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The problem of the center for cubic differential systems with the line at infinity and an affine real invariant straight line of total algebraic multiplicity five

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Abstract. In this article, we study the real planar cubic differential systems with a non-degenerate monodromic critical point M_0. In the cases when the algebraic multiplicity $m(Z) = 5$ or $m(l_1) + m(Z) \geq 5$, where $Z = 0$ is the line at infinity and $l_1 = 0$ is an affine real invariant straight line, we prove that the critical point M_0 is of the center type if and only if the first Lyapunov quantity vanishes. More over, if $m(Z) = 5$ (respectively, $m(l_1) + m(Z) \geq 5$, $m(l_1) \geq j$, $j = 2, 3$) then M_0 is a center if the cubic systems have a polynomial first integral (respectively, an integrating factor of the form $1/l_1^j$).

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1 Introduction and statement of main results

We consider the real polynomial differential systems

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad \gcd(P, Q) = 1 \tag{1}$$

and the vector fields $\mathbb{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$ associated to systems (1).

Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 2$ ($n = 3$) then the system (1) is called quadratic (cubic).

An algebraic curve $f(x, y) = 0$, $f \in \mathbb{C}[x, y]$ (a function $f = \exp(g/h)$, $g, h \in \mathbb{C}[x, y]$) is called an invariant algebraic curve (exponential factor) of the system (1) if there exists a polynomial $K_f \in \mathbb{C}[x, y]$, $\deg(K) \leq n - 1$ such that the identity $\mathbb{X}(f) \equiv f(x, y)K_f(x, y)$, $(x, y) \in \mathbb{R}^2$ $((x, y) \in \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 \,|\, h(x, y) = 0\})$ holds. In particular, a straight line $l \equiv \alpha x + \beta y + \gamma = 0$, $\alpha, \beta, \gamma \in \mathbb{C}$, is invariant for (1) if there exists a polynomial $K_l \in \mathbb{C}[x, y]$ such that the identity $\alpha P(x, y) + \beta Q(x, y) \equiv (\alpha x + \beta y + \gamma)K_l(x, y)$, $(x, y) \in \mathbb{R}^2$, holds. The polynomial $K_f(x, y)$ is called the cofactor of the invariant algebraic curve (exponential factor) f. If m is the greatest natural number such that f^m divides $\mathbb{X}(f)$, then we say that f has parallel multiplicity m.

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An invariant algebraic curve f of degree d for the vector field $\mathbb{X}$ has algebraic multiplicity k when k is the greatest positive integer such that the k-th power of f divides $E_d(\mathbb{X})$, where

$$E_d(\mathbb{X}) = det \begin{pmatrix} v_1 & v_2 & ... & v_l \\ \mathbb{X}(v_1) & \mathbb{X}(v_2) & ... & \mathbb{X}(v_l) \\ ... & ... & ... & ... \\ \mathbb{X}^{l-1}(v_1) & \mathbb{X}^{l-1}(v_2) & ... & \mathbb{X}^{l-1}(v_l) \end{pmatrix},$$

and $v_1, v_2, ..., v_l$ is a basis of $\mathbb{C}_d[x, y]$ [6]. If $d = 1$ then $v_1 = 1$, $v_2 = x$, $v_3 = y$ and $E_1(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P)$.

The polynomial $E_d(\mathbb{X})$ has in x and y the degree $d(d+1)(d+2)[8+3(d+3)(n-1)]/24$ (see [16]). In the case of cubic systems ($n = 3$) and invariant straight lines we have $\deg(E_1(\mathbb{X})) = 8$.

Denote by $m(f)$ the algebraic multiplicity of the algebraic curve f.

We say that an invariant affine straight line f (respectively, the line at infinity $Z = 0$) of a cubic vector field $\mathbb{X}$ has geometric multiplicity m if there exists a sequence of cubic vector fields $(\mathbb{X}_r)_{r\geq 1}$ converging to $\mathbb{X}$, such that each $\mathbb{X}_r$ has m (respectively, $m-1$) distinct invariant affine straight lines $f_{1r} = 0, ..., f_{mr} = 0$, converging to f = 0 as $r \to \infty$, and this does not occur for m + 1 (respectively, m).

Let $f_1, \ldots, f_r$ $(f_{r+1} = \exp(g_{r+1}/h_{r+1}), \ldots, f_s = \exp(g_s/h_s))$ be invariant algebraic curves (exponential factors) of (1) and let K_{f_j}, $j = \overline{1, s}$, be its cofactors. The system (1) is called *Darboux integrable* if (1) has a first integral (an integrating factor) of the form $F(x, y) = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$ $(\mu(x, y) = f_1^{\alpha_1} \cdots f_s^{\alpha_s})$, $\alpha_j \in \mathbb{C}$, $j = \overline{1, s}$ (on the theory of Darboux, presented in the context of planar polynomial differential systems on the affine plane, see [22]). Note that the constants $\alpha_1, \ldots, \alpha_s$ are not all equal to zero.

It is easy to show that $F(x, y)$ $(\mu(x, y))$ is a Darboux first integral (a Darboux integrating factor) if and only if the following identity

$$\alpha_1 K_{f_1} + \alpha_2 K_{f_2} + \cdots + \alpha_s K_{f_s} \equiv 0$$
$$\left(\alpha_1 K_{f_1} + \alpha_2 K_{f_2} + \cdots + \alpha_s K_{f_s} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \equiv 0\right)$$

holds in x and y.

By present a great number of works have been dedicated to the investigation of polynomial differential systems with invariant straight lines.

The problem of estimating the number of invariant straight lines which a polynomial differential system can have, was considered in [2]; the problem of coexistence of the invariant straight lines and limit cycles was examined in $\{[21] : n = 2\}$, $\{[11], n = 3\}$, [10].

The classification of all cubic systems with the maximum number of invariant straight lines, including the line at infinity, and taking into account their multiplicities, is given in [13].

In [2] it was proved that the cubic system (1) can have at most eight affine invariant straight lines. The cubic systems with exactly eight and exactly seven distinct affine invariant straight lines have been studied in [13], [15]; with invariant straight lines of total geometric (parallel) multiplicity eight (seven) – in [3], [4], [5] ([28]), and with six real invariant straight lines along two (three) directions – in [17], [18]. The cubic systems with degenerate infinity and invariant straight lines of total parallel multiplicity six and total parallel multiplicity five were investigated in [19], [26], [27]. In [30] it was shown that in the class of cubic differential systems the maximal multiplicity of an affine real straight line (of the line at infinity) is seven. The cubic systems with two affine real non-parallel invariant straight lines of maximal multiplicity are classified in [31].

In this work we consider the cubic systems of the form

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \\ \qquad \gcd(P,Q) = 1. \end{cases} \tag{2}$$

We suppose that the infinity is not degenerate for (2), i.e. $\kappa(x,y) = yP_3(x,y) - xQ_3(x,y) \not\equiv 0$, where $P_3(x,y) = kx^3 + mx^2y + pxy^2 + ry^3$, $Q_3(x,y) = -(sx^3 + qx^2y + nxy^2 + ly^3)$.

The critical point $(0,0)$ of system (2) is either a focus or a center, i.e. is monodromic. The problem of distinguishing between a center and a focus is called *the problem of the center* or *the center-focus problem.*

A critical point $(0,0)$ is a center for (2) if and only if the system has a nonconstant analytic first integral $F(x,y)$ (an analytic integrating factor of the form $\mu(x,y) = 1 + \sum \mu_j(x,y)$) in a neighborhood of $(0,0)$ [14] ([1]).

It is known there exists a formal power series $F(x,y) = x^2 + y^2 + \sum_{j\geq 3} F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (2) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$, i.e. $\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j$. The quantities L_j, $j = \overline{1,\infty}$, are polynomials with respect to the coefficients of system (2) called to be *the Lyapunov quantities.* For example, the first Lyapunov quantity looks as

$$L_1 = (bd - ac + 2bf - 2ag + dg - cf + 3k - 3l + p - q)/4.$$

The origin $(0,0)$ is a center for (2) if and only if $L_j = 0$, $j = \overline{1,\infty}$.

The problem of the center is completely solved for quadratic systems ($k = l = m = n = p = q = r = s = 0$) [9] and for symmetric cubic systems ($a = b = c = d = f = g = 0$) [24]. For other polynomial differential systems the necessary and sufficient conditions for the monodromic critical point to be a center were obtained in some particular cases (see, for example, [7], [20]).

The problem of coexistence in cubic systems of the distinct invariant straight lines and critical points of center type was studied in [7], [8], [25]. In [8] (see also [7]) it was proved that if the cubic system (2) has four distinct invariant straight lines of the form $1 + \alpha_j x + \beta_j y = 0$, $j = 1,2,3,4$ $(y \pm ix = 0, 1 + \alpha_j x + \beta_j y = 0,\ j = 1,2)$ and the Lyapunov quantity vanishes: $L_1 = 0$ $(L_1 = L_2 = 0)$, then the origin is a center.

In this article we investigate the problem of the center for (2) with two invariant straight lines of total multiplicity five, including the line at infinity. Our main result is the following one:

Main Theorem. *Let the cubic system have a non-degenerate monodromic critical point $M_0(x_0, y_0)$, i.e. the eigenvalues $\lambda_{1,2}$ are purely imaginary. If $m(Z) = 5$ or $m(Z) \geq 2, m(l_1) + m(Z = 0) \geq 5$, where $Z = 0$ is the line at infinity and $l_1 = 0$ is an affine real invariant straight line, then M_0 is a center if and only if the first Lyapunov quantity vanishes ($L_1 = 0$). Moreover, if $m(Z) = 5$ (respectively, $m(l_1) + m(Z) \geq 5$, $m(l_1) \geq j$, $j = 2, 3$) then M_0 is a center if the cubic system has a polynomial first integral (respectively, an integrating factor of the form $1/l_1^j$).*

2 Cubic systems with a non-degenerate monodromic critical point and the line at infinity $Z = 0$ of multiplicity $m(Z) \geq 5$

Remark 1. Via an affine transformation of coordinates and time rescaling any cubic system with a non-degenerate monodromic critical point can be written in the form (2).

In this section we will determine the conditions under which the cubic system (2) has the line at infinity of algebraic multiplicity two (respectively: three, four, five) and in the case of configuration a) from Fig. 2.1 we will solve the problem of the center.

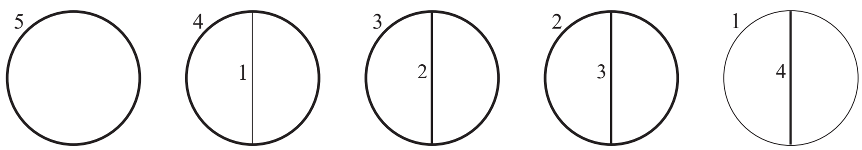

Fig. 2.1

2.1 Classification of cubic systems (2) with multiple line at infinity

We consider the homogenized system of (2):

$$\begin{cases} \dot{x} = yZ^2 + ax^2Z + cxyZ + fy^2Z + kx^3 + mx^2y + pxy^2 + ry^3, \\ \dot{y} = -(xZ^2 + gx^2Z + dxyZ + by^2Z + sx^3 + qx^2y + nxy^2 + ly^3). \end{cases} \tag{3}$$

For (3), $E_1(\mathbb{X})$ looks as

$$E_1(\mathbb{X}) = C_0(x, y) + C_1(x, y)Z + C_2(x, y)Z^2 + ... + C_8(x, y)Z^8,$$

where $C_i(x, y), i = \overline{0, 8}$, are polynomials in x and y. If $C_i(x, y) \equiv 0, i = \overline{0, j}$, then the algebraic multiplicity of the line at infinity $Z = 0$ is $j + 2$.

We have $C_0(x, y) = C_{01}(x, y)C_{02}(x, y)$, where $C_{01}(x, y) = sx^4 + kx^3y + qx^3y + mx^2y^2 + nx^2y^2 + lxy^3 + pxy^3 + ry^4$ and $C_{02}(x, y) = (kq - ms)x^4 + 2(kn - ps)x^3y +$

$(3kl + mn - pq - 3rs)x^2y^2 + 2(lm - qr)xy^3 + (lp - nr)y^4$. If $C_{01}(x, y) \equiv 0$ then the infinity is degenerate. Let $C_{01}(x, y) \not\equiv 0$. The identity $C_{02}(x, y) \equiv 0$ gives us the following five series of conditions:

$$k = m = n = p = q = s = 0,\ r \neq 0; \tag{4}$$

$$k = m = p = r = 0; \tag{5}$$

$$k = m = q = s = 0,\ l = \frac{nr}{p}; \tag{6}$$

$$k = s = 0,\ l = \frac{qr}{m},\ n = \frac{pq}{m}; \tag{7}$$

$$l = \frac{rs}{k},\ n = \frac{ps}{k}, q = \frac{ms}{k}. \tag{8}$$

Solving the identities $C_1(x, y) \equiv 0$, $\{C_1(x, y) \equiv 0,\ C_2(x, y) \equiv 0\}$ and $\{C_1(x, y) \equiv 0,\ C_2(x, y) \equiv 0,\ C_3(x, y) \equiv 0\}$ for each set of conditions (4)−(8), we obtain the following three Lemmas, respectively.

