B_6+2B_3B_5+B_4^2\Big)+2\frac{m}{f}B_8+3\frac{u}{f}B_7\Big],$$

$$B_{10}=-\Big[\frac{n}{f}\Big(2B_2B_8+2B_3B_7+2B_4B_6+B_5^2\Big)+3\frac{w}{f}\Big(B_2^2B_6+2B_2B_3B_5+B_2B_4^2+B_3^2B_4\Big)+$$
$$+6\frac{v}{f}\Big(B_2B_7+B_3B_6+B_4B_5\Big)+2\frac{m}{f}B_9+3\frac{u}{f}B_8\Big],$$

$$B_{11}=-\Big[2\frac{n}{f}\Big(B_2B_9+B_3B_8+B_4B_7+B_5B_6\Big)+3\frac{w}{f}\Big(B_2^2B_7+2B_2B_3B_6+2B_2B_4B_5+$$
$$+B_3^2B_5+B_3B_4^2\Big)+3\frac{v}{f}\Big(2B_2B_8+2B_3B_7+2B_4B_6+B_5^2\Big)+2\frac{m}{f}B_{10}+3\frac{u}{f}B_9\Big],$$

$$B_{12}=-\Big[\frac{n}{f}\Big(2B_2B_{10}+2B_3B_9+2B_4B_8+2B_5B_7+B_6^2\Big)+\frac{w}{f}\Big(3B_2^2B_8+6B_2B_3B_7+$$
$$+6B_2B_4B_6+3B_2B_5^2+3B_3^2B_6+6B_3B_4B_5+B_4^3\Big)+6\frac{v}{f}\Big(B_2B_9+B_3B_8+B_4B_7+B_5B_6\Big)+$$
$$+2\frac{m}{f}B_{11}+3\frac{u}{f}B_{10}\Big],$$

$$B_{13}=-\Big[2\frac{n}{f}\Big(B_2B_{11}+B_3B_{10}+B_4B_9+B_5B_8+B_6B_7\Big)+3\frac{w}{f}\Big(B_2^2B_9+2B_2B_3B_8+$$
$$+2B_2B_4B_7+2B_2B_5B_6+B_3^2B_7+2B_3B_4B_6+B_3B_5^2+B_4^2B_5\Big)+3\frac{v}{f}\Big(2B_2B_{10}+$$
$$+2B_3B_9+2B_4B_8+2B_5B_7+B_6^2\Big)+2\frac{m}{f}B_{12}+3\frac{u}{f}B_{11}\Big],$$

$$B_{14}=-\Big[\frac{n}{f}\Big(2B_2B_{12}+2B_3B_{11}+2B_4B_{10}+2B_5B_9+2B_6B_8+B_7^2\Big)+3\frac{w}{f}\Big(B_2^2B_{10}+$$
$$+2B_2B_3B_9+2B_2B_4B_8+2B_2B_5B_7+B_2B_6^2+B_3^2B_8+2B_3B_4B_7+2B_3B_5B_6+$$
$$+B_4^2B_6+B_4B_5^2\Big)+6\frac{v}{f}\Big(B_2B_{11}+B_3B_{10}+B_4B_9+B_5B_8+B_6B_7\Big)+2\frac{m}{f}B_{13}+3\frac{u}{f}B_{12}\Big],$$

$$B_{15}=-\Big[2\frac{n}{f}\Big(B_2B_{13}+B_3B_{12}+B_4B_{11}+B_5B_{10}+B_6B_9+B_7B_8\Big)+\frac{w}{f}\Big(3B_2^2B_{11}+$$
$$+6B_2B_3B_{10}+6B_2B_4B_9+6B_2B_5B_8+6B_2B_6B_7+3B_3^2B_9+6B_3B_4B_8+6B_3B_5B_7+$$
$$+3B_3B_6^2+3B_4^2B_7+6B_4B_5B_6+B_5^3\Big)+3\frac{v}{f}\Big(2B_2B_{12}+2B_3B_{11}+$$
$$+2B_4B_{10}+2B_5B_9+2B_6B_8+B_7^2\Big)+2\frac{m}{f}B_{14}+3\frac{u}{f}B_{13}\Big],\ldots \tag{5}$$

Substituting (4) into the right-hand sides of the critical differential equations (2), we get the following identity

$$gx^2+2hx(B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+$$
$$+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+\cdots)+k(B_2x^2+$$

$$+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+$$
$$+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+\cdots)^2+px^3+3qx^2(B_2x^2+B_3x^3+$$
$$+B_4x^4+B_5x^5+B_6x^6+B_7x^7+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+$$
$$+B_{13}x^{13}+B_{14}x^{14}+B_{15}x^{15}+\cdots)+3rx(B_2x^2+B_3x^3+B_4x^4+B_5x^5+$$
$$+B_6x^6+B_7x^7+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+$$
$$+B_{14}x^{14}+B_{15}x^{15}+\cdots)^2+s(B_2x^2+B_3x^3+B_4x^4+B_5x^5+B_6x^6+B_7x^7+$$
$$+B_8x^8+B_9x^9+B_{10}x^{10}+B_{11}x^{11}+B_{12}x^{12}+B_{13}x^{13}+B_{14}x^{14}+$$
$$+B_{15}x^{15}+\cdots)^3=A_2x^2+A_3x^3+A_4x^4+A_5x^5+A_6x^6+A_7x^7+A_8x^8+$$
$$+A_9x^9+A_{10}x^{10}+A_{11}x^{11}+A_{12}x^{12}+A_{13}x^{13}+A_{14}x^{14}+A_{15}x^{15}+\cdots$$