Lemma 1. *The line at infinity $Z = 0$ of the system (2) has algebraic multiplicity at least three if and only if one of the following ten series of conditions holds:*

$$a = c = f = k = m = p = r = 0; \tag{9}$$

$$c = aq/s,\ f = an/s,\ k = l = m = p = r = 0; \tag{10}$$

$$a = 0,\ c = fq/n,\ k = l = m = p = r = s = 0; \tag{11}$$

$$a = f = k = l = m = n = p = r = s = 0,\ q \neq 0; \tag{12}$$

$$a = gp/n,\ c = dp/n,\ k = 0,\ l = p,\ m = q = 0,\ r = p^2/n,\ s = 0; \tag{13}$$

$$b = fq/m,\ d = cq/m,\ g = aq/m,\ k = 0,\ l = qr/m,\ n = pq/m,\ s = 0,\ q \neq 0; \tag{14}$$

$$\begin{gathered} d = cq/m + m(bm - fq)/(qr),\ g = aq/m,\ k = 0,\ l = qr/m, \\ n = m + q^2r/m^2,\ p = m^2/q + qr/m,\ s = 0; \end{gathered} \tag{15}$$

$$b = fq/m,\ g = aq/m,\ k = l = 0,\ n = m,\ p = m^2/q,\ r = s = 0; \tag{16}$$

$$b = fs/k,\ d = cs/k,\ g = as/k,\ l = rs/k,\ n = ps/k,\ q = ms/k; \tag{17}$$

$$\begin{gathered} b = (fk^2s + gkrs - ars^2)/k^3,\ d = s(ck^3 + gk^2p - akps - gkrs + ars^2)/k^4, \\ l = rs/k,\ m = (k^4 + kps^2 - rs^3)/(k^2s),\ n = ps/k, q = (k^4 + kps^2 - rs^3)/k^3. \end{gathered} \tag{18}$$

Lemma 2. *The line at infinity $Z = 0$ of the system (2) has algebraic multiplicity at least four if and only if one of the following seven series of conditions holds:*

$$a = c - b = f = g = k = l = m = n = p = r = s = 0, q \neq 0; \tag{19}$$

$$a = c = f = k = l = m = n = p = r = s = 0,\ q \neq 0; \tag{20}$$

$$\begin{gathered} b = c = aq/s, d = (agq + a^2s + s^2)/(as), \\ f = k = l = m = n = p = r = 0; \end{gathered} \tag{21}$$

$$a=-m^3/(q(fq-bm)), c=(-m^4-b^2mq^2+bfq^3)/(q^2(-bm+fq)),$$
$$d=(m^5+2b^2m^2q^2-3bfmq^3+f^2q^4)/(m^2q(bm-fq)), \quad (22)$$
$$g=m^2/(bm-fq), k=0, l=m^2/q, n=2m, p=2m^2/q, r=m^3/q^2, s=0;$$

$$b=fs/k, d=cs/k, g=as/k, l=-k, m=(2k^2-s^2)/s,$$
$$n=(k^2-2s^2)/s, p=k(k^2-2s^2)/s^2, q=(2k^2-s^2)/k, r=-k^2/s; \quad (23)$$

$$b=s(k^4-agkrs+k^2rs+a^2rs^2)/\delta,$$
$$c=(g^2k^6-3agk^5s+2a^2k^4s^2+k^4s^3-agkrs^4+k^2rs^4+a^2rs^5)/(s^2\delta),$$
$$d=-(agk^4-a^2k^3s-k^3s^2+g^2krs^2-agrs^3-krs^3)/\delta, \quad (24)$$
$$f=k(k^4+g^2k^2r-3agkrs+k^2rs+2a^2rs^2)/\delta,$$
$$l=rs/k,\ m=(2k^4+rs^3)/(k^2s),\ n=(k^4+2rs^3)/(k^2s),$$
$$p=(k^4+2rs^3)/(ks^2),\ q=(2k^4+rs^3)/k^3,\ \delta=k^3(as-gk).$$

Lemma 3. *In the class of cubic differential systems* (2) *the maximal multiplicity of the line at infinity is five. Modulo the transformations $x\to y, y\to x, t\to -t$, the coefficients of any cubic system which has the line at infinity of multiplicity 5 satisfies one of the following six sets of conditions:*

$$a=b=c=f=g=k=l=0,\ m=n=p=r=s=0,\ q\neq 0; \quad (25)$$

$$b=-as/k,\ c=a(k^2-s^2)/(ks),\ d=a(k^2-s^2)/k^2,$$
$$f=-a, g=as/k,\ l=-k,\ m=(2k^2-s^2)/s,\ n=(k^2-2s^2)/s, \quad (26)$$
$$p=k(k^2-2s^2)/s^2,\ q=(2k^2-s^2)/k,\ r=-k^2/s;$$

$$b=c=0,\ d=2a,\ f=k=l=m=n=p=q=r=0,\ s=a^2,\ a\neq 0; \quad (27)$$

$$a=0,\ b=-gk^2/s^2,\ c=-2gk^2/s^2,\ d=0,\ f=-2gk^3/s^3,\ l=k^3/s^2,$$
$$m=n=3k^2/s,\ p=3k^3/s^2,\ q=3k,\ r=k^4/s^3,\ g^2k^2-k^2s-s^3=0; \quad (28)$$

$$c=2b=-2as/k,\ d=2a,\ f=-a(k^2+2s^2)/s^2,\ g=(k^2+a^2s)/(ak),$$
$$m=n=3k^2/s,\ p=3l=3k^3/s^2, q=3k,\ r=k^4/s^3,\ k^4-a^2k^2s-a^2s^3=0; \quad (29)$$

$$b=k(-agk+k^2+a^2s+s^2)/(s(as-gk)),\ c=2k(2as-gk)/s^2,$$
$$d=2a,\ f=k^2(3as-2gk)/s^3,\ l=k^3/s^2,\ m=3k^2/s,\ n=3k^2/s, \quad (30)$$
$$p=3k^3/s^2,\ q=3k,\ r=k^4/s^3,\ g^2k^2-2agks-k^2s+a^2s^2-s^3=0.$$

2.2 Integrability of the cubic systems {(2),(25)}–{(2),(30)}

For system {(2),(25)} (respectively, {(2),(26)}) the first Lyapunov quantity $L_1=-q/4$ (respectively, $L_1=(k^2+s^2)^2/(4ks^2)$) does not vanish, therefore {(2),(25)} (respectively, {(2),(26)}) has a focus at (0,0).

The divergence of the vector field associated to the systems {(2),(27)} − {(2),(30)} is identically zero, i.e. $\frac{\partial P(x,y)}{\partial x}+\frac{\partial Q(x,y)}{\partial y}\equiv 0$, and {(2),(27)} − {(2),(30)} have the following first integrals, respectively:

$$\begin{aligned}
&F(x,y)=6(x^2+y^2)+4gx^3+12ax^2y+3a^2x^4;\\
&F(x,y)=6s^3(x^2+y^2)+4g(sx-2ky)(sx+ky)^2+3(sx+ky)^4;\\
&F(x,y)=6aks^3(x^2+y^2)+4s^3(k^2+a^2s)x^3+12a^2s^3xy(kx-sy)\\
&\qquad -4a^2ks(k^2+2s^2)y^3+3ak(sx+ky)^4;\\
&F(x,y)=2(as-gk)(3s^3(x^2+y^2)+2gs^3x^3+6as^3x^2y+6aks^2xy^2-4gk^3y^3\\
&\qquad +6ak^2sy^3)+12ks^2(k^2+s^2)xy^2-3(gk-as)(sx+ky)^4.
\end{aligned}$$

Therefore, the following theorem holds:

Theorem 1. *Cubic differential systems* (2) *with the line at infinity of algebraic multiplicity five have a center at origin* $(0,0)$ *if and only if the divergence of the vector fields associated to these systems vanishes or, equivalently, each of these systems has a polynomial first integral.*

3 Cubic systems (2) with the line at infinity $Z=0$ and an affine real invariant straight line l_1 of multiplicities $m(Z)\geq 4$, $m(l_1)\geq 1$

In this section we will solve the problem of the center for cubic systems (2) in the case of configuration b) of Fig. 2.1. For systems {(2),(19)}−{(2),(24)} we will determine the conditions for the existence of an invariant real affine straight line l_1 and under these conditions the problem of the center will be solved.

Denote

$$\begin{aligned}
&X_1(x)=A\cdot P(x,Ax+B)-Q(x,Ax+B),\\
&Y_1(y)=P(Ay+B,y)-A\cdot Q(Ay+B,y),\\
&X_2(x)=\Big(E_1(\mathbb{X})/(y-Ax-B)\Big)\Big|_{y=Ax+B},\\
&X_3(x)=\Big(E_1(\mathbb{X})/(y-Ax-B)^2\Big)\Big|_{y=Ax+B},\\
&Y_2(y)=\Big(E_1(\mathbb{X})/(x-Ay-B)\Big)\Big|_{x=Ay+B},\\
&Y_3(y)=\Big(E_1(\mathbb{X})/(x-Ay-B)^2\Big)\Big|_{x=Ay+B}.
\end{aligned}$$

The straight line $l_1\equiv y-Ax-B=0$ $\;(l_1\equiv x-Ay-B=0)$ is invariant for cubic systems (2) if and only if the identity $X_1(x)\equiv 0$ $\;(Y_1(y)\equiv 0)$ holds. In particular, the straight line $y-\gamma=0$ $(x-\gamma=0)$ is invariant for (2) if $Q(x,\gamma)\equiv 0$ $(P(\gamma,y)\equiv 0)$. The invariant straight line $y-Ax-B=0$ has algebraic multiplicity two (three) if $X_2(x)\equiv 0$ $\;(\{X_2(x)\equiv 0, X_3(x)\equiv 0\})$. Analogously, the invariant straight line $x-Ay-B=0$ has algebraic multiplicity two (three) if $Y_2(y)\equiv 0$ $(\{Y_2(y)\equiv 0, Y_3(y)\equiv 0\})$.

System {(2),(19)}. This system has a single invariant straight line $bx+1=0$.

System {(2),(20)}. Because $\kappa(x,y)=qx^3y$ and $P(\gamma,y)\not\equiv 0$, $\forall\gamma\in\mathbb{R}$, the system {(2),(20)} can have only invariant straight lines parallel to the axis Ox of coordinates. The identity $Q(x,\gamma)\equiv 0$ implies $\gamma=-1/d$,

$$b=0,\ g=q/d \tag{31}$$

and the system {(2),(20)} with conditions (31) has the invariant straight line $l_1 = dy + 1$.

System {(2),(21)}.

We have $\kappa(x, y) = x^3(sx + qy)$, $P(\gamma, y) = (sy + aqy\gamma + as\gamma^2)/s \not\equiv 0$, $\forall \gamma \in \mathbb{C}$. If the straight line is described by the equation $x = -(qy + \gamma)/s$, then $Y_1(y) = (a\gamma(qs - gq\gamma + as\gamma + q\gamma^2) + y(aq^2s + as^3 - agq^2\gamma + 2a^2qs\gamma + qs^2\gamma + 2aq^2\gamma^2) + q^2y^2(s^2 + aq\gamma))/(as^2)$. The identity $Y_1(y) \equiv 0$ gives us

$$g = (a^2s^2 - a^2q^2 - s^3)/(aqs). \tag{32}$$

Under the conditions $\{(21), (32)\}$ the system $\{(2)\}$ has the invariant straight line $l_1 = aqsx + aq^2y - s^2$.

System {(2),(22)}.

In this case $\kappa(x, y) = y(qx + my)^3/q^2$ and $X_1(x) = -(\gamma(mq + bm\gamma - fq\gamma) + [m^3 + mq^2 + bmq\gamma - fq^2\gamma]x)/m^2 \not\equiv 0$ if $y = (-qx - \gamma)/m$. The identity $Q(x, \gamma) = (m^2(bm - fq)\gamma^2(bq + m^2\gamma) + [bm^3q - fm^2q^2 + m^5\gamma + 2b^2m^2q^2\gamma - 3bfmq^3\gamma + f^2q^4\gamma + 2bm^4q\gamma^2 - 2fm^3q^2\gamma^2]x + m^2q[m^2 + bmq\gamma - fq^2\gamma]x^2)/(m^2q(fq - bm)) \equiv 0$ yields $\gamma = -bq/m^2$,

$$f = m(b^2q^2 - m^3)/(bq^3) \tag{33}$$

and the invariant straight line is $l_1 = bq + m^2y$.

System {(2),(23)}.

We have $\kappa(x, y) = (sx+ky)^3(kx-sy)/(ks^2)$ and $X_1(x) = -([k^2+s^2]x+s\gamma)/k \not\equiv 0$ if $y = -(sx + \gamma)/k$. If $y = (kx + \gamma)/s$, then $X_1(x) = -(\gamma(-k^2s^3 - fk^2s^2\gamma - fs^4\gamma + k^4\gamma^2 + k^2s^2\gamma^2) + [(k^2 + s^2)(-ks^3 - 2fks^2\gamma - cs^3\gamma + 2k^3\gamma^2 + 2ks^2\gamma^2)]x - [(k^2 + s^2)(fk^2s^2 + cks^3 + as^4 - k^4\gamma - 2k^2s^2\gamma - s^4\gamma)]x^2)/(ks^4)$ and the identity $X_1(x) \equiv 0$ gives us $\gamma = s^2(fk^2s^2 + cks + as^2)/(k^2 + s^2)^2$,

$$\begin{aligned} &a = -(2k^4 + f^2k^2s + 2k^2s^2 + f^2s^3 \pm fs^{1/2}\Delta)/((fk^2s^{1/2} + fs^{5/2} \pm \Delta)s^{1/2}),\\ &c = (3fs^{5/2} - fk^2s^{1/2} \mp \Delta)/(2ks^{3/2}), \end{aligned} \tag{34}$$

where $\Delta = \sqrt{(k^2 + s^2)(4k^4 + f^2k^2s + f^2s^3)}$. Under the conditions {(23), (34)} the system (2) has the affine invariant straight lines $l_1 = -2k^2s^{3/2} + fk^3s^{1/2}x + fks^{5/2}x \pm k\Delta x - fk^2s^{3/2}y - fs^{7/2}y \mp s\Delta y$.