From here, it follows that the coefficients of the Lyapunov series A_i $(i=2,3,...)$ can be written in the form

$$A_2=g,\ \ A_3=2hB_2+p,\ \ A_4=kB_2^2+2hB_3+3qB_2,$$
$$A_5=2kB_2B_3+2hB_4+3rB_2^2+3qB_3,$$
$$A_6=sB_2^3+k(2B_2B_4+B_3^2)+2hB_5+6rB_2B_3+3qB_4;$$
$$A_7=3sB_2^2B_3+2k(B_2B_5+B_3B_4)+2hB_6+3r(2B_2B_4+B_3^2)++3qB_5,$$
$$A_8=3s(B_2^2B_4+B_2B_3^2)+k(2B_2B_6+2B_3B_5+B_4^2)+2hB_7+$$
$$+6r(B_2B_5+B_3B_4)+3qB_6,$$
$$A_9=s(3B_2^2B_5+6B_2B_3B_4+B_3^3)+2k(B_2B_7+B_3B_6+B_4B_5)+2hB_8+$$
$$+3r(2B_2B_6+2B_3B_5+B_4^2)+3qB_7,$$
$$A_{10}=3s(B_2^2B_6+2B_2B_3B_5+B_2B_4^2+B_3^2B_4)+k(2B_2B_8+$$
$$+2B_3B_7+2B_4B_6+B_5^2)+2hB_9+6r(B_2B_7+B_3B_6+B_4B_5)+3qB_8,$$
$$A_{11}=3s(B_2^2B_7+2B_2B_3B_6+2B_2B_4B_5+B_3^2B_5+B_3B_4^2)+2k(B_2B_9+B_3B_8+$$
$$+B_4B_7+B_5B_6)+2hB_{10}+3r(2B_2B_8+2B_3B_7+2B_4B_6+B_5^2)+3qB_9,$$
$$A_{12}=s(3B_2^2B_8+6B_2B_3B_7+6B_2B_4B_6+3B_2B_5^2+3B_3^2B_6+6B_3B_4B_5+B_4^3)+$$
$$+k(2B_2B_{10}+2B_3B_9+2B_4B_8+2B_5B_7+B_6^2)+2hB_{11}+$$
$$+6r(B_2B_9+B_3B_8+B_4B_7+B_5B_6)+3qB_{10},$$
$$A_{13}=3s(B_2^2B_9+2B_2B_3B_8+2B_2B_4B_7+2B_2B_5B_6+B_3^2B_7+2B_3B_4B_6+$$
$$+B_3B_5^2+B_4^2B_5)+2k(B_2B_{11}+B_3B_{10}+B_4B_9+B_5B_8+B_6B_7)+$$
$$+2hB_{12}+3r(2B_2B_{10}+2B_3B_9+2B_4B_8+2B_5B_7+B_6^2)+3qB_{11},$$
$$A_{14}=3s(B_2^2B_{10}+2B_2B_3B_9+2B_2B_4B_8+2B_2B_5B_7+B_2B_6^2+B_3^2B_8+2B_3B_4B_7+$$
$$+2B_3B_5B_6+B_4^2B_6+B_4B_5^2)+k(2B_2B_{12}+2B_3B_{11}+2B_4B_{10}+2B_5B_9+$$
$$+2B_6B_8+B_7^2)+2hB_{13}+6r(B_2B_{11}+B_3B_{10}+B_4B_9+B_5B_8+B_6B_7)+3qB_{12},$$
$$A_{15}=s(3B_2^2B_{11}+6B_2B_3B_{10}+6B_2B_4B_9+6B_2B_5B_8+6B_2B_6B_7+3B_3^2B_9+$$
$$+6B_3B_4B_8+6B_3B_5B_7+3B_3B_6^2+3B_4^2B_7+6B_4B_5B_6+B_5^3)+$$

$$
\begin{aligned}
&+2k(B_2B_{13}+B_3B_{12}+B_4B_{11}+B_5B_{10}+B_6B_9+B_7B_8)+2hB_{14}+\\
&+3r(2B_2B_{12}+2B_3B_{11}+2B_4B_{10}+2B_5B_9+2B_6B_8+B_7^2)+3qB_{13},...
\end{aligned}
\tag{6}
$$

2 Conditions of stability of unperturbed motion governed by the critical system (2)

2.1. Assume that $fl \neq 0$. In this case, by a center-affine transformation $\bar{x} = lx$, $\bar{y} = ly$, we obtain the system (2) with $l = 1$. Taking into account this equality and the relations (5)–(6), we consider the following expressions:

$$
\begin{gathered}
A = 2h - fp;\quad B = -k - 4hm + 3fq + 2fht;\\
C = -(8hm^2 - 2hn - 6fmq + 3fr - 12fhmt + 3f^2qt + 4f^2ht^2 + 6fhu);\\
D = 4hmn - 3fnq + fs + 16fhm^2t - 4fhnt - 6f^2mqt - 20f^2hmt^2+\\
+3f^3qt^2 + 6f^3ht^3 - 24fhmu + 9f^2qu + 18f^2htu + 6fhv;\\
E = -32hm^3t + 12hmnt + 12fm^2qt - 3fnqt + 80fhm^2t^2 - 8fhnt^2 - 18f^2mqt^2-\\
-60f^2hmt^3 + 6f^3qt^3 + 14f^3ht^4 + 48hm^2u - 6hnu - 18fmqu - 108fhmtu+\\
+18f^2qtu + 48f^2ht^2u + 18fhu^2 - 24hmv + 9fqv + 18fhtv + 2hw;\\
F = 64hm^4t - 8hm^2nt - 2hn^2t - 24fm^3qt - 144fhm^3t^2 + 36f^2m^2qt^2+\\
+112f^2hm^2t^3 + 2f^2hnt^3 - 18f^3mqt^3 - 36f^3hmt^4 + 3f^4qt^4 + 4f^4ht^5 - 96hm^3u-\\
-12hmnu + 36fm^2qu + 9fnqu + 120fhm^2tu + 24fhntu - 18f^2mqtu-\\
-12f^2hmt^2u - 12f^3ht^3u + 72fhmu^2 - 27f^2qu^2 - 72f^2htu^2 + 48hm^2v + 6hnv-\\
-18fmqv - 36fhmtv + 9f^2qtv + 6f^2ht^2v - 36fhuv - 3fqw;\\
G = 4m^2t - nt - 6fmt^2 + 2f^2t^3 - 6mu + 6ftu + 3v.
\end{gathered}
\tag{7}
$$

Lemma 1. *The stability of unperturbed motion governed by system of perturbed motion* (2) *with $f < 0$ and $l = 1$ is characterized by one of the following sixteen possible cases:*