System {(2),(24)}

has $\kappa(x, y) = (k^3x+rs^2y)(sx+ky)^3/(k^3s^2)$ and for it the straight line $k^3x+rs^2y+\gamma = 0$ is invariant if $Y_1(y) = (s^2(as - gk)\gamma(-k^6rs^2 - ak^6\gamma + gk^3rs^2\gamma + k^4\gamma^2 - rs^3\gamma^2) + y(gk^13s^2 - ak^12s^3 + gk^7r^2s^6 - ak^6r^2s^7 + g^2k^12\gamma - 3agk^11s\gamma + 2a^2k^10s^2\gamma + k^10s^3\gamma + 2agk^7rs^4\gamma + k^8rs^4\gamma - 2a^2k^6rs^5\gamma - k^6rs^6\gamma - g^2k^4r^2s^6\gamma + agk^3r^2s^7\gamma - k^4r^2s^7\gamma + 2gk^9s\gamma^2 - 2ak^8s^2\gamma^2 - 4gk^5rs^4\gamma^2 + 4ak^4rs^5\gamma^2 + 2gkr^2s^7\gamma^2 - 2ar^2s^8\gamma^2) - (k^4 - rs^3)^2y^2(k^6s^2 + k^4rs^3 + gk^5\gamma - ak^4s\gamma - gkrs^3\gamma + ars^4\gamma))/(k^9s^2(gk - as)) \equiv 0$, i.e. if

$$\begin{aligned} &a = ((k^4 - rs^3)^2\gamma^2 - k^8rs^2(k^2 + s^2))/(k^2(k^4 - rs^3)^2\gamma),\\ &g = s((k^4 - rs^3)^2\gamma^2 - k^6s(k^6 + 2k^4rs - r^2s^4))/(k^3(k^4 - rs^3)^2\gamma). \end{aligned} \tag{35}$$

In the case of the straight line of the form $sx + ky + \gamma = 0$ the identity $X_1(x) = -(s^2\gamma(k^3 + gkr\gamma - ars\gamma) + x(k^5s + k^3s^3 - (gk - as)(k^4 - rs^3)\gamma))/(k^4s) \equiv 0$ yields

$r=-k^2/s$, $\gamma=ks/(gk-as)$ and the invariant straight line is $l_1\equiv ks+(gk-as)(sx+ky)=0$.

In this way we have proved the following lemma:

Lemma 4. *The systems* (2) *with an affine real invariant straight line have the line at infinity $Z=0$ of multiplicity at least four if and only if one of the following eight sets of conditions holds:*

$$a=c-b=f=g=k=l=m=n=p=r=s=0, q\neq 0; \tag{36}$$

$$a=b=c=f=k=l=m=n=p=r=s=0,\ g=q/d,\ q\neq 0; \tag{37}$$

$$\begin{array}{l} d=a(2s^2-q^2)/s^2,\ g=(a^2s^2-a^2q^2-s^3)/(aqs),\\ b=c=aq/s,\ f=k=l=m=n=p=r=0; \end{array} \tag{38}$$

$$\begin{array}{l} a=bq/m, c=2b,\ d=(m^3+2b^2q^2)/(bmq), f=(-m^4+b^2mq^2)/(bq^3),\\ g=bq^2/m^2, k=0, l=m^2/q, n=2m,\ p=2m^2/q,\ r=m^3/q^2,\ s=0; \end{array} \tag{39}$$

$$\begin{array}{l} a=(-k^4s^2-f^2k^2s^3/2-k^2s^4-f^2s^5/2\pm fs^{5/2}\Delta/2)/(fk^2s^3/2+fs^5/2\\ \mp s^{5/2}\Delta/2), b=fs/k,\ c=(-fk^3s^2+3fks^4\pm ks^{3/2}\Delta)/(2k^2s^3),\\ d=(-fk^2s^{1/2}+3fs^{5/2}\pm\Delta)/(2k^2s^{1/2}), g=(fs(s^2-k^2)\pm s^{1/2}\Delta)/(2k^3),\\ l=-k,\ m=(2k^2-s^2)/s, n=(k^2-2s^2)/s,\ p=k(k^2-2s^2)/s^2,\\ q=(2k^2-s^2)/k,\ r=-(k^2/s), \Delta=\sqrt{(k^2+s^2)(4k^4+f^2k^2s+f^2s^3)}\ ; \end{array} \tag{40}$$

$$\begin{array}{l} a=((k^4-rs^3)^2\gamma^2-k^8rs^2(k^2+s^2))/(k^2(k^4-rs^3)^2\gamma),\\ b=((k^4-rs^3)^2\gamma^2-k^4r^2s^5(k^2+s^2))/(ks(k^4-rs^3)^2\gamma),\\ c=(2(k^4-rs^3)^2\gamma^2+k^4s(k^8-2k^4rs^3-2k^2r^2s^4-r^2s^6))/(ks(k^4-rs^3)^2\gamma),\\ d=(2(k^4-rs^3)^2\gamma^2-k^2rs^2(k^8+2k^6s^2+2k^4rs^3-r^2s^6))/(k^2(k^4-rs^3)^2\gamma),\\ f=((k^4-rs^3)^2\gamma^2+k^2rs^4(k^6-2k^2rs^3-r^2s^4))/(s^2(k^4-rs^3)^2\gamma),\\ g=s((k^4-rs^3)^2\gamma^2-k^6s(k^6+2k^4rs-r^2s^4))/(k^3(k^4-rs^3)^2\gamma),\\ l=rs/k,\ m=(2k^4+rs^3)/(k^2s),\ n=(k^4+2rs^3)/(k^2s),\\ p=(k^4+2rs^3)/(ks^2),\ q=(2k^4+rs^3)/k^3; \end{array} \tag{41}$$

$$\begin{array}{l} b=-as/k,\ c=(2ak^2s-gk^3-as^3)/(ks^2),\ d=(ak-gs)/k,\\ f=(gk-2as)/s,\ l=-k,\ m=(2k^2-s^2)/s,\ n=(k^2-2s^2)/s,\\ p=k(k^2-2s^2)/s^2,\ q=(2k^2-s^2)/k,\ r=-k^2/s. \end{array} \tag{42}$$

3.1 Integrability of the cubic systems {(2),(36)}–{(2),(42)}

If for system {(2),(36)} the first Lyapunov quantity $L_1=(bd-q)/4$ vanishes, i.e. $q=bd$, then {(2),(36)} has integrating factor

$$\mu(x,y)=(1+bx)\exp[-d(3dx^2+2bdx^3+6y+6bxy)/6].$$

The system {(2),(37)} has the first integral

$$F(x,y)=\exp(3d^2x^2+2dqx^3+6dy)/(1+dy)^6],$$

and, therefore, for each of the systems $\{(2),(36),\ q=bd\}$ and $\{(2),(37)\}$ the critical point $(0,0)$ is a center.

The *systems* $\{(2),(38)\}$, $\{(2),(39)\}$, $\{(2),(40)\}$ and $\{(2),(41)\}$ are integrable and they have the following integrating factors, respectively:

$$\mu(x,y) = 1/(-s^2 + aqsx + aq^2y),$$

$$\mu(x,y) = 1/(bq + m^2y),$$

$$\mu(x,y) = 1/(-2k^2s^{3/2} + fk^3s^{1/2}x + fks^{5/2}x \mp k\Delta x - fk^2s^{3/2}y - fs^{7/2}y \pm s\Delta y),$$

where $\Delta = \sqrt{(k^2+s^2)(4k^4+f^2k^2s+f^2s^3)}$,

$$\mu(x,y) = 1/(k^3x + rs^2y + \gamma).$$

System $\{(2),(42)\}$. We consider the function

$$\mu(x,y) = l_1^{\alpha_1} \cdot \exp[\alpha_2G_2 + \alpha_3G_3 + \alpha_4G_4],$$

where

$$\begin{aligned}
l_1 &= ks + (gk-as)(sx+ky),\\
G_2 &= sx+ky,\\
G_3 &= 2(as-gk)y + (sx+ky)^2,\\
G_4 &= 3s(g^2k^2 - 3agks + k^2s + 2a^2s^2)x + 3ksx(gk-as)(sx+ky) + k^2(sx+ky)^3
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1 = {} & (g^4k^6 - 6ag^3k^5s + 2g^2k^6s + 13a^2g^2k^4s^2 - 6agk^5s^2 + k^6s^2 - 12a^3gk^3s^3\\
& +4a^2k^4s^3 + 4a^4k^2s^4 + a^2g^2k^2s^4 - 2agk^3s^4 + k^4s^4 - 2a^3gks^5\\
& +2a^2k^2s^5 + a^4s^6)/(s^2(gk-as)^4),\\
\alpha_2 = {} & (4ag^3k^8s - g^4k^9 + g^6k^7s + g^2k^9s - 8ag^5k^6s^2 - 5a^2g^2k^7s^2 - 4agk^8s^2\\
& +k^9s^2 + 26a^2g^4k^5s^3 - g^6k^5s^3 + 2a^3gk^6s^3 + ag^3k^6s^3 + 3a^2k^7s^3 - g^2k^7s^3\\
& -44a^3g^3k^4s^4 + 6ag^5k^4s^4 - 2a^2g^2k^5s^4 - g^4k^5s^4 + k^7s^4 + 41a^4g^2k^3s^5\\
& -14a^2g^4k^3s^5 + a^3gk^4s^5 + 3ag^3k^4s^5 + a^2k^5s^5 - 20a^5gk^2s^6 + 16a^3g^3k^2s^6\\
& -3a^2g^2k^3s^6 + 4a^6ks^7 - 9a^4g^2ks^7 + a^3gk^2s^7 + 2a^5gs^8)/(k^4s^3(as-gk)^3),\\
\alpha_3 = {} & (2g^2k^7 - g^4k^5s - 6agk^6s + k^7s + 6ag^3k^4s^2 + 4a^2k^5s^2 - 13a^2g^2k^3s^3\\
& +g^4k^3s^3 - 2agk^4s^3 + k^5s^3 + 12a^3gk^2s^4 - 4ag^3k^2s^4 + 2a^2k^3s^4 - 4a^4ks^5\\
& +5a^2g^2ks^5 - 2a^3gs^6)/(2k^3s^3(gk-as)^2),\\
\alpha_4 = {} & (k^5 - g^2k^3s + 3agk^2s^2 + k^3s^2 - 2a^2ks^3 + g^2ks^3 - ags^4)/(3k^4s^3(as-gk)).
\end{aligned}$$

The first Lyapunov quantity L_1 computed for $\{(2),(42)\}$ looks as $L_1 = (k^2 + s^2)(g^2k^3 - 3agk^2s + k^3s + 2a^2ks^2 - g^2ks^2 + ags^3 + ks^3)/(4k^2s^3)$. If $L_1 = 0$ then the function $\mu(x,y)$ is an integrating factor for system (42) and the critical point $(0,0)$ is a center.

4 Cubic systems (2) with infinite line and an affine invariant real line l_1 of multiplicities $m(Z) \geq 3$, $m(l_1) \geq 2$

In this section we will solve the problem of the center for cubic systems (2) in the case of configuration c) of Fig. 2.1. For systems {(2),(9)}−{(2),(18)} we will determine the conditions for the existence of an invariant affine real straight line l_1 of multiplicity two and under these conditions the problem of the center will be solved.

Lemma 5. *Let $y - Ax - B = 0$ $(x - Ay - B = 0)$ $A, B \in \mathbb{R}$, $B \neq 0$ be an invariant straight line of the system (2). Then $X_2(x) = X_{2d}(x) \cdot \zeta(x)$ $(Y_2(y) = Y_{2d}(y) \cdot \eta(y))$, where $X_{2d}(x), \zeta(x)$ $(Y_{2d}(y), \eta(y))$ are polynomials in x (y). If $X_{2d}(x) \equiv 0$ $(Y_{2d}(y) \equiv 0)$, then $y - Ax - B$ $(x - Ay - B)$ divides $P(x, y)$ and $Q(x, y)$, i.e. the system (2) is degenerate.*

Proof. The straight line $y - Ax - B = 0$ $\Big(x - Ay - B = 0\Big)$ is invariant for system (2) if the following set of conditions holds:

$$\begin{aligned}
&b = -(A + ABf + B^2l + AB^2r)/B,\\
&d = -(1 - A^2 + ABc + AB^2l + B^2n + AB^2p + A^2B^2r)/B,\\
&g = -(-A + aAB + A^2B^2l + AB^2m + AB^2n + A^2B^2p + B^2q + A^3B^2r)/B,\\
&s = -A(k + A^2l + Am + An + A^2p + q + A^3r)
\end{aligned}$$

$$\begin{aligned}
&\Big(a = -(A + ABg + B^2k + AB^2s)/B,\\
&c = -(1 - A^2 + ABd + AB^2k + B^2m + AB^2q + A^2B^2s)/B,\\
&f = -(-A + AbB + A^2B^2k + AB^2m + AB^2n + B^2p + A^2B^2q + A^3B^2s)/B,\\
&r = -A(A^2k + l + Am + An + p + A^2q + A^3s)\Big).
\end{aligned}$$

Under these conditions we have:

$$\begin{aligned}
&X_{2d}(x) = B(1 + Bf + B^2r) + (A + Bc + 2ABf + B^2p + 3AB^2r)x+\\
&\quad (a + Ac + A^2f + Bm + 2ABp + 3A^2Br)x^2 + (k + Am + A^2p + A^3r)x^3 \equiv 0\\
&\quad \Rightarrow \{a = -B(m + Ap + A^2r),\ c = (A - B^2p - AB^2r)/B),\\
&\quad f = -(1 + B^2r)/B,\ k = -A(m + Ap + A^2r)\} \Rightarrow\\
&P(x, y) = (y - Ax - B)(-y + B(m + Ap + A^2r)x^2 + B(p + Ar)xy + Bry^2)/B,\\
&Q(x, y) = (y - Ax - B)(x - B(A^2l + An + q)x^2 - B(Al + n)xy - Bry^2)/B
\end{aligned}$$

$$\begin{aligned}
&\Big(Y_{2d}(y) = B(1 + Bg + B^2s) + (A + Bd + 2ABg + B^2q + 3AB^2s)y+\\
&\quad (b + Ad + A^2g + Bn + 2ABq + 3A^2Bs)y^2 + (l + An + A^2q + A^3s)y^3 \equiv 0\\
&\quad \Rightarrow \{b = -B(n + Aq + A^2s),\ d = (A - B^2q - AB^2s)/B,\\
&\quad g = -(1 + B^2s)/B,\ l = -A(n + Aq + A^2s)\} \Rightarrow\\
&P(x, y) = (x - Ay - B)(-y + Bkx^2 + B(Ak + m)xy + B(A^2k + Am + p)y^2)/B,\\
&Q(x, y) = (x - Ay - B)(x - Bsx^2 - B(q + As)xy - B(n + Aq + A^2s)y^2)/B\Big).
\end{aligned}$$

□

System $\{(2),(9)\}$.