I. $g \neq 0$, *then the unperturbed motion is unstable;*

II. $g = 0,\ A > 0$, *then the unperturbed motion is unstable;*

III. $g = 0,\ A < 0$, *then the unperturbed motion is stable;*

IV. $g = A = 0,\ B \neq 0$, *then the unperturbed motion is unstable;*

V. $g = A = B = 0,\ C > 0$, *then the unperturbed motion is unstable;*

VI. $g = A = B = 0,\ C < 0$, *then the unperturbed motion is stable;*

VII. $g = A = B = C = 0,\ D \neq 0$, *then the unperturbed motion is unstable;*

VIII. $g = A = B = C = D = 0,\ E > 0$, *then the unperturbed motion is unstable;*

IX. $g = A = B = C = D = 0,\ E < 0$, *then the unperturbed motion is stable;*

X. $g = A = B = C = D = E = 0,\ F \neq 0$, *then the unperturbed motion is unstable;*

XI. $g = A = B = C = D = E = F = 0, \ \ h < 0, \ \ G \neq 0,$ *then the unperturbed motion is unstable;*

XII. $g = A = B = C = D = E = F = 0, \ \ h > 0, \ \ G \neq 0,$ *then the unperturbed motion is stable;*

XIII. $g = A = B = C = D = E = F = h = 0, \ \ qG \neq 0,$ *then the unperturbed motion is unstable;*

XIV. $g = A = B = C = D = E = F = h = q = 0,$ *then the unperturbed motion is stable;*

XV. $g = A = B = C = D = E = F = G = h = 0, \ \ q \neq 0,$ *then the unperturbed motion is stable;*

XVI. $g = A = B = C = D = E = F = G = 0, \ \ h \neq 0,$ *then the unperturbed motion is stable.*

In the last three cases, the unperturbed motion belongs to some continuous series of stabilized motions. Moreover, for sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series.

Proof. According to Theorem from [12, p.118], we analyze the coefficients of the Lyapunov series A_i of (6) and the expressions (7). If $A_2 \neq 0$, then we obtain the Case I of Lemma 1.

If $A_2 = 0$, then $g = 0$ and $A_3 = -f^{-1}A$. Depending on the sign of the expression A we get the Cases II and III.

Let $g = A = 0$. Then $A_4 = -f^{-2}B$. When $B \neq 0$ we obtain the Case IV.

If $g = A = B = 0$, then $A_5 = -f^{-3}C$. Taking into account the sign of the expression C we get the Cases V and VI.

Suppose that $g = A = B = C = 0$. Then $A_6 = -f^{-4}D$. When $D \neq 0$ we obtain the Case VII.

If $g = A = B = C = D = 0$, then $A_7 = f^{-4}E$. Depending on the sign of the expression E we get the Cases VIII and IX.

Assume that $g = A = B = C = D = E = 0$. Then $A_8 = f^{-5}F$. When $F \neq 0$ we obtain the Case X.

If $g = A = B = C = D = E = F = 0$, then $A_9 = 2f^{-5}hG^2$. Taking into account the sign of the expression h and $G \neq 0$ we get the Cases XI and XII.

Let $g = A = B = C = D = E = F = h = 0$. Then $A_{10} = 3f^{-5}qG^2$. When $qG \neq 0$ we obtain the Case XIII.

If $g = A = B = C = D = E = F = h = q = 0$, then $A_i = 0$ for all i, we get the Case XIV.

If $g = A = B = C = D = E = F = G = h = 0, \ q \neq 0$, then $A_i = 0$ for all i, we obtain the Case XV.

If $g = A = B = C = D = E = F = G = 0, \ h \neq 0$, then $A_i = 0$ for all i, we have the Case XVI. Lemma 1 is proved. □

2.2. Let us examine the case $f \neq 0, \ l = 0$. In this case, the coefficients (5) can be written in the form

$$B_2=0,\quad B_3=-\frac{t}{f},\quad B_4=-2\frac{m}{f}B_3,\quad B_5=-\Big(2\frac{m}{f}B_4+3\frac{u}{f}B_3\Big),$$
$$B_6=-\Big(\frac{n}{f}B_3^2+2\frac{m}{f}B_5+3\frac{u}{f}B_4\Big),\quad B_7=-\Big(2\frac{n}{f}B_3B_4+3\frac{v}{f}B_3^2+2\frac{m}{f}B_6+3\frac{u}{f}B_5\Big),$$
$$B_8=-\Big[\frac{n}{f}\Big(2B_3B_5+B_4^2\Big)+6\frac{v}{f}B_3B_4+2\frac{m}{f}B_7+3\frac{u}{f}B_6\Big],$$
$$B_9=-\Big[2\frac{n}{f}\Big(B_3B_6+B_4B_5\Big)+\frac{w}{f}B_3^3+3\frac{v}{f}\Big(2B_3B_5+B_4^2\Big)+2\frac{m}{f}B_8+3\frac{u}{f}B_7\Big],$$
$$B_{10}=-\Big[\frac{n}{f}\Big(2B_3B_7+2B_4B_6+B_5^2\Big)+3\frac{w}{f}B_3^2B_4+6\frac{v}{f}\Big(B_3B_6+B_4B_5\Big)+$$
$$+2\frac{m}{f}B_9+3\frac{u}{f}B_8\Big],$$
$$B_{11}=-\Big[2\frac{n}{f}\Big(B_3B_8+B_4B_7+B_5B_6\Big)+3\frac{w}{f}\Big(B_3^2B_5+B_3B_4^2\Big)+$$
$$+3\frac{v}{f}\Big(2B_2B_8+2B_3B_7+2B_4B_6+B_5^2\Big)+2\frac{m}{f}B_{10}+3\frac{u}{f}B_9\Big],$$
$$B_{12}=-\Big[\frac{n}{f}\Big(2B_3B_9+2B_4B_8+2B_5B_7+B_6^2\Big)+\frac{w}{f}\Big(3B_3^2B_6+6B_3B_4B_5+B_4^3\Big)+$$
$$+6\frac{v}{f}\Big(B_3B_8+B_4B_7+B_5B_6\Big)+2\frac{m}{f}B_{11}+3\frac{u}{f}B_{10}\Big],$$
$$B_{13}=-\Big[2\frac{n}{f}\Big(B_3B_{10}+B_4B_9+B_5B_8+B_6B_7\Big)+3\frac{w}{f}\Big(B_3^2B_7+2B_3B_4B_6+$$
$$+B_3B_5^2+B_4^2B_5\Big)+3\frac{v}{f}\Big(2B_3B_9+2B_4B_8+2B_5B_7+B_6^2\Big)+2\frac{m}{f}B_{12}+3\frac{u}{f}B_{11}\Big],$$
$$B_{14}=-\Big[\frac{n}{f}\Big(2B_3B_{11}+2B_4B_{10}+2B_5B_9+2B_6B_8+B_7^2\Big)+$$
$$+3\frac{w}{f}\Big(B_3^2B_8+2B_3B_4B_7+2B_3B_5B_6+B_4^2B_6+B_4B_5^2\Big)+$$
$$+6\frac{v}{f}\Big(B_3B_{10}+B_4B_9+B_5B_8+B_6B_7\Big)+2\frac{m}{f}B_{13}+3\frac{u}{f}B_{12}\Big],$$
$$B_{15}=-\Big[2\frac{n}{f}\Big(B_3B_{12}+B_4B_{11}+B_5B_{10}+B_6B_9+B_7B_8\Big)+\frac{w}{f}\Big(3B_3^2B_9+$$
$$+6B_3B_4B_8+6B_3B_5B_7+3B_3B_6^2+3B_4^2B_7+6B_4B_5B_6+B_5^3\Big)+$$
$$+3\frac{v}{f}\Big(2B_3B_{11}+2B_4B_{10}+2B_5B_9+2B_6B_8+B_7^2\Big)+2\frac{m}{f}B_{14}+3\frac{u}{f}B_{13}\Big],\ldots, \tag{8}$$

and the coefficients of the Lyapunov series A_i $(i=2,3,...)$ from (6) are given by the expressions

$$A_2=g,\quad A_3=p,\quad A_4=2hB_3,\quad A_5=2hB_4+3qB_3,$$
$$A_6=kB_3^2+2hB_5+3qB_4;\quad A_7=2kB_3B_4+2hB_6+3rB_3^2+3qB_5,$$
$$A_8=k(2B_3B_5+B_4^2)+2hB_7+6rB_3B_4+3qB_6,$$

$$
\begin{gathered}
A_9 = sB_3^3 + 2k(B_3B_6 + B_4B_5) + 2hB_8 + 3r(2B_3B_5 + B_4^2) + 3qB_7, \\
A_{10} = 3s(B_3^2B_4) + k(2B_3B_7 + 2B_4B_6 + B_5^2) + 2hB_9 + 6r(B_3B_6 + B_4B_5) + 3qB_8, \\
A_{11} = 3s(B_3^2B_5 + B_3B_4^2) + 2k(B_3B_8 + B_4B_7 + B_5B_6) + 2hB_{10} + 3r(2B_3B_7 + \\
+2B_4B_6 + B_5^2) + 3qB_9, \\
A_{12} = s(3B_3^2B_6 + 6B_3B_4B_5 + B_4^3) + k(2B_3B_9 + 2B_4B_8 + 2B_5B_7 + B_6^2) + 2hB_{11} + \\
+6r(B_3B_8 + B_4B_7 + B_5B_6) + 3qB_{10}, \\
A_{13} = 3s(B_3^2B_7 + 2B_3B_4B_6 + B_3B_5^2 + B_4^2B_5) + 2k(B_3B_{10} + B_4B_9 + B_5B_8 + B_6B_7) + \\
+2hB_{12} + 3r(2B_3B_9 + 2B_4B_8 + 2B_5B_7 + B_6^2) + 3qB_{11}, \\
A_{14} = 3s(B_3^2B_8 + 2B_3B_4B_7 + 2B_3B_5B_6 + B_4^2B_6 + B_4B_5^2) + k(2B_3B_{11} + 2B_4B_{10} + \\
+2B_5B_9 + 2B_6B_8 + B_7^2) + 2hB_{13} + 6r(B_3B_{10} + B_4B_9 + B_5B_8 + B_6B_7) + 3qB_{12}, \\
A_{15} = s(3B_3^2B_9 + 6B_3B_4B_8 + 6B_3B_5B_7 + 3B_3B_6^2 + 3B_4^2B_7 + 6B_4B_5B_6 + B_5^3) + \\
+2k(B_3B_{12} + B_4B_{11} + B_5B_{10} + B_6B_9 + B_7B_8) + 2hB_{14} + \\
+3r(2B_3B_{11} + 2B_4B_{10} + 2B_5B_9 + 2B_6B_8 + B_7^2) + 3qB_{13}, \ldots
\end{gathered}
\tag{9}
$$

Lemma 2. *The stability of unperturbed motion governed by system of perturbed motion* (2) *with* $f < 0$ *and* $l = 0$ *is characterized by one of the following thirteen possible cases:*

I. $g \neq 0$, *then the unperturbed motion is unstable;*

II. $g = 0$, $p > 0$, *then the unperturbed motion is unstable;*

III. $g = 0$, $p < 0$, *then the unperturbed motion is stable;*

IV. $g = p = 0$, $ht \neq 0$, *then the unperturbed motion is unstable;*

V. $g = p = h = 0$, $qt > 0$, *then the unperturbed motion is unstable;*

VI. $g = p = h = 0$, $qt < 0$, *then the unperturbed motion is stable;*

VII. $g = p = h = q = 0$, $kt \neq 0$, *then the unperturbed motion is unstable;*

VIII. $g = p = h = q = k = 0$, $t \neq 0$, $r > 0$, *then the unperturbed motion is unstable;*

IX. $g = p = h = q = k = 0$, $t \neq 0$, $r < 0$, *then the unperturbed motion is stable;*

X. $g = p = h = q = k = r = 0$, $st > 0$, *then the unperturbed motion is unstable;*

XI. $g = p = h = q = k = r = 0$, $st < 0$, *then the unperturbed motion is stable;*

XII. $g = p = h = q = k = r = s = 0$, $t \neq 0$, *then the unperturbed motion is stable;*

XIII. $g = p = t = 0$, *then the unperturbed motion is stable.*

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motions. Moreover, for sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series.

Proof. According to Theorem from [12, p.118] we analyze the coefficients of the Lyapunov series A_i of (9). If $A_2 \neq 0$, then we obtain the Case I of Lemma 2.

Let $g = 0$. Then $A_3 = p$. Depending on the sign of the expression p we get the Cases II and III.

Assume that $g = p = 0$. Then $A_4 = -2f^{-1}ht$. When $ht \neq 0$, we get the Case IV of Lemma 2.

If $g = p = t = 0$, then $A_i = 0$ for all i, we get the Case XIII.