The function $\kappa(x,y)$ looks as $\kappa(x,y) = x(sx^3+qx^2y+nxy^2+ly^3)$ and $P(\gamma,y) = y \not\equiv 0$. So, we are looking for the straight line $l_1 \equiv y-Ax-B = 0$, $A, B \in \mathbb{R}$, $B \neq 0$. The identity $(sx^3+qx^2y+nxy^2+ly^3)\big|_{y=Ax} = (A^3l+A^2n+Aq+s)x^3 \equiv 0$ holds if $s = -(A^3l+A^2n+Aq)$. Then $X_1(x) = B(A+bB+B^2l)+(1+A^2+2AbB+Bd+3AB^2l+B^2n)x+(A^2b+Ad+g+3A^2Bl+2ABn+Bq)x^2 \equiv 0$ if $\{d = (-1+b^2B^2+3bB^3l+2B^4l^2-B^2n)/B$, $g = -b-Bl-b^2B^3l-2bB^4l^2-B^5l^3+bB^2n+B^3ln-Bq$, $A = -B(b+Bl)\}$. We determine the conditions under which the invariant straight line l_1 has the multiplicity two. Here, $X_2(x) = (1-bx-Blx)X_{21}(x)/B$, where $X_{21}(x) = B^2(1+b^2B^2+6bB^3l+6B^4l^2-B^2n)-2B^2(b+b^3B^2+2Bl+10b^2B^3l+20bB^4l^2+11B^5l^3-3bB^2n-4B^3ln+Bq)x+(1+2b^2B^2+b^4B^4+10bB^3l+19b^3B^5l+8B^4l^2+66b^2B^6l^2+79bB^7l^3+31B^8l^4-2B^2n-8b^2B^4n-26bB^5ln-18B^6l^2n+B^4n^2+3bB^3q+5B^4lq)x^2-2B(1+b^2B^2+5bB^3l+4B^4l^2-B^2n)(3b^2B^2l+6bB^3l^2+3B^4l^3-2bBn-2B^2ln+q)x^3+B^2(3b^2B^2l+6bB^3l^2+3B^4l^3-2bBn-2B^2ln+q)^2x^4$. The identity $X_2(x) \equiv 0$ holds if $X_{21}(x) \equiv 0 \Rightarrow \{l = 0$, $n = (1+b^2B^2)/B^2$, $q = 2b(1+b^2B^2)/B\}$ or $\{n = (8l^2-b^4)/(2b^2)$, $q = (b^4-32l^2)/(16l)$, $B = -b/(2l)\}$.

For the *System* $\{(2),(10)\}$ we have $\kappa(x,y) = x^2(sx^2+qxy+ny^2)$ and $P(\gamma,y) = ((s+aq)y\gamma+any^2+as\gamma^2)/s \not\equiv 0$.

We will examine the second factor from $\kappa(x,y)$. Denote $q^2-4ns = u^2 \Rightarrow n = (q^2-u^2)/(4s) \Rightarrow \kappa(x,y) = x^2(2sx+qy-uy)(2sx+qy+uy)/(4s)$.

Let $l_{1,2} \equiv 2sx+qy \mp uy+\gamma = 0$, $s\gamma(q \mp u) \neq 0$. For straight lines $x = (-qy \pm uy-\gamma)/(2s)$ we get $Y_1(y) = (\gamma(4qs \mp 4su-2gq\gamma+4as\gamma \pm 2gu\gamma+q\gamma^2 \mp u\gamma^2)+(4q^2s+16s^3 \mp 8qsu+4su^2-4gq^2\gamma+4dqs\gamma \pm 8gqu\gamma \mp 8asu\gamma \mp 4dsu\gamma-4gu^2\gamma+q^2\gamma^2 \mp 4qu\gamma^2+3u^2\gamma^2)y-2(q \mp u)(gq^2-2dqs+4bs^2 \mp 2gqu \pm 2dsu+gu^2 \pm qu\gamma-u^2\gamma)y^2)/(8s^2) \equiv 0 \Rightarrow \{a = -(q \mp u)(4s-2g\gamma+\gamma^2)/(4s\gamma)$, $b = (-16s^3+4su^2+u^2\gamma(\gamma-2g)-q^2(4s+\gamma(\gamma-2g)))/(8s^2\gamma)$, $d = gq/s+(\pm u-q)\gamma/(4s)-(q^2+4s^2-u^2)/(q\gamma \mp u\gamma)\}$. In these conditions $X_2(x) = -1024s^7(q \mp u)\gamma^3X_{2d}(x)\zeta_1(x)$, where $X_{2d}(x) = -(q \mp u)\gamma(4s-2g\gamma+\gamma^2)+2(8s^3 \pm 4qsu-4su^2 \mp 2gqu\gamma+2gu^2\gamma \pm qu\gamma^2-u^2\gamma^2)x$ and $\zeta_1(x) = 2\gamma^2(16q^2s^2+64s^4 \mp 16qs^2u-4gq^2s\gamma-16gs^3\gamma+4gsu^2\gamma-4q^2s\gamma^2 \pm 12qsu\gamma^2-8su^2\gamma^2+gq^2\gamma^3 \mp 2gqu\gamma^3+gu^2\gamma^3)+2(q \mp u)\gamma(16q^2s^2+64s^4 \mp 32qs^2u+16s^2u^2-8q^2s\gamma^2-16s^3\gamma^2 \pm 32qsu\gamma^2-24su^2\gamma^2 \mp 4gqu\gamma^3+4gu^2\gamma^3+q^2\gamma^4 \mp 2qu\gamma^4+u^2\gamma^4)x+(16q^4s^2+128q^2s^4+256s^6 \mp 64q^3s^2u \mp 256qs^4u+96q^2s^2u^2+128s^4u^2 \mp 64qs^2u^3+16s^2u^4-8q^4s\gamma^2-32q^2s^3\gamma^2 \pm 64q^3su\gamma^2 \pm 96qs^3u\gamma^2-160q^2su^2\gamma^2-64s^3u^2\gamma^2 \pm 160qsu^3\gamma^2-56su^4\gamma^2+8gq^2u^2\gamma^3 \mp 16gqu^3\gamma^3+8gu^4\gamma^3+q^4\gamma^4 \mp 12q^3u\gamma^4+30q^2u^2\gamma^4 \mp 28qu^3\gamma^4+9u^4\gamma^4)x^2 \pm 4(q \mp u)^2u\gamma(4q^2s+16s^3 \mp 8qsu+4su^2-q^2\gamma^2 \pm 4qu\gamma^2-3u^2v^2)x^3+4(q \mp u)^4u^2\gamma x^4$. If $X_{2d}(x) \equiv 0$, then the system (2) is degenerate, therefore $X_{2d} \not\equiv 0$ (see Lemma 5). The identity $\zeta_1(x) \equiv 0$ gives us $\{u = 0$, $\gamma = \pm 2\sqrt{q^2s+4s^3}/q\}$.

System $\{(2),(11)\}$.

For this system we have $\kappa(x,y) = x^2y(qx+ny)$, $P(\gamma,y) = y(n+fny+fq\gamma)/n \not\equiv 0$ and $Q(x,\gamma) = -b\gamma^2-(1+d\gamma+n\gamma^2)x-(g+q\gamma)x^2 \equiv 0 \Rightarrow \{b = 0$, $g = -q\gamma$, $d = (-1-n\gamma^2)/\gamma\}$. Under these conditions the polynomial $X_2(x) = \gamma(n+fqx+fn\gamma)(-n\gamma^2(1+f\gamma)(-1+n\gamma^2)-2nq\gamma^3(1+f\gamma)x+(n-2n^2\gamma^2-fq^2\gamma^3+n^3\gamma^4)x^2+2nq\gamma(-1+n\gamma^2)x^3+nq^2\gamma^2x^4)/(n^2\gamma^2)$ is identically zero when $\{q = 0$, $n = 1/\gamma^2\}$.

Let $l_1 \equiv qx+ny+\gamma = 0$, $nq \neq 0$. Then $y = (-qx-\gamma)/n$ and $X_1(x) =$

$-(\gamma(nq+bn\gamma-fq\gamma)+(n^3+nq^2-dn^2\gamma+2bnq\gamma-fq^2\gamma+n^2\gamma^2)x+n(gn^2-dnq+bq^2+nq\gamma)x^2)/n^2\equiv 0\Rightarrow\{\gamma=q/g,\ b=(fq-gn)/n,\ d=(g^2n^3-g^2nq^2+n^2q^2+fgq^3)/(gn^2q)\}$. The straight line l_1, i.e. $y=(-qx-\gamma)/n$, does not have the multiplicity two, as $X_2(x)=-q(gn-fq)(1+gx)(q^2(g^3n^2-fg^2nq+g^3q^2-gnq^2+fq^3)+2gq^2(g^3n^2+g^3q^2-2gnq^2+fq^3)x+g(g^4n^4+2g^4n^2q^2-2g^2n^3q^2+g^4q^4-5g^2nq^4+n^2q^4+fgq^5)x^2-2g^2nq^2(g^2n^2+g^2q^2-nq^2)x^3+g^3n^2q^4x^4)/(g^4n^2)\not\equiv 0$.

System $\{(2),(12)\}$.

For this system we have $\kappa(x,y)=qx^3y$. If $c=0$, then the line at infinity has the multiplicity four and this case was investigated above. If $c\neq 0$ then this system has also the invariant straight line $l\equiv cx+1=0$. Taking into account that $\gcd(P,Q)=1$, $Y_2(y)=(c-g+cdy-qy-bc^2y^2)(c-g+bc^2y^2+c^3y^2)/c^4\equiv 0$ if $\{b=-c,\ g=c\}$. We are looking for an invariant straight line $y-\gamma=0$. For this straight line the algebraic and parallel multiplicities are equal. So $Q(x,\gamma)=-b\gamma^2-(1+d\gamma)x-(g+q\gamma)x^2\equiv 0$ $\Rightarrow\{b=0,\ \gamma=-1/d,\ q=dg\}$. In these conditions $Q(x,y)$ has the form: $Q(x,y)=-x(1+gx)(1+dy)$, and therefore the invariant straight line $1+dy=0$ does not have the parallel multiplicity two.

System $\{(2),(13)\}$.

The function $\kappa(x,y)$ looks as $\kappa(x,y)=y^2(nx+py)^2/n$. If $y=\gamma$ then $Q(x,\gamma)=-\gamma^2(b+p\gamma)-(1+d\gamma+n\gamma^2)x-gx^2\equiv 0\Rightarrow\{g=0,\ b=-p\gamma,\ d=(-1-n\gamma^2)/\gamma\}$ and in these conditions $X_2(x)=(-1+n\gamma^2)(-px+n\gamma+fn\gamma^2+p^2\gamma^3)(-x^2-\gamma^2+nx^2\gamma^2-f\gamma^3+2px\gamma^3)/(n\gamma^2)\equiv 0\Rightarrow n=1/\gamma^2$.
For $x=(-py-\gamma)/n$ we have $Y_1(y)=p\gamma+(n^2+p^2)y+n(fn-bp)y^2\not\equiv 0$.

System $\{(2),(14)\}$ has $\kappa(x,y)=y(qx+my)(mx^2+pxy+ry^2)/m$. We are looking for the straight line $y+\gamma=0$. For system $\{(2),(14)\}$ the algebraic and parallel multiplicity of the straight line $y+\gamma=0$ are the same. So, $X_1(x)=q\gamma^2(f+r\gamma)+(m+cq\gamma+pq\gamma^2)x+q(a+m\gamma)x^2\equiv 0\Rightarrow\{f=-r\gamma,\ a=(c+p\gamma)q\gamma^2,\ m=-(c+p\gamma)q\gamma\}$ $\Rightarrow X_2(x)=-r\gamma^2-(c+2p\gamma)x+q\gamma(c+p\gamma)x^2\not\equiv 0$.

Let $l_1\equiv x-Ay-B=0, B\neq 0$ be a straight line for system $\{(2),(14)\}$. Therefore $Y_1(y)=B(Am+aBm+aABq)+(m+A^2m+2aABm+Bcm+B^2m^2+2aA^2Bq+ABcq+AB^2mq)y+(aA^2+Ac+f+2ABm+Bp)(m+Aq)y^2+(m+Aq)(A^2m+Ap+r)y^2)/m\equiv 0\Rightarrow\{a=-Am/(B(m+Aq)),\ c=-m(1-A^2+B^2m+AB^2q)/(B(m+Aq)),\ f=-(-Am+AB^2m^2+B^2mp+A^2B^2mq+AB^2pq)/(B(m+Aq)),\ r=-A(Am+p)\}$ and $Y_2(y)=(Bm+(Am-q)y)(-B^2m^2(-1-A^2+B^2m)+2Bm^2(A+A^3-3AB^2m-B^2p-A^2B^2q)y+m(m+2A^2m+A^4m-2B^2m^2-8A^2B^2m^2+B^4m^3-3AB^2mp-6A^3B^2mq+2AB^4m^2q+B^2pq-2A^2B^2pq+A^2B^4mq^2)y^2-2Bm(2Am+p)(m+Aq)(1+A^2-B^2m-AB^2q)y^3+B^2(2Am+p)^2(m+Aq)^2)/(B^2m^2(m+Aq))\equiv 0$ if $\{m=1/B^2,\ p=A=0\}$.