If $g = p = h = 0$, $t \neq 0$, then $A_5 = -3f^{-1}qt$. Taking into account the sign of the expression qt we get the Cases V and VI.

Suppose that $g = p = h = q = 0$, $t \neq 0$. Then $A_6 = f^{-2}kt^2$. Under condition $kt \neq 0$ we get the Case VII.

If $g = p = h = q = k = 0$, $t \neq 0$, then $A_7 = 3f^{-2}rt^2$. Depending on the sign of the expression r we get the Cases VIII and IX.

If $g = p = h = q = k = r = 0$, $t \neq 0$, then $A_8 = 0$ and $A_9 = -f^{-3}st^3$. Taking into account the sign of the expression st we obtain the Cases X and XI.

If $g = p = h = q = k = r = s = 0$, $t \neq 0$, then $A_i = 0$ for all i, we get the Case XII. Lemma 2 is proved. □

3 Comitants and transvectants of system (1) with applications to the conditions of Lemmas 1 and 2

We consider the system (1), written in the form

$$\dot{x} = \sum_{i=1}^{3} P_i(x, y), \quad \dot{y} = \sum_{i=1}^{3} Q_i(x, y), \tag{10}$$

where P_i and Q_i are homogeneous polynomials of degree $i = 1, 2, 3$ in the phase variables x and y. Denote by q the transformations of the center-affin group $GL(2, \mathbb{R})$

$$\begin{gathered} \bar{x} = \alpha x + \beta y, \quad \bar{y} = \gamma x + \delta y, \\ (\Delta = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0). \end{gathered} \tag{11}$$

Observe that system (1) (or (10)) by transformation (11) preserves the form. Its new coefficients are linear functions with respect to the coefficients of system (1) and rational functions with respect to the parameters of transformations (11).

Denote by a the set of coefficients of (1) and by $\bar{a}$ the set of coefficients of system obtained after transformation (11).

Definition 1. *The polynomial $K(a, x, y)$ in the coefficients of system (1) and the phase variables x, y is called the center-affine comitant, or comitant with respect to the center-affine group $GL(2, \mathbb{R})$ of system (1), if the following identity holds*

$$K(\bar{a}, \bar{x}, \bar{y}) = \Delta_q^{-\varkappa} K(a, x, y) \tag{12}$$

for any coefficients of system (1)*, any phase variables* x, y *and any transformations* q *from* (11) *of the group* $GL(2,\mathbb{R})$.

If K *does not depend on* x *and* y*, then it is called the center-affine invariant of the system* (1).

In some cases, for simplicity, the center-affine comitant (invariant) of system $GL(2,\mathbb{R})$ will be called comitant (invariant).

Remark 1. The number $\varkappa$ is called the weight of the comitant K. If $K(\alpha, x, y)$ is a polynomial with well-defined sign and $\varkappa$ is even or zero, then this polynomial by any center-affine transformation (11) keeps the sign. This is not true when $\varkappa$ is odd.

We consider two center-affine comitants φ and ψ of system (1) of degree r and ρ, respectively, in the phase variables x and y. Then according to [2] and [13] the polynomial

$$(\varphi,\psi)^{(k)} = \frac{(r-k)(\rho-k)}{r!\rho!}\sum_{h=0}^{k}(-1)^h \begin{pmatrix} k \\ h \end{pmatrix} \frac{\partial^k\varphi}{\partial x^{k-h}\partial y^h}\frac{\partial^k\psi}{\partial x^h\partial y^{h-k}} \tag{13}$$

is a center-affine comitant of system (1) and it is called a transvectant of order k with respect to polynomials φ and ψ.

Remark 2. [14] If the weight of comitant K_1 is $\varkappa_1$, and the weight of comitant K_2 is $\varkappa_2$, then the weight of comitant $(K_1, K_2)^{(k)}$ is equal to $\varkappa_1 + \varkappa_2 + k$.

Using (10) we will examine the comitants of system (1) written in the form [14,15]

$$\begin{gathered} R_i = P_i(x,y)y - Q_i(x,y)x \quad (i=1,2,3), \\ S_i = \frac{1}{i}(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}) \ \ (i=1,2,3), \end{gathered} \tag{14}$$

where R_i and S_i are polynomials of the first degree with respect to the coefficients of the system (1).

Taking into account the comitants (14) and the transvectant (13), the following invariants and comitants of system (1) were constructed in [14, 15]:

$$\begin{gathered} K_1[0] = S_1, \ \ K_2[-1] = R_1, \ \ K_3[0] = (R_1, R_1)^{(2)}, \ \ K_4[-1] = R_2, \\ K_5[0] = S_2, \ \ K_6[-1] = (R_2, R_1)^{(1)}, \ \ K_7[0] = (R_2, R_1)^{(2)}, \\ K_8[-1] = R_3, \ \ K_9[-1] = (R_3, R_1)^{(1)}, \ \ K_{10}[0] = (R_3, R_1)^{(2)}, \\ K_{11}[0] = (K_{10}, R_1)^{(1)}, \ \ K_{12}[1] = (K_{10}, R_1)^{(2)}, \\ K_{13}[0] = (K_7, R_1)^{(1)}, \ \ K_{14}[0] = (S_2, R_1)^{(1)}, \ K_{15}[0] = S_3, \\ K_{16}[0] = (S_3, R_1)^{(1)}, \ \ K_{17}[1] = (S_3, R_1)^{(2)}, \end{gathered} \tag{15}$$

where in square brackets the weight of the comitant (invariant) is indicated.

Remark 3. The invariants and the comitants (15) form an algebraic basis of the comitants of system (1), i.e. any center-affine comitant of this system is a solution of a polynomial equation, whose coefficients are polynomials from (15).