System $\{(2),(15)\}$. In this case $\kappa(x,y)=y(qx+my)^2(m^2x+qry)/(m^2q)$. If $y=\gamma$ then $X_1(x)=mqr\gamma^2(bm+qr\gamma)+(m^2qr+bm^4\gamma-fm^3q\gamma+cmq^2r\gamma+m^3qr\gamma^2+q^3r^2\gamma^2)x+mq^2r(a+m\gamma)\equiv 0\Rightarrow\{a=-m\gamma,\ b=-qr\gamma/m,\ f=r(m^2+cmq\gamma+q^2r\gamma^2)/(m^3\gamma)\}$. In these conditions the straight line $y=\gamma$ does not have the parallel multiplicity two because $Q(x,\gamma)-mqr\gamma^3+(m^2-m^3\gamma^2-q^2r\gamma^2)x-m^2q\gamma x^2\not\equiv 0$.

For $y=(-qx-\gamma)/m\Rightarrow X_1(x)=(-qr\gamma(mq+bm\gamma-fq\gamma)-(m^3qr+mq^3r-bm^4\gamma+fm^3q\gamma+2bmq^2r\gamma-2fq^3r\gamma)x+q(bm-fq)(m^3-q^2r)x^2)/(m^2qr)\not\equiv 0$ and for

$y=(-m^2x-\gamma)/(qr)$ the polynomial $X_1(x)=(-m\gamma(m^3q^2r-fm^3q\gamma+bmq^2r\gamma+m^3\gamma^2-q^2r\gamma^2)-(m^6q^2r+m^2q^4r^3-2fm^6q\gamma+bm^4q^2r\gamma+cm^4q^2r\gamma+fm^3q^3r\gamma-cmq^4r^2\gamma+2m^6\gamma^2-3m^3q^2r\gamma^2+q^4r^2\gamma^2)x+m(m^3-q^2r)(fm^4q-cm^2q^2r+aq^3r^2-m^4\gamma+mq^2r\gamma)x^2)/(m^2q^3r^2)\equiv 0$ if $\{\gamma=-m^3qr/(a(m^3-q^2r)),\ b=(a^2m^3+afm^3+m^3r-a^2q^2r)/(amqr),\ c=(afm^4+m^4r+a^2q^2r^2)/(am^2qr)\}$. In this case $X_2(x)=-(-m^2qr(a^2m^6+afm^6+m^6r-2a^2m^3q^2r-afm^3q^2r+a^2q^4r^2)+a(m^3-q^2r)(a^2m^7+afm^7+m^7r-2a^2m^4q^2r-afm^4q^2r-a^2m^3q^2r^2+a^2mq^4r^2+a^2q^4r^3)x)(m^6q^4r^2(a^3m^7-3am^7r-fm^7r-a^3m^4q^2r+2a^3m^3q^2r^2+a^2fm^3q^2r^2+am^4q^2r^2-a^3q^4r^3)-2m^5q^3r(m^3-q^2r)(a^4m^7-4a^2m^7r-afm^7r-a^4m^4q^2r+m^7r^2+a^4m^3q^2r^2+2a^2m^4q^2r^2-a^2m^3q^2r^3-a^4q^4r^3)x+aq^2(m^3-q^2r)^2(a^4m^11-7a^2m^11r-afm^11r-a^4m^8q^2r+5m^11r^2+2a^4m^7q^2r^2+5a^2m^8q^2r^2-3a^2m^7q^2r^3-m^8q^2r^3-2a^4m^4q^4r^3+a^4m^3q^4r^4+2a^2m^4q^4r^4-a^4q^6r^5)x^2+2a^2m^3q(m^3-q^2r)^3(a^2m^7-2m^7r-a^2m^4q^2r+a^2m^3q^2r^2+m^4q^2r^2-a^2q^4r^3)x^3+a^3m^6(m^3-q^2r)^5x^4)/(a^7m^5q^8r^6(m^3-q^2r)^5)$ is not equivalent zero.

System $\{(2),(16)\}$.

We have $\kappa(x,y)=xy(qx+my)^2/q$ and $P(\gamma,y)=a\gamma^2+(1+c\gamma+m\gamma^2)y+(fq+m^2\gamma)y^2/\equiv 0\Rightarrow\{a=0,\ f=-m^2\gamma/q,\ c=(-1-m\gamma^2)/\gamma\}$. Under these conditions $P(x,y)=y(x-\gamma)(-q+mqx\gamma+m^2y\gamma)/(q\gamma)$ and the straight line $x-\gamma=0$ does not have the parallel multiplicity two. If $y=\gamma$, then $Q(x,\gamma)=-(fq\gamma^2+m(1+d\gamma+m\gamma^2)x+q(a+m\gamma)x^2)/m\equiv 0\Rightarrow\{f=0,\ a=-m\gamma,\ d=(-1-m\gamma^2)/\gamma\}\Rightarrow Q(x,y)=-x(y-\gamma)(-1+qx\gamma+my\gamma)/\gamma$ whence we see that the invariant straight line $y-\gamma=0$ does not have the parallel multiplicity two. The system $\{(2),(16)\}$ does not have the invariant straight line of the form $qx+my+\gamma=0$, $qm\gamma\neq 0$ because $X_1(x)=(-q\gamma-(m^2+q^2-dm\gamma+cq\gamma)x+q(dm-cq)x^2)/m\not\equiv 0$.

For the *System* $\{(2),(17)\}$ we have $\kappa(x,y)=(sx+ky)(kx^3+mx^2y+pxy^2+ry^3)/k$ and $X_1(x)=-(k^2x+s^2x+s\gamma)/k\not\equiv 0$ if $y=(-sx-\gamma)/k$.

Let $l_1\equiv y-Ax-B=0$, $A,B\neq 0$. The identity $(kx^3+mx^2y+pxy^2+ry^3)\,|_{y=Ax}=(k+Am+A^2p+A^3r)x^3\equiv 0$ holds if $k=-A(m+Ap+A^2r)$. Therefore $X_1(x)=(-B(A^2m+A^2Bfm+A^3p+A^3Bfp+A^4r+A^4Bfr+A^2B^2mr+A^3B^2pr+A^4B^2r^2-Bfs-B^2rs)-(Am+A^3m+A^2Bcm+2A^3Bfm+A^2p+A^4p+A^3Bcp+2A^4Bfp+A^2B^2mp+A^3B^2p^2+A^3r+A^5r+A^4Bcr+2A^5Bfr+3A^3B^2mr+4A^4B^2pr+3A^5B^2r^2-Bcs-2ABfs-B^2ps-3AB^2rs)x-(a+Ac+A^2f+Bm+2ABp+3A^2Br)(A^2m+A^3p+A^4r-s)x^2)/(A(m+Ap+A^2r))\equiv 0$ if $\{a=(m+Ap+A^2r)(A^2-A^2B^2m-A^3B^2p-A^4B^2r+B^2s)/(B(A^2m+A^3p+A^4r-s)),\ c=(A(m+Ap+A^2r)(1+AB^2p+A^2(B^2r-1))+B^2(p+Ar)s)/(Bs-A^2B(m+Ap+A^2r)),\ f=(A^2(1+B^2r)(m+Ap+A^2r)+B^2rs)/(Bs-A^2B(m+Ap+A^2r))\}$. The polynomial $X_2(x)=(Bs+(Am+A^2p+A^3r+As)x)X_{21}(x)/(A^2B^2(m+Ap+A^2r)^2(A^2m+A^3p+A^4r-s))$, where $X_{21}(x)=B^2(A^2m^2+A^4m^2+2A^3mp+2A^5mp+A^4p^2+A^6p^2+2A^4mr+2A^6mr-A^2B^2m^2r-2A^4B^2m^2r+2A^5pr+2A^7pr-2A^3B^2mpr-4A^5B^2mpr-A^4B^2p^2r-2A^6B^2p^2r+A^6r^2+A^8r^2-2A^4B^2mr^2-4A^6B^2mr^2+A^4B^4m^2r^2-2A^5B^2pr^2-4A^7B^2pr^2+2A^5B^4mpr^2+A^6B^4p^2r^2-A^6B^2r^3-2A^8B^2r^3+2A^6B^4mr^3+2A^7B^4pr^3+A^8B^4r^4+AB^2mps+A^2B^2p^2s+4A^2B^2mrs+5A^3B^2prs+4A^4B^2r^2s-2A^2B^4mr^2s-2A^3B^4pr^2s-2A^4B^4r^3s+B^4r^2s^2)+2B(A^3m^2+A^5m^2+2A^4mp+2A^6mp-A^4B^2m^2p+A^5p^2+A^7p^2-2A^5B^2mp^2-A^6B^2p^3+2A^5mr+2A^7mr-A^3B^2m^2r-4A^5B^2m^2r+2A^6pr+2A^8pr-2A^4B^2mpr-10A^6B^2mpr+A^4B^4m^2pr-$

$A^5B^2p^2r - 6A^7B^2p^2r + 2A^5B^4mp^2r + A^6B^4p^3r + A^7r^2 + A^9r^2 - 2A^5B^2mr^2 - 8A^7B^2mr^2 + 3A^5B^4m^2r^2 - 2A^6B^2pr^2 - 9A^8B^2pr^2 + 8A^6B^4mpr^2 + 5A^7B^4p^2r^2 - A^7B^2r^3 - 4A^9B^2r^3 + 6A^7B^4mr^3 + 7A^8B^4pr^3 + 3A^9B^4r^4 + AB^2m^2s + 4A^2B^2mps + 3A^3B^2p^2s + AB^2mrs + 8A^3B^2mrs + A^2B^2prs + 10A^4B^2prs - 2A^2B^4mprs - 2A^3B^4p^2rs + A^3B^2r^2s + 7A^5B^2r^2s - 6A^3B^4mr^2s - 8A^4B^4pr^2s - 6A^5B^4r^3s + B^4prs^2 + 3AB^4r^2s^2)x + (A^2m^2 + 2A^4m^2 + A^6m^2 + A^2B^2m^3 - 2A^4B^2m^3 + 2A^3mp + 4A^5mp + 2A^7mp + 2A^3B^2m^2p - 10A^5B^2m^2p + A^4p^2 + 2A^6p^2 + A^8p^2 + A^4B^2mp^2 - 14A^6B^2mp^2 + A^4B^4m^2p^2 - 6A^7B^2p^3 + 2A^5B^4mp^3 + A^6B^4p^4 + 2A^4mr + 4A^6mr + 2A^8mr - A^4B^2m^2r - 16A^6B^2m^2r + 2A^4B^4m^3r + 2A^5pr + 4A^7pr + 2A^9pr - 4A^5B^2mpr - 40A^7B^2mpr + 14A^5B^4m^2pr - 3A^6B^2p^2r - 24A^8B^2p^2r + 24A^6B^4mp^2r + 12A^7B^4p^3r + A^6r^2 + 2A^8r^2 + A^{10}r^2 - 5A^6B^2mr^2 - 26A^8B^2mr^2 + 19A^6B^4m^2r^2 - 6A^7B^2pr^2 - 30A^9B^2pr^2 + 54A^7B^4mpr^2 + 36A^8B^4p^2r^2 - 3A^8B^2r^3 - 12A^{10}B^2r^3 + 32A^8B^4mr^3 + 40A^9B^4pr^3 + 15A^{10}B^4r^4 + 3A^2B^2m^2s + 2AB^2mps + 11A^3B^2mps + 2A^2B^2p^2s + 8A^4B^2p^2s - 2A^2B^4mp^2s - 2A^3B^4p^3s + 6A^2B^2mrs + 18A^4B^2mrs - 4A^2B^4m^2rs + 8A^3B^2prs + 23A^5B^2prs - 24A^3B^4mprs - 22A^4B^4p^2rs + 6A^4B^2r^2s + 15A^6B^2r^2s - 34A^4B^4mr^2s - 50A^5B^4pr^2s - 30A^6B^4r^3s + B^4p^2s^2 + 2B^4mrs^2 + 10AB^4prs^2 + 15A^2B^4r^2s^2)x^2 + 2B(m + 2Ap + 3A^2r)(A^2m + A^3p + A^4r - s)(-Am - A^3m - A^2p - A^4p + A^2B^2mp + A^3B^2p^2 - A^3r - A^5r + 3A^3B^2mr + 4A^4B^2pr + 3A^5B^2r^2 - B^2ps - 3AB^2rs)x^3 + B^2(m + 2Ap + 3A^2r)^2(A^2m + A^3p + A^4r - s)^2x^4$. The identity $X_2(x) \equiv 0$ implies $X_{21}(x) \equiv 0 \Rightarrow \{m = (A^2 - 2)r,\ p = (r - 2A^2r)/A,\ s = -A^2(B^2r - 1)/B^2\}$ or $\{p = -3A(1 + A^2)u^2 \pm (1 + A^2)u/B,\ r = u^2(1 + A^2)\}$, where u is a parameter.

System $\{(2),(18)\}$ has $\kappa(x, y) = (sx + ky)^2(k^3x^2 + kpsxy - rs^2xy + krsy^2)/(sk^3)$. From $ks \neq 0$ it results that the system can have an invariant straight line of the form $y - Ax - B = 0$, $B \neq 0$. For this it is necessary that $X_1(x) = (Bks(Ak^3 + ABfk^3 + AB^2k^3r + Bfk^2s + Bgkrs + B^2k^2rs - aBrs^2) + s(k^4 + A^2k^4 + ABck^4 + 2A^2Bfk^4 + AB^2k^4p + 3A^2B^2k^4r + Bck^3s + 2ABfk^3s + Bgk^2ps + B^2k^3ps + 2ABgk^2rs + 3AB^2k^3rs - aBkps^2 - 2aABkrs^2 - Bgkrs^2 + aBrs^3)x + (ABk^6 + aAk^4s + A^2ck^4s + A^3fk^4s + gk^4s + Bk^5s + 2A^2Bk^4ps + 3A^3Bk^4rs + Ack^3s^2 + A^2fk^3s^2 + Agk^2ps^2 + 3ABk^3ps^2 + A^2gk^2rs^2 + 3A^2Bk^3rs^2 - aAkps^3 + Bk^2ps^3 - aA^2krs^3 - Agkrs^3 - ABk^2rs^3 + aArs^4 - Bkrs^4)x^2 + k(Ak + s)^2(k^3 + Akps + A^2krs - Ars^2)x^3)/(k^4s) \equiv 0 \Rightarrow A \neq 0$ and two set of conditions:

$$g = s(aB - 1)/(Bk),\ p = k(k^2 - 2s^2)/s^2,\ r = -k^2/s; \tag{43}$$

$$\begin{aligned} c &= -(1/(A^2Bk^2s(Ak + s)))(-A^2B^2k^5 - A^4k^3s + aA^2Bk^3s - 2AB^2k^4s \\ &\quad + aABk^2s^2 - B^2k^3s^2 - A^4krs^2 + aA^4Bkrs^2 + aA^3Brs^3), \\ f &= -(1/(ABk^2(Ak + s)))(A^2k^3 + A^2B^2k^3r + A^2krs - aA^2Bkrs \\ &\quad + 2AB^2k^2rs - aABrs^2 + B^2krs^2), \\ g &= (A^2 - aA^2B + AB^2k + B^2s)/(AB), \\ p &= -(k^3 + A^2krs - Ars^2)/(Aks). \end{aligned} \tag{44}$$

Under conditions (43) $A = -s/k$, $X_2(x) = -X_{2d}(x)X_{22}(x)/(B^2k^6s^2)$, where

$X_{2d}(x) = Bk^2s(B^2k^2 - s - Bfs) - k(B^2k^4 + Bcks^2 + B^2k^2s^2 - s^3 - 2Bfs^3)x - s^2(ak^2 - cks + fs^2)x^2$ and $X_{22}(x) = B^2k^2(k^2 + Bfk^2 + Bcks + s^2 - Bfs^2) + (aBk^2 - k^2 - Bcks - s^2 + Bfs^2)x(2Bks - (k^2 + s^2)x)$. If $X_{2d}(x) \equiv 0$, then the cubic system is degenerate, and if $X_{22}(x) \equiv 0$, then

$$a = -f, \quad c = (Bfs^2 - k^2 - Bfk^2 - s^2)/(Bks).$$

Under conditions (44) $X_2(x) = -X_{2d}(x)X_{22}(x)/(A^5B^2k^6s^2(Ak+s)^2)$, where
$X_{2d}(x) = ABs(Ak(k-Ar)+B(aA-Bk)r(Ak+s)) + (A^3k(Ar-k)s + B(aA-Bk)(Ak+s)(A^2rs-k^2))x$ and
$X_{22}(x) = A^2B^2s^2(A^4k^5 + aA^2Bk^5 - A^2B^2k^5r - 2A^4B^2k^5r + A^4B^4k^5r^2 + Ak^4s + A^3k^4s + aABk^4s - A^2k^3rs + aA^2Bk^3rs - 2AB^2k^4rs - 5A^3B^2k^4rs + A^4B^2k^3r^2s - aA^4B^3k^3r^2s + 4A^3B^4k^4r^2s + aABk^2rs^2 - B^2k^3rs^2 - 4A^2B^2k^3rs^2 + 2A^3B^2k^2r^2s^2 - 3aA^3B^3k^2r^2s^2 + 6A^2B^4k^3r^2s^2 - AB^2k^2rs^3 + A^2B^2kr^2s^3 - 3aA^2B^3kr^2s^3 + 4AB^4k^2r^2s^3 - aAB^3r^2s^4 + B^4kr^2s^4) + 2ABs(Ak+s)(A^3B^2k^6 - A^3B^4k^6r + A^3k^4s + A^5k^4s + 2A^2B^2k^5s - 2A^3B^2k^4rs - 3A^5B^2k^4rs + aA^3B^3k^4rs - 3A^2B^4k^5rs + 2A^5B^4k^4r^2s + AB^2k^4s^2 - 2A^2B^2k^3rs^2 - 5A^4B^2k^3rs^2 + 2aA^2B^3k^3rs^2 - 3AB^4k^4rs^2 + A^5B^2k^2r^2s^2 - aA^5B^3k^2r^2s^2 + 6A^4B^4k^3r^2s^2 - 2A^3B^2k^2rs^3 + aAB^3k^2rs^3 - B^4k^3rs^3 + A^4B^2kr^2s^3 - 2aA^4B^3kr^2s^3 + 6A^3B^4k^2r^2s^3 - aA^3B^3r^2s^4 + 2A^2B^4kr^2s^4)x + (Ak+s)(A^3B^4k^8 + 2A^3B^2k^6s + 4A^5B^2k^6s - aA^3B^3k^6s + 3A^2B^4k^7s - 6A^5B^4k^6rs + A^3k^4s^2 + 2A^5k^4s^2 + A^7k^4s^2 + 2A^2B^2k^5s^2 + 7A^4B^2k^5s^2 - 2aA^2B^3k^5s^2 + 3AB^4k^6s^2 - 5A^5B^2k^4rs^2 - 6A^7B^2k^4rs^2 + 2aA^5B^3k^4rs^2 - 18A^4B^4k^5rs^2 + 6A^7B^4k^4r^2s^2 + 3A^3B^2k^4s^3 - aAB^3k^4s^3 + B^4k^5s^3 - 7A^4B^2k^3rs^3 - 11A^6B^2k^3rs^3 + 4aA^4B^3k^3rs^3 - 18A^3B^4k^4rs^3 + A^7B^2k^2r^2s^3 - aA^7B^3k^2r^2s^3 + 19A^6B^4k^3r^2s^3 - 2A^3B^2k^2rs^4 - 5A^5B^2k^2rs^4 + 2aA^3B^3k^2rs^4 - 6A^2B^4k^3rs^4 + A^6B^2kr^2s^4 - 2aA^6B^3kr^2s^4 + 21A^5B^4k^2r^2s^4 - aA^5B^3r^2s^5 + 9A^4B^4kr^2s^5 + A^3B^4r^2s^6)x^2 + 2AB(Ak+s)^3(-k^2 + A^2rs)(-AB^2k^4 - Ak^2s - A^3k^2s - B^2k^3s + 2A^3B^2k^2rs + 3A^2B^2krs^2 + AB^2rs^3)x^3 + AB^2(Ak+s)^5(-k^2+A^2rs)^2x^4$.

If $X_{2d}(x) \equiv 0$, then the cubic system is degenerate, and if $X_{22}(x) \equiv 0$, then

$$r = k^2/(A^2s), \quad A^2B^2k^2 - A^2s - A^4s + 2AB^2ks + B^2s^2 = 0.$$

In this way we have proved the following lemma:

Lemma 6. *The systems* (2) *with an affine real invariant straight line of multiplicity two have the line at infinity* $Z = 0$ *of multiplicity at least three if and only if one of the following ten series of conditions holds:*

$$\begin{array}{l} a = c = f = k = l = m = p = r = 0,\ d = -2/B,\ g = -b(2+b^2B^2), \\ n = (1+b^2B^2)/B^2,\ q = 2b(1+b^2B^2)/B,\ s = b^2(1+b^2B^2); \end{array} \tag{45}$$

$$\begin{array}{l} a = c = f = k = m = p = r = 0,\ d = -(b^4-16l^2)/(4bl), \\ g = -b,\ n = -(b^4-8l^2)/(2b^2),\ q = (b^4-32l^2)/(16l),\ s = b^2/4; \end{array} \tag{46}$$

$$\begin{array}{l} a = (\pm 2q^2s \pm 4s^3 + gq\sqrt{s(q^2+4s^2)})/(2s\sqrt{s(q^2+4s^2)}), \\ b = q(\pm 2q^2s \pm 8s^3 + gq\sqrt{s(q^2+4s^2)})/(4s^2\sqrt{s(q^2+4s^2)}), \\ c = q(\pm 2q^2s \pm 4s^3 + gq\sqrt{s(q^2+4s^2)})/(2s^2\sqrt{s(q^2+4s^2)}), \\ d = (\pm q^2s \pm 4s^3 + gq\sqrt{s(q^2+4s^2)})/(s\sqrt{s(q^2+4s^2)}), \\ f = q^2(\pm 2q^2s \pm 4s^3 + gq\sqrt{s(q^2+4s^2)})/(8s^3\sqrt{s(q^2+4s^2)}), \\ n = q^2/(4s),\ k = l = m = p = r = 0; \end{array} \tag{47}$$

$$a = b = c = 0,\ d = -2/\gamma,\ n = 1/\gamma^2,\ g = k = l = m = p = q = r = s = 0; \tag{48}$$

$$a = f = k = l = m = n = p = r = s = 0,\ b = -c,\ g = c,\ q \neq 0; \tag{49}$$

$$\begin{aligned} &a = 0,\ b = -p\gamma,\ c = -2p\gamma,\ d = -2/\gamma,\ g = k = 0,\ n = 1/\gamma^2, \\ &l = p,\ m = q = 0,\ r = p^2\gamma^2,\ s = 0; \end{aligned} \tag{50}$$

$$\begin{aligned} &a = b = 0,\ c = -2/B,\ d = -2Bq,\ f = g = 0, \\ &k = l = 0,\ m = 1/B^2,\ n = p = r = s = 0,\ q \neq 0; \end{aligned} \tag{51}$$

$$\begin{aligned} &a = 2Br,\ b = (2A(B^2r - 1))/B,\ c = (2(A^2 - 1)Br)/A, \\ &d = (2(1 - A^2)(B^2r - 1))/B,\ f = -2Br,\ m = (A^2 - 2)r, \\ &g = (2A(1 - B^2r))/B,\ k = Ar,\ l = (A(1 - B^2r))/B^2, \\ &n = ((2A^2 - 1)(B^2r - 1))/B^2,\ p = ((1 - 2A^2)r)/A, \\ &q = (A(2 - A^2)(B^2r - 1))/B^2,\ s = (A^2(1 - B^2r))/B^2; \end{aligned} \tag{52}$$

$$\begin{aligned} &a = -Au(ABu + A^3Bu \pm A^2 \pm 2), \\ &b = (ABu \pm 1)(Bu + A^2Bu \mp A)/B, \\ &c = 2u(ABu + A^3Bu \pm 1), \\ &d = -2(ABu \pm 1)(ABu + A^3Bu \pm 1)/B, \\ &f = -u(Bu + A^2Bu \mp A), \\ &g = A(ABu \pm 1)(ABu + A^3Bu \pm 2 \pm A^2)/B, \\ &k = -A^2(1 + A^2)u(ABu \pm 1)/B, \\ &l = -(1 + A^2)u(ABu \pm 1)/B, \\ &m = A(1 + A^2)u(3ABu \pm 2)/B, \\ &n = (1 + A^2)(ABu \pm 1)(3ABu \pm 1)/B^2, \\ &p = -(1 + A^2)u(3ABu \pm 1)/B,\ r = u^2(1 + A^2), \\ &q = -A(1 + A^2)(ABu \pm 1)(3ABu \pm 2)/B^2, \\ &s = A^2(1 + A^2)(ABu \pm 1)^2/B^2; \end{aligned} \tag{53}$$

$$\begin{aligned} &a = -f,\ b = -g = s(1 + Bf)/(Bk),\ c = (Bfs^2 - k^2 - Bfk^2 - s^2)/(Bks), \\ &d = (Bfs^2 - 2k^2 - Bfk^2)/(Bk^2),\ l = -k,\ m = (2k^2 - s^2)/s, \\ &n = (k^2 - 2s^2)/s,\ p = k(k^2 - 2s^2)/s^2,\ q = (2k^2 - s^2)/k,\ r = -k^2/s; \end{aligned} \tag{54}$$

$$\begin{aligned} &b = (Ak - aABk - A^2s - aBs)/(AB(Ak + s)), \\ &c = 2(Ak + A^3k - aABk - aBs)/(AB(Ak + s)), \\ &d = 2(aB - 1)/B, f = (aABk + aBs - 2Ak - 2A^3k)/(A^2B(Ak + s)), \\ &g = (A^2 - aA^2B + AB^2k + B^2s)/(AB),\ l = k/A^2,\ m = k(Ak - 2s)/(As), \\ &n = (s - 2Ak)/A^2,\ p = k(s - 2Ak)/(A^2s),\ q = (Ak - 2s)/A, \\ &r = k^2/(A^2s),\ A^2B^2k^2 - A^2s - A^4s + 2AB^2ks + B^2s^2 = 0. \end{aligned} \tag{55}$$

4.1 Integrability of the cubic systems {(2),(45)}–{(2),(55)}

The *Systems* {(2),(45)} and {(2),(47)} are integrable and they have the following integrating factors, respectively

$$\mu(x,y) = 1/(-B + bBx + y)^2,$$

$$\mu(x,y) = 1/(2\sqrt{s(q^2+4s^2)} \pm 2qsx \pm q^2 y)^2.$$

The *Systems* {(2),(52)}, {(2),(53)}, {(2),(55)} have the integrating factor

$$\mu(x,y) = 1/(-B - Ax + y)^2,$$

and the *Systems* {(2),(48)}, {(2),(50)} have the integrating factor

$$\mu(x,y) = 1/(y-\gamma)^2,$$

and for them the critical point $(0,0)$ is a center.

For *System* {(2),(46)} (respectively, {(2),(54)}) the first Lyapunov quantity $L_1 = -(b^4+16l^2)/(64l)$ (respectively, $L_1 = (k^2+s^2)^2/(4ks^2)$) is not equal to zero and therefore {(2),(46)} (respectively, {(2),(54)}) has a focus at $(0,0)$.

The *Systems* {(2),(49)} and {(2),(51)} have the first Lyapunov quantity $L_1 = -q/4, q \neq 0$ which is not equal to zero.