Later on, the following comitants and invariants of system (1) will be used:

$$M_1[0] = 3K_1K_7 - 4K_1K_{14} + 2K_1^2K_5 - 6K_{13},$$
$$M_2[0] = -8K_1K_{11} + 6K_1K_2K_{17} + 4K_1^2K_{10} - 6K_1^2K_{16} + 3K_1^3K_{15} + 8K_2K_{12},$$
$$M_3[-1] = 6K_1K_2K_7 + 4K_1K_2K_{14} - 2K_1^2K_2K_5 + 6K_1^2K_6 + 3K_1^3K_4 + 12K_2K_{13},$$
$$M_4[0] = -K_1K_5 + 2K_{14}, \quad M_5[0] = 3K_1K_5 + 6K_7 - 2K_{14},$$
$$M_6[-1] = 4K_1K_2K_{11} - 4K_1^2K_2K_{10} + 2K_1^3K_9 - K_1^4K_8 - 2K_2^2K_{12},$$
$$M_7[-1] = K_1K_{17} + 2K_{12}, \quad M_8[-1] = -3K_1K_2K_7 + 2K_1^2K_6 - K_1^3K_4 + 2K_2K_{13},$$
$$M_9[0] = -3K_1K_7 + 4K_1K_{14},$$
$$M_{10}[0] = 2K_1K_2K_{17} + 8K_1K_{11} + 4K_1^2K_{10} + 2K_1^2K_{16} + K_1^3K_{15} + 8K_2K_{12},$$
$$M_{11}[0] = 2K_1K_2K_{17} + 8K_1K_{11} - 4K_1^2K_{10} - 2K_1^2K_{16} + K_1^3K_{15} - 8K_2K_{12},$$
$$M_{12}[-1] = 4K_1K_2K_{11} + 4K_1^2K_2K_{10} + 2K_1^3K_9 + K_1^4K_8 + 2K_2^2K_{12},$$
$$M_{13}[1] = K_1K_{17} - 2K_{12},$$
$$M_{14}[0] = -8K_1K_{11} + 6K_1K_2K_{17} - 4K_1^2K_{10} + 6K_1^2K_{16} + 3K_1^3K_{15} - 8K_2K_{12},$$
$$M_{15}[0] = -8K_1K_{11} + 6K_1K_2K_{17} + 4K_1^2K_{10} + 3K_1^3K_{15} - 6K_1^2K_{16} + 8K_2K_{12}, \qquad (16)$$

where K_i $(i = \overline{1, 17})$ are from (15), and in square brackets the weight of the comitant (invariant) is indicated.

Note that for the system (2) we have

$$S_1[0] = f, \;\; M_8[-1] = 2f^3lx^3, \qquad (17)$$

where S_1 is from (14) and M_8 is from (16).

It was shown that if $l \neq 0$ in system (2), then by a center-affine transformation, we can obtain $l = 1$. Taking this into account for system (2), we have

$$M_1[0] = 6f^2gx, \qquad (18)$$

where $M_1[0]$ is from (16).

Let us consider $l = 1, g = 0$ in system (2), then by (15)–(16) we obtain

$$K_2[-1] = -fxy, \quad M_5[0] = 6fhy, \quad M_7[1] = 2f^2q, \quad M_8[-1] = 2f^3x^3,$$
$$N_1[-1] \equiv 3K_1K_2M_2 + 4M_5M_8 = 24f^4(2h - fp)x^3y,$$
$$N_2[-3] \equiv 4K_1^3K_2^2M_5M_6 + 4K_1^2K_2^2M_4M_5M_8 - 18K_1^2K_2^3M_7M_8 - M_3M_8^2 =$$
$$= 24f^9(-k - 4hm + 3fq + 2fht)x^6y^3,$$
$$N_3[-4] \equiv -3K_1^2M_7K_2^2M_4M_8^3 + \frac{1}{4}K_1^2K_2M_{11}M_5M_8^3 + \frac{2}{3}K_1^2K_2M_4^2M_5M_8^3 -$$
$$-\frac{3}{8}K_1M_{10}M_8^4 + \frac{1}{9}M_9M_5M_8^4 - 3K_1^3M_7K_2^2M_8^2M_6 +$$
$$+2K_1^3K_2M_4M_5M_8^2M_6 + \frac{4}{3}K_1^4K_2M_5M_8M_6^2 =$$
$$= -16f^{15}(8hm^2 - 2hn - 6fmq + 3fr - 12fhmt + 3f^2qt + 4f^2ht^2 + 6fhu)x^{12}y^2,$$

$$N_4[-6] \equiv -\frac{9}{8}K_1^3M_7K_2^4M_{11}M_8^3 + \frac{1}{2}K_1^3K_2^3M_{11}M_4M_5M_8^3 - \frac{1}{2}K_1M_7K_2^3M_9M_8^4 +$$
$$+\frac{1}{2}K_1^2M_{13}K_2^3M_5M_8^4 + \frac{1}{9}K_1K_2^2M_4M_9M_5M_8^4 - 3K_1^4M_7K_2^4M_4M_8^2M_6 +$$
$$+\frac{3}{4}K_1^4K_2^3M_{11}M_5M_8^2M_6 + \frac{4}{3}K_1^4K_2^3M_4^2M_5M_8^2M_6 + \frac{2}{9}K_1^2K_2^2M_9M_5M_8^3M_6 -$$
$$-3K_1^5M_7K_2^4M_8M_6^2 + \frac{10}{3}K_1^5K_2^3M_4M_5M_8M_6^2 + 2K_1^6K_2^3M_5M_6^3 - \frac{1}{4}K_1M_8^5M_{12} =$$
$$= -16f^{19}(4hmn - 3fnq + fs + 16fhm^2t - 4fhnt - 6f^2mqt - 20f^2hmt^2 +$$
$$+3f^3qt^2 + 6f^3ht^3 - 24fhmu + 9f^2qu + 18f^2htu + 6fhv)x^{15}y^4,$$