In this way we proved:

Theorem 2. *Cubic differential system* (2) *with the line at infinity of algebraic multiplicity at least three and an affine real invariant straight line* l_1 *of the multiplicity two has at the origin* $(0,0)$ *a center if and only if it has an integrating factor of the form* $\mu(x,y) = 1/l_1^2$.

5 Cubic systems (2) with infinite line and an affine invariant real line l_1 of multiplicities $m(Z) \geq 2$, $m(l_1) \geq 3$

It is easy to prove the following two Remarks:

Remark 2. In the class of differential systems (2), the maximal parallel multiplicity of an invariant affine real straight line is two.

Remark 3. Let the invariant straight line $Ax + By + C = 0$ of system (2) have the algebraic multiplicity three. Then, $(Ax+By)^3$ divides $\kappa(x,y)$ if the parallel multiplicity of $Ax+By+C=0$ is exactly one and $(Ax+By)^2$ divides $\kappa(x,y)$ if the parallel multiplicity of $Ax+By+C=0$ is exactly two.

In this section we will solve the problem of the center for cubic systems (2) in the case of configuration d) of Fig. 2.1. For systems {(2),(4)}– {(2),(8)} we will determine the conditions for the existence of an invariant affine real straight line l_1 of multiplicity three and under these conditions the problem of the center will be solved.

System {(2),(4)}.

For this system the function $\kappa(x,y)$ looks as $\kappa(x,y) = y^3(lx+ry)$. According to Remark 3 the system {(2),(4)} can not have the invariant straight lines of the form $lx+ry+\gamma=0$, $l\neq 0$ with algebraic multiplicity three. So, {(2),(4)} can have only the real invariant straight line of the form $y-\gamma=0: \quad Q(x,\gamma) = -\gamma^2(b+l\gamma)-(1+d\gamma)x-gx^2\equiv 0 \quad \Rightarrow \{g=0,\, l=bd,\, \gamma=-1/d\}$. Under these conditions $X_2(x)=X_{2d}(x)\zeta(x)/d^6$, where $X_{2d}(x)=d^2-df+r+cd^2x-ad^3x^2$ and $\zeta(x)=b^2-bc+d^2-df+r+2bd(a+d)x+d^3(a+d)x^2$. If $X_{2d}(x)\equiv 0$, the system {(2),(4)} is degenerate and if $\zeta(x)\equiv 0$, then $\{a=-d,\, r=bc-b^2-d^2+df\}$. Taking into account that $r\neq 0$ and $\gcd(P,Q)=1$, the polynomial $X_3(x)=(c-2b)(b+d^2x)(2d-f+2bdx+d^3x^2)/d^5$ is identically zero if and only if $c=2b$.

System {(2),(5)}.

The function $\kappa(x,y)$ is of the form $\kappa(x,y)=x(sx^3+qx^2y+nxy^2+ly^3)$. If $x=\gamma$, then $P(\gamma,y)=a\gamma^2+(1+c\gamma)y+fy^2\equiv 0\Rightarrow\{a=0,\, f=0,\, \gamma=-1/c\}$. According to Remark 3 the monomial x^2 must divide $\kappa(x,y)$, i.e. $l=0$. Therefore, $Y_2(y)\equiv (E_1(\mathbb{X})/l_1)|_{x=\gamma}=-Y_{2d}(y)\eta(y)/c^6$, where $Y_{2d}(y)=cg-c^2-s-c(cd-q)y+c^2(bc-n)y^2$ and $\eta(y)=c^2-cg+s+c^2(bc+c^2-n)y^2$. If $Y_{2d}(y)\equiv 0$, then the system is degenerate.

Assume that $Y_{2d}(y)\not\equiv 0$ and let $\eta(y)\equiv 0$. Then $\{n=c(b+c);\, s=c(g-c)\}$. Under these conditions $Y_3(y)=-((2c-g)(cd-q)+c^2(bcd+2c^2d-bq-3cq)y^2+2c^5(b+c)y^3)/c^4$ is identically zero if $\{b=-c,\, d=q=0\}$ or $\{b=-c,\, g=2c,\, q=cd/2\}$.

Now we are looking for the straight lines of the form $l_1=y-Ax-B, B\neq 0$. According to Remark 3 the polynomial $(y-Ax)^2$ must divide $\kappa(x,y)$, i.e. $\{s=A^2(2Al+n),\, q=-A(3Al+2n)\}$. We obtain $\varphi(x)=-B(A+bB+ABf+B^2l)-(1+A^2+2AbB+ABc+Bd+2A^2Bf+3AB^2l+B^2n)x-(aA+A^2b+A^2c+Ad+A^3f+g)x^2\equiv 0\Rightarrow\{b=-(A+ABf+B^2l)/B,\, d=-(1-A^2+ABc+AB^2l+B^2n)/B,\, g=A(1-aB+2AB^2l+B^2n)/B\}$. The cofactor of l_1 is $K_{l_1}(x,y)=(x+Ay+AB(2Al+n)x^2-B(Al+n)xy-Bly^2)/B$. The invariant straight line l_1 has the parallel multiplicity two if $K_{l_1}(x,Ax-B)\equiv 0$, i.e. if $\{A=B^2l,\, n=(1-2B^4l^2)/B^2\}$. Under these conditions $X_3(x)=X_{31}(x)X_{32}(x)$, where $X_{31}(x)=B(1+Bf)+B(c+Bl+2B^2fl)x+(a+B^2cl+B^4fl^2)x^2$ and $X_{32}(x)=B(-1-Bf+B^3cl+B^5fl^2)+2B^3l(a+B^2cl+B^4fl^2)x+(1+B^4l^2)(a+B^2cl+B^4fl^2)x^2$. When $X_{31}(x)\equiv 0$, the system is degenerate.

Assume that $X_{31}(x)\not\equiv 0$ and let $X_{32}(x)\equiv 0$. Then $\{a=0,\, B=-1/f,\, l=0\}$ or $\{a=-(1+Bf)/B,\, c=(1+Bf-B^5fl^2)/(B^3l)\}$.

Suppose that the invariant straight line l_1 has the parallel multiplicity equal to one. If the algebraic multiplicity of the line l_1 is three then $(y-Ax)^3$ divides $\kappa(x,y)$, i.e. $\{n=-3Al,\, q=3A^2l,\, s=-A^3l\}$. Under these conditions $X_2(x)=X_{2d}(x)\zeta(x)/B^2$, where $X_{2d}(x)=B(1+Bf)+(A+Bc+2ABf)x+(a+Ac+A^2f)x^2$ and $\zeta(x)=B^2(1+A^2-ABc+Bf-A^2Bf-AB^2l+B^3cl+2AB^3fl+B^4l^2)+(1+A^2-aB-ABc-A^2Bf)x(2AB-2B^3l+x+A^2x)$. If $X_{2d}(x)\equiv 0$, then the system is degenerate.

Let $X_{2d}(x)\not\equiv 0$ and $\zeta(x)\equiv 0$. Then we have two cases:

Case 1. $a=-(f^4+cfl+2l^2)/f^3$, $A=l/f^2$, $B=-1/f$ or

Case 2. $a=(Af+Bl)(1+AB^2l)/(B^2l-A)$, $c=(1+A^2+Bf-A^2Bf-AB^2l+$

$2AB^3fl + B^4l^2)/(B(A - B^2l))$.

In Case 1, $X_3(x) = ((cf+l)f^2(f^4+cfl+2l^2)+(f^4+l^2)^2x^2)/f^{13} \equiv 0 \Rightarrow l = -cf$.

Consider now the Case 2. Let $B(A - B^2l) \neq 0$. The polynomial $X_3(x)$ looks as: $X_3(x) = -X_{31}(x)X_{32}(x)X_{33}/(B^3(B^2l - A)^2)$, where $X_{31}(x) = B(A - B^2l) + (1 + A^2)x \not\equiv 0$, $X_{32}(x) = B^2(A-2B^2l-B^3fl)+2B(A-B^2l)^2x+(1+A^2)(A-B^2l)x^2 \not\equiv 0$ and $X_{33} = 1 + Bf + A^2Bf + 2AB^2l - B^4l^2$. The equality $X_{33} = 0$ yields $f = (B^4l^2 - 1 - 2AB^2l)/(B(1 + A^2))$.

System {(2),(6)}.

For this system we have $\kappa(x, y) = y^2(nx + py)(px + ry)/p$. The straight line $l_1 = y - \gamma$ is an invariant straight line for {(2),(6)} if $Q(x, \gamma) = -\gamma^2(bp + nr\gamma)/p - (1 + d\gamma + n\gamma^2)x - gx^2 \equiv 0$, i.e. $\{g = 0,\ b = -nr\gamma/p,\ d = (-1 - n\gamma^2)/\gamma\}$. The invariant straight line has the parallel multiplicity two if $(y - \gamma)^2$ divides $Q(x, y) = -(y-\gamma)(-px+npxy\gamma+nry^2\gamma)/(p\gamma)$, i.e. $\{r = 0,\ n = 1/\gamma^2\}$. Under these conditions $X_3(x) = (-\gamma(1+f\gamma)+ax^2)(\gamma(1+f\gamma)+\gamma(c+p\gamma)+ax^2)/\gamma^2$. The identity $X_3(x) \equiv 0$ gives us $\{a = 0,\ \gamma = -1/f\}$.

Let the parallel multiplicity of the straight line l_1 be one and the algebraic multiplicity be three. According to Remark 3 the polynomial y^3 must divide $\kappa(x, y)$, i.e. $n = 0$. The polynomial $X_2(x)$ has the form $X_2(x) = X_{2d}(x)\zeta(x)/\gamma^2$, where $X_{2d}(x) = \gamma(1+f\gamma+r\gamma^2)+\gamma(c+p\gamma)x+ax^2$ and $\zeta(x) = \gamma^2(1+f\gamma+r\gamma^2)+(1-a\gamma)x^2$. If $X_{2d}(x) \equiv 0$, then the system is degenerate. If $\zeta(x) \equiv 0$, then $\{a = 1/\gamma,\ f = -(1+r\gamma^2)/\gamma\}$. Under these conditions, $X_3(x) = \gamma^2(c+p\gamma)(r\gamma^2-1)x-(c+2p\gamma)x^3/\gamma^2 \equiv 0 \Rightarrow \{c = -2p\gamma,\ r = 1/\gamma^2\}$.

Let $l_2 = nx + py + \gamma$. According to Remark 3 the polynomial $(nx + py)^2$ must divide $\kappa(x, y)$, i.e. $n = p$, $r = p$. The polynomial $\kappa(x, y)$ looks as $\kappa(x, y) = py^2(x+y)^2$ and we take the straight line in the form $l_2 = x+y+\gamma$. This straight line is invariant if $\varphi(x) = -\gamma(1+b\gamma-f\gamma)-(2+2b\gamma+c\gamma-d\gamma-2f\gamma)x+(a-b-c+d+f-g)x^2 \equiv 0 \Rightarrow \{d = c,\ f = (1 + b\gamma)/\gamma,\ g = (1 + a\gamma)/\gamma\}$. The cofactor of l_2 is $K_{l_2}(x, y) = -(x-y)(x+y+\gamma)/\gamma$. It is easy to show that $(x+y+\gamma)^2$ does not divide $K_{l_2}(x; y)$. If the straight line l_2 has the parallel multiplicity one and the algebraic multiplicity three then the polynomial $(x + y)^3$ must divide $\kappa(x, y)$. This is impossible because $\kappa(x, y) = py^2(x + y)^2$.

System {(2),(7)}.

For this system the following assertion holds.

Lemma 7. *In the class of systems* {(2), (7)} *the maximal parallel multiplicity of the real invariant straight line of the form* $\alpha x + \beta y + \gamma = 0$, $\beta\gamma \neq 0$ *is one.*

Proof. Let l_1 be a real invariant straight line of the system {(2), (7)}. If l_1 is described by the equation $y - \gamma = 0$, $\gamma \in \mathbb{R} \setminus \{0\}$, then $Q\ (x, \gamma) \equiv 0$, i.e. $b = -qr\gamma/m$, $d = -(m + pq\gamma^2)/(m\gamma)$, $g = -q\ \gamma$. The cofactor of l_1 is $K_{l_1}(x, y) = (mx - mq\gamma x^2 - pq\gamma xy - qr\gamma y^2)/(m\gamma)$. It is easy to show that $y - \gamma$ does not divide $K_{l_1}(x, y)$.

Let $l_1 = x - Ay - B$, $AB \neq 0$. The straight line l_1 is invariant for {(2), (7)} if the following set of conditions holds:

$$\begin{aligned}&a=-A(1+Bg)/B,\ c=-(1-A^2+ABd+B^2m+AB^2q)/B,\\&f=(Am-AbB\ m-AB^2m^2-B^2mp-A^2B^2mq-AB^2pq)/(mB),\\&\quad(m+Aq)(A^2m+Ap+r)=0.\end{aligned}\tag{56}$$

If $m=-Aq$, then the cofactor of l_1 is $K_{l_1}(x,y)=-(Ax+y)/B$ and therefore l_1 has not parallel multiplicity greater than one.

Assume that $mAB(m+Aq)\neq 0$ and let $r=-A(Am+p)$. Then $K_{l_1}(Ay+B,y)=-A-(1+A^2-B^2m-AB^2q)y/B+(2Am+p)(m+Aq)y^2/m\not\equiv 0$. □

According to Lemma 7, the real straight line $y-\gamma=0$ does not have the algebraic multiplicity three because y^3 does not divide $\kappa(x,y)=y(qx+my)(mx^2+pxy+ry^2)/m$.