$$N_5[-6] \equiv \frac{9}{8}K_1^2M_7K_2^3M_{11}M_4M_8^4 - \frac{3}{32}K_1^2K_2^2M_{11}^2M_5M_8^4 - \frac{1}{2}K_1^2K_2^2M_{11}M_4^2M_5M_8^4 +$$
$$+\frac{9}{4}K_1M_{13}M_7K_2^3M_8^5 - K_1M_{13}K_2^2M_4M_5M_8^5 - \frac{1}{24}K_2M_{11}M_9M_5M_8^5 -$$
$$-\frac{1}{48}M_{14}M_5M_8^6 + \frac{9}{4}K_1^3M_7K_2^3M_{11}M_8^3M_6 + 3K_1^3M_7K_2^3M_4^2M_8^3M_6 -$$
$$-\frac{9}{4}K_1^3K_2^2M_{11}M_4M_5M_8^3M_6 - \frac{4}{3}K_1^3K_2^2M_4^3M_5M_8^3M_6 + \frac{1}{2}K_1M_7K_2^2M_9M_8^4M_6 -$$
$$-\frac{3}{2}K_1^2M_{13}K_2^2M_5M_8^4M_6 - \frac{1}{3}K_1K_2M_4M_9M_5M_8^4M_6 + 9K_1^4M_7K_2^3M_4M_8^2M_6^2 -$$
$$-2K_1^4K_2^2M_{11}M_5M_8^2M_6^2 - \frac{20}{3}K_1^4K_2^2M_4^2M_5M_8^2M_6^2 - \frac{4}{9}K_1^2K_2M_9M_5M_8^3M_6^2 +$$
$$+6K_1^5M_7K_2^3M_8M_6^3 - 10K_1^5K_2^2M_4M_5M_8M_6^3 - \frac{14}{3}K_1^6K_2^2M_5M_6^4 =$$
$$= -32f^{22}(-32hm^3t + 12hmnt + 12fm^2qt - 3fnqt + 80fhm^2t^2 - 8fhnt^2 -$$
$$-18f^2mqt^2 - 60f^2hmt^3 + 6f^3qt^3 + 14f^3ht^4 + 48hm^2u - 6hnu - 18fmqu -$$
$$-108fhmtu + 18f^2qtu + 48f^2ht^2u + 18fhu^2 - 24hmv + 9fqv +$$
$$+18fhtv + 2hw)x^{18}y^3,$$

$$N_6[-7] \equiv \frac{4}{3}K_1^8K_2^2M_5M_6^5 + 6K_1^7K_2^2M_5^2M_6^4M_8 - 3K_1^7K_2^3M_6^4M_7M_8 -$$
$$-\frac{1}{2}K_1^6K_2^2M_{11}M_5M_6^3M_8^2 + \frac{28}{3}K_1^6K_2^2M_4^2M_5M_6^3M_8^2 - 9K_1^6K_2^3M_4M_6^3M_7M_8^2 +$$
$$+\frac{1}{4}K_1^5K_2^2M_{11}M_4M_5M_6^2M_8^3 + 6K_1^5K_2^2M_4^3M_5M_6^2M_8^3 -$$
$$-9K_1^5K_2^3M_4^2M_6^2M_7M_8^3 - \frac{3}{8}K_1^4K_2^2M_{11}^2M_5M_6M_8^4 + \frac{5}{4}K_1^4K_2^2M_{11}M_4^2M_5M_6M_8^4 +$$
$$+\frac{4}{3}K_1^4K_2^2M_4^4M_5M_6M_8^4 + \frac{1}{2}K_1^4K_2^2M_{13}M_5M_6^2M_8^4 - \frac{9}{8}K_1^4K_2^3M_{11}M_4M_6M_7M_8^4 -$$
$$-3K_1^4K_2^3M_4^3M_6M_7M_8^4 - \frac{3}{16}K_1^3K_2^2M_{11}^2M_4M_5M_8^5 + \frac{1}{2}K_1^3K_2^2M_{11}M_4^3M_5M_8^5 +$$
$$+\frac{3}{2}K_1^3K_2^2M_{13}M_4M_5M_6M_8^5 + \frac{27}{64}K_1^3K_2^3M_{11}^2M_7M_8^5 - \frac{9}{8}K_1^3K_2^3M_{11}M_4^2M_7M_8^5 -$$
$$-\frac{9}{4}K_1^3K_2^3M_{13}M_6M_7M_8^5 - \frac{3}{8}K_1^2K_2^2M_{11}M_{13}M_5M_8^6 + K_1^2K_2^2M_{13}M_4^2M_5M_8^6 -$$
$$-\frac{9}{4}K_1^2K_2^3M_{13}M_4M_7M_8^6 + \frac{3}{32}K_1K_2M_{14}M_7M_8^7 - \frac{1}{9}K_1^4K_2M_5M_6^3M_8^3M_9 -$$

$$-\frac{1}{6}K_1^2K_2M_{11}M_5M_6M_8^5M_9+\frac{1}{9}K_1^2K_2M_4^2M_5M_6M_8^5M_9-\frac{1}{24}K_1K_2M_{11}M_4M_5M_8^6M_9+$$
$$+\frac{3}{16}K_1K_2^2M_{11}M_7M_8^6M_9-\frac{1}{12}K_2M_{13}M_5M_8^7M_9-\frac{1}{54}M_5M_6M_8^6M_9^2=$$
$$=64f^{27}(64hm^4t-8hm^2nt-2hn^2t-24fm^3qt-144fhm^3t^2+36f^2m^2qt^2+$$
$$+112f^2hm^2t^3+2f^2hnt^3-18f^3mqt^3-36f^3hmt^4+3f^4qt^4+4f^4ht^5-96hm^3u-$$
$$-12hmnu+36fm^2qu+9fnqu+120fhm^2tu+24fhntu-18f^2mqtu-12f^2hmt^2u-$$
$$-12f^3ht^3u+72fhmu^2-27f^2qu^2-72f^2htu^2+48hm^2v+6hnv-18fmqv-$$
$$-36fhmtv+9f^2qtv+6f^2ht^2v-36fhuv-3fqw)x^{22}y^3,$$

$$N_7[-4]\equiv 48K_1^4K_2M_6^3+72K_1^3K_2M_4M_6^2M_8+18K_1^2K_2M_{11}M_6M_8^2+$$
$$+24K_1^2K_2M_4^2M_6M_8^2+9K_1K_2M_{11}M_4M_8^3+18K_2M_{13}M_8^4+4M_6M_8^3M_9= \qquad (19)$$
$$=-192f^{15}(4m^2t-nt-6fmt^2+2f^2t^3-6mu+6ftu+3v)x^{13}y,$$

where K_i $(i=\overline{1,17})$ and M_i $(i=\overline{0,14})$ are from (15)–(16), written without the weight and square brackets.

Theorem 1. *The stability of unperturbed motion governed by system of perturbed motion $s(1,2,3)$ from (1) with $S_1<0$, $S_1^2+2K_3=0$ and $M_8\not\equiv 0$ is characterized by one of the following sixteen possible cases:*

I. $M_1\not\equiv 0$, *then the unperturbed motion is unstable;*

II. $M_1\equiv 0$, $K_2N_1>0$, *then the unperturbed motion is unstable;*

III. $M_1\equiv 0$, $K_2N_1<0$, *then the unperturbed motion is stable;*

IV. $M_1\equiv N_1\equiv 0$, $N_2\neq 0$, *then the unperturbed motion is unstable;*

V. $M_1\equiv N_1\equiv N_2\equiv 0$, $N_3<0$, *then the unperturbed motion is unstable;*

VI. $M_1\equiv N_1\equiv N_2\equiv 0$, $N_3>0$, *then the unperturbed motion is stable;*

VII. $M_1\equiv N_1\equiv N_2\equiv N_3\equiv 0$, $N_4\not\equiv 0$, *then the unperturbed motion is stable;*

VIII. $M_1\equiv N_1\equiv N_2\equiv N_3\equiv N_4\equiv 0$, $K_2M_8N_5>0$, *then the unperturbed motion is unstable;*

IX. $M_1\equiv N_1\equiv N_2\equiv N_3\equiv N_4\equiv 0$, $K_2M_8N_5<0$, *then the unperturbed motion is stable;*

X. $M_1\equiv N_1\equiv N_2\equiv N_3\equiv N_4\equiv N_5\equiv 0$, $N_6\not\equiv 0$, *then the unperturbed motion is unstable;*

XI. $M_1\equiv N_1\equiv N_2\equiv N_3\equiv N_4\equiv N_5\equiv N_6\equiv 0$, $K_2M_5M_8<0$, $N_7\not\equiv 0$, *then the unperturbed motion is unstable;*

XII. $M_1\equiv N_1\equiv N_2\equiv N_3\equiv N_4\equiv N_5\equiv N_6\equiv 0$, $K_2M_5M_8>0$, $N_7\not\equiv 0$, *then the unperturbed motion is stable;*

XIII. $M_1\equiv N_1\equiv N_2\equiv N_3\equiv N_4\equiv N_5\equiv N_6\equiv M_5\equiv 0$, $M_7N_7\not\equiv 0$, *then the unperturbed motion is unstable;*

XIV. $M_1\equiv N_1\equiv N_2\equiv N_3\equiv N_4\equiv N_5\equiv N_6\equiv M_5\equiv 0$, $M_7=0$, *then the unperturbed motion is stable;*

XV. $M_1\equiv N_1\equiv N_2\equiv N_3\equiv N_4\equiv N_5\equiv N_6\equiv M_5\equiv N_7\equiv 0$, $M_7\neq 0$, *then the unperturbed motion is stable;*

XVI. $M_1 \equiv N_1 \equiv N_2 \equiv N_3 \equiv N_4 \equiv N_5 \equiv N_6 \equiv N_7 \equiv 0$, $M_5 \not\equiv 0$, *then the unperturbed motion is stable.*

In the last three cases, the unperturbed motion belongs to some continuous series of stabilized motions. Moreover, for sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. The expressions K_2, M_1, M_5, M_7, M_8 *and* N_i $(i = \overline{1,7})$ *are from* (15)*,* (16) *and* (19)*, respectively.*

The proof of Theorem 1 follows from Lemma 1, using the expressions S_1 from (14), K_3 from (15), M_8 from (18) and those from (19). An important role in proving the theorem is played by the weights of these comitants and invariants, the properties of which are mentioned in Remark 1.

If in system (2) we suppose $l = 0$, then for M_1 from (16) we get (18). But if we put $l = g = 0$ in system (2), then for the expressions from (16) we have

$$\begin{gathered} K_2[-1] = -fxy, \quad M_3[-1] = 6f^3ky^3, \quad M_5[0] = 6fhy, \quad M_6[-1] = 2f^4tx^4, \\ M_7[1] = 2f^2q, \quad M_8[-1] = 2f^3x^3, \quad M_{10}[0] = 8f^3ry^2, \\ M_{12}[-1] = 2f^4sy^4, \quad M_{15}[0] = 8f^3p\,x^2. \end{gathered} \tag{20}$$

In this case we use the expressions S_1 from (14), K_3 from (15), M_8 from (18) and (19). Then by Lemma 2 and taking into account the weights of the comitants characterized in Remark 1, we have the following theorem.

Theorem 2. *The stability of the unperturbed motion governed by system of perturbed motion* $s(1,2,3)$ *from* (1) *with* $S_1 < 0$*,* $S_1^2 + 2K_3 = 0$ *and* $M_8 \equiv 0$ *is characterized by one of the following thirteen possible cases:*

I. $M_1 \not\equiv 0$, *then the unperturbed motion is unstable;*

II. $M_1 \equiv 0$, $M_{15} < 0$, *then the unperturbed motion is unstable;*

III. $M_1 \equiv 0$, $M_{15} > 0$, *then the unperturbed motion is stable;*

IV. $M_1 \equiv M_{15} \equiv 0$, $M_5M_6 \not\equiv 0$, *then the unperturbed motion is unstable;*

V. $M_1 \equiv M_{15} \equiv M_5 \equiv 0$, $M_6M_7 > 0$, *then the unperturbed motion is unstable;*

VI. $M_1 \equiv M_{15} \equiv M_5 \equiv 0$, $M_6M_7 < 0$, *then the unperturbed motion is stable;*

VII. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv 0$, $M_3M_6 \not\equiv 0$, *then the unperturbed motion is unstable;*

VIII. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv 0$, $M_6 \not\equiv 0$, $M_{10} < 0$, *then the unperturbed motion is unstable;*

IX. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv 0$, $M_6 \not\equiv 0$, $M_{10} > 0$, *then the unperturbed motion is stable;*

X. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv M_{10} \equiv 0$, $M_6M_{12} > 0$, *then the unperturbed motion is unstable;*

XI. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv M_{10} \equiv 0$, $M_6M_{12} < 0$, *then the unperturbed motion is stable;*

XII. $M_1 \equiv M_{15} \equiv M_5 \equiv M_7 \equiv M_3 \equiv M_{10} \equiv M_{12} \equiv 0$, $M_6 \not\equiv 0$, *then the unperturbed motion is stable;*

XIII. $M_1 \equiv M_{15} \equiv M_6 \equiv 0$, *then the unperturbed motion is stable.*

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motions. Moreover, for sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. The expressions $S_1, K_3, M_1, M_3, M_5, M_6, M_7, M_{10}, M_{12}, M_{15}$ *are from* (14), (15) *and* (16).

Acknowledgments. This research was partially supported by grants 15.817.02.03F, 15.817.02.18F and 16.80012.02.01F.

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Received September 6, 2019

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