Let the straight line $l_1\equiv x-\gamma=0$, $\gamma\in\mathbb{R}\setminus\{0\}$ be invariant for nondegenerate system $\{(2),\ (7)\}$. We will show that if $L_1=0$ then the maximal multiplicity of this invariant straight line is at most two. We have $P(\gamma,y)\equiv 0\ \Rightarrow\ \{a=0, r=0, f=-p\gamma, c=-(1+m\gamma^2)/\gamma\}$. Under these conditions $\kappa(x,y)=xy(qx+my)(mx+py)/m$ and according to Remark 3, x^2 must divide $\kappa(x,y)$, i.e. $p=0$. Then $Y_2(y)\equiv(E_1(\mathbb{X})/l_1)|_{x=\gamma}=\eta_0\eta_1(y)Y_{2d}(y)/\gamma^2$, where $\eta_0=1-m\gamma^2$, $\eta_1(y)=\gamma^2(1+g\gamma)+(1-b\gamma-m\gamma^2)y^2$, $Y_{2d}(y)=\gamma(1+g\gamma)+\gamma(d+q\gamma)y+by^2$.

If $Y_{2d}(y)\equiv 0$, then the cubic system is degenerate.

Assume that $Y_{2d}(y)\not\equiv 0$ and let $\eta_1(y)\equiv 0$. Then $b=(1-m\gamma^2)/\gamma$, $g=-1/\gamma$ and $Y_3(y)\equiv(E_1(\mathbb{X})/l_1^2)|_{x=\gamma}=y(m\gamma^2-1)(\gamma^2(d+q\gamma)+(d+2q\gamma+dm\gamma^2)y^2-2m(m\gamma^2-1)y^3)/\gamma^2\Rightarrow Y_3(y)\equiv 0\Rightarrow m=1/\gamma^2\Rightarrow L_1=-\varphi_1(y)/(4\gamma^2y)\not\equiv 0$.

Let now $Y_{2d}\not\equiv 0$, $\eta_1(y)\not\equiv 0$ and $\eta_0=0\ \Rightarrow\ m=1/\gamma^2$. Then $Y_3(y)=-Y_{2d}(y)\eta_1(y)/\gamma^3\not\equiv 0$.

Finally, we consider the case when the real straight line has the form $l_1\equiv x-Ay-B=0$, $AB\neq 0$. This line is invariant for system $\{(2),(7)\}$ if the conditions (56) hold. Taking into account Lemma 7, if the real straight line l_1 has algebraic multiplicity three then $(x-Ay)^3$ divides $\kappa(x,y)=y(qx+my)(mx^2+pxy+ry^2)/m$, i.e. $m=-Aq$, $p=2A^2q$, $r=-A^3q$, $q\neq 0$. We have $Y_2(y)\equiv(E_1(\mathbb{X})/l_1)|_{x=Ay+B}=Y_{2d}(y)\eta(y)/B^2$, where $Y_{2d}(y)=B(1+Bg)+(A+Bd+2ABg+B^2q)y+(b+Ad+A^2g)y^2$, $\eta(y)=B^2(1+A^2-ABd+Bg-A^2Bg-AB^2q)+(1+A^2-bB-ABd-A^2Bg)(2AB+y+A^2y)y$.

When $Y_{2d}(y)\equiv 0$, then the cubic system is degenerate.

Let $\eta(y)\equiv 0$, i.e. $b=ABq-g$, $d=(1+A^2+Bg-A^2Bg-AB^2q)/(AB)$. Then $Y_3(y)\equiv(E_1(\mathbb{X})/l_1^2)|_{x=Ay+B}=-(AB+y+A^2y)((1+B\ g+A^2Bg+AB^2q)(B^2+2ABy+(1+A^2)y^2)+AB^5gq)/(AB^3)\equiv 0\Rightarrow\{g=0,\ q=-1/(AB^2)\}\Rightarrow L_1=-(1+A^2)/(4AB^2)\not\equiv 0$.

System$\{(2),(8)\}$.

For this system we have $\kappa(x,y)=(sx+ky)(kx^3+mx^2y+pxy^2+ry^3)/k$. We are looking for a straight line of the form $l_1=sx+ky+\gamma$, $\gamma\in\mathbb{R}\setminus\{0\}$. According to Remark 2, the polynomial $(sx+ky)^2$ must divide $\kappa(x,y)$, i.e. $r=(k^4-k^2ms+kps^2)/s^3$. Then $\varphi(x)=-\gamma(ks+bk\gamma-fs\gamma)-(k^3+ks^2-dk^2\gamma+2bks\gamma+cks\gamma-$

$2fs^2\gamma)x-(gk^3-ak^2s-dk^2s+bks^2+cks^2-fs^3)x^2 \equiv 0 \Rightarrow \{b=(-ks+fs\gamma)/(k\gamma),\ d=(k^2-s^2+cs\gamma)/(k\gamma),\ g=s(k+a\gamma)/(k\gamma)\}$. The invariant straight line l_1 does not have the parallel multiplicity two because its cofactor is $K_{l_1}(x,y)=(-kx+sy)/\gamma$ and $(sx+ky)^2$ does not divide $K_{l_1}(x,y)$. The algebraic multiplicity of the invariant straight line l_1 is three if $(sx+ky)^3$ divides $\kappa(x,y)$, i.e. $\{p=(-3k^3+2kms)/s^2,\ r=k^2(-2k^2+ms)/s^3\}$. Under these conditions $X_2(x)=X_{2d}(x)\zeta(x)/k^4$, where $X_{2d}(x)=-\gamma(-ks^3+fs^3\gamma+2k^3\gamma^2-kms\gamma^2)-s(-ks^3-cks^2\gamma+2fs^3\gamma+3k^3\gamma^2-kms\gamma^2)x-s^3(ak^2-cks+fs^2)x^2$ and $\zeta(x)=-\gamma^2(-k^3s^3-ks^5+fk^2s^3\gamma+cks^4\gamma-fs^5\gamma+2k^5\gamma^2-k^3ms\gamma^2-k^3s^2\gamma^2)+2s^4\gamma(k^3+ks^2+ak^2\gamma-cks\gamma+fs^2\gamma)x+s^3(k^2+s^2)(k^3+ks^2+ak^2\gamma-cks\gamma+fs^2\gamma)x^2$. If $X_{2d}(x)\equiv 0$ then the system $\{(2),(8)\}$ is degenerate.

Assume that $X_{2d}(x)\not\equiv 0$ and let $\zeta(x)\equiv 0$. Then $\{a=-(fs^3+2k^3\gamma-kms\gamma-ks^2\gamma)/s^3,\ c=(k^3s^3+ks^5-fk^2s^3\gamma+fs^5\gamma-2k^5\gamma^2+k^3ms\gamma^2+k^3s^2\gamma^2)/(ks^4\gamma)\}$. Therefore $X_3(x)=(-\gamma^2(-k^3s^6+fk^2s^6\gamma+fs^8\gamma+5k^3s^5\gamma^2-2kms^6\gamma^2+2fk^4s^3\gamma^3-fk^2ms^4\gamma^3-fk^2s^5\gamma^3+4k^7\gamma^4-4k^5ms\gamma^4-2k^5s^2\gamma^4+k^3m^2s^2\gamma^4+k^3ms^3\gamma^4)-2s^4\gamma(-k^3s^3+fk^2s^3\gamma+fs^5\gamma+2k^5\gamma^2-k^3ms\gamma^2+5k^3s^2\gamma^2-2kms^3\gamma^2)x-s^3(k^2+s^2)(-k^3s^3+fk^2s^3\gamma+fs^5\gamma+2k^5\gamma^2-k^3ms\gamma^2+5k^3s^2\gamma^2-2kms^3\gamma^2)x^2)/k^3\equiv 0 \Rightarrow \{f=k^3(s^3+k^2\gamma^2+s^2\gamma^2)/(s^3(k^2+s^2)\gamma),\ m=3k^2/s\}$ or $\{f=2k(s-\gamma^2)/(s\gamma),\ m=(s^3+2k^2\gamma^2-s^2\gamma^2)/(s\gamma^2)$.

Now we are looking for a straight line of the form $l_2=x-Ay-B$, $kB(sA+k)\neq 0$. According to Remark 2, $(x-Ay)^2$ must divide $\kappa(x,y)$, i.e. $\{p=-(3A^2k+2Am),\ r=A^2(2Ak+m)\}\Rightarrow \psi(y)=Bk(A+aB+ABg+B^2k+AB^2s)+(k+A^2k+2aABk+Bck+ABdk+2A^2Bgk+3AB^2k^2+B^2km+3A^2B^2ks+AB^2ms)y+(aA^2+Ab+Ac+A^2d+f+A^3g)ky^2\equiv 0\Rightarrow \{a=-(A+ABg+B^2k+AB^2s)/B,\ c=-(k-A^2k+ABdk+AB^2k^2+B^2km+A^2B^2ks+AB^2ms)/(Bk),\ f=A(k-bBk+2AB^2k^2+B^2km+2A^2B^2ks+AB^2ms)/(Bk)\}$. The cofactor of l_2 is $K_{l_2}(x,y)=(B-x+Ay)(Akx-Bk(k+As)x^2+ky-B(Ak^2+km+A^2ks+Ams)xy+AB(2Ak^2+km+2A^2ks+Ams)y^2)$ and the parallel multiplicity of the straight line l_2 is two if $(B-x+Ay)^2$ divides $K_{l_2}(x,y)$, i.e. $\{m=(k-2A^2k)/A,\ s=(A-B^2k)/(AB^2)\}$. Under these conditions $Y_3(y)=Y_{31}(y)Y_{32}(y)/(A^2B^2)$, where $Y_{31}(y)=AB(2A+ABg-B^2k)+(A+2A^3+A^2Bd+2A^3Bg-B^2k-A^2B^2k)y+A^2(b+Ad+A^2g)y^2$ and $Y_{32}(y)=B(-1+ABd-Bg+A^2Bg)+2AB(b+Ad+A^2g)y+(1+A^2)(b+Ad+A^2g)y^2$. If $Y_{31}(y)\equiv 0$, then the system $\{(2),(8)\}$ is degenerate.

Assume that $Y_{31}(y)\not\equiv 0$ and let $Y_{32}(y)\equiv 0$. From $Y_{32}(y)\equiv 0$ it results $\{b=(-1-Bg)/B,\ d=(1+Bg-A^2Bg)/(AB)\}$.

We suppose that the parallel multiplicity of the straight line l_2 is only one and the algebraic multiplicity is three. Therefore, the expression $(x-Ay)^3$ must divide $\kappa(x,y)$, i.e.$\{m=-3Ak,\ p=3A^2k,\ r=-A^3k\}$. In these conditions $Y_2(y)=Y_{2d}(y)\eta(y)/B^2$, where $Y_{2d}(y)=B(1+Bg+B^2s)+(A+Bd+2ABg)y+(b+Ad+A^2g)y^2$ and $\eta(y)=B^2(1+A^2-ABd+Bg-A^2Bg-AB^2k+B^3dk+2AB^3gk+B^4k^2+B^2s+AB^3ds+2A^2B^3gs+2AB^4ks+A^2B^4s^2)+2B(-1-A^2+bB+ABd+A^2Bg)(-A+B^2k+AB^2s)y-(1+A^2)(-1-A^2+bB+ABd+A^2Bg)y^2$. If $Y_{2d}(y)\equiv 0$, then the system $\{(2),(8)\}$ is degenerate.

Let $Y_{2d}(y)\not\equiv 0$ and $\eta(y)\equiv 0$. From $\eta(y)\equiv 0\Rightarrow$:

Case 1. $\{b=(1+2A^2-ABd+A^2B^2s)/B,\ g=-(1+B^2s)/B,\ k=-A(-1+$

$B^2s)/B^2\}$;

Case 2. $\{b = (Ag+Bk+A^2B^2gk+AB^3k^2+2ABs+A^3Bs+A^3B^2gs+2A^2B^3ks+A^3B^3s^2)/(-A+B^2k+AB^2s)$, $d = (-1-A^2-Bg+A^2Bg+AB^2k-2AB^3gk-B^4k^2-B^2s-2A^2B^3gs-2AB^4ks-A^2B^4s^2)/(B(-A+B^2k+AB^2s))\}$.

In the Case 1: $Y_3(y) = (A-Bd+2AB^2s)y(B^2+2A^2B^2-AB^3d-B^4s+A^2B^4s+y^2+2A^2y^2+A^4y^2)/B^3 \equiv 0 \Rightarrow d = (A+2AB^2s)/B$.

In the Case 2: $Y_3(y) = -Y_{33}(y)Y_{34}(y)Y_{35}/(B^3(-A+B^2k+AB^2s)^2)$, where $Y_{33}(y) = AB-B^3k-AB^3s+(1+A^2)y \not\equiv 0$, $Y_{34}(y) = B^2(-A+2B^2k+B^3gk+3AB^2s+AB^3gs)-2B(-A+B^2k+AB^2s)^2y+(1+A^2)(-A+B^2k+AB^2s)y^2 \not\equiv 0$ and $Y_{35} = -1-Bg-A^2Bg-2AB^2k+B^4k^2-B^2s-3A^2B^2s+2AB^4ks+A^2B^4s^2 = 0$ $\Rightarrow g = (-1-2AB^2k+B^4k^2-B^2s-3A^2B^2s+2AB^4ks+A^2B^4s^2)/(1+A^2)B$.

In this way we have proved the following lemma:

Lemma 8. *The systems* (2) *with the line at infinity of multiplicity at least two have an affine real invariant straight line of multiplicity three if and only if one of the following fourteen sets of conditions holds:*

$$\begin{array}{l} a=-d,\ c=2b,\ g=n=k=m=s=p=q=0,\\ l=bd,\ r=b^2-d^2+df,\ b^2-d^2+df\neq 0; \end{array} \tag{57}$$

$$\begin{array}{l} a=d=f=k=l=m=n=p=q=r=0,\\ b=-c,\ s=c(g-c),\ c\neq 0; \end{array} \tag{58}$$

$$\begin{array}{l} a=f=k=l=m=n=p=r=0,\ g=2c,\\ q=cd/2,\ s=c^2,\ c\neq 0; \end{array} \tag{59}$$

$$\begin{array}{l} a=b=g=k=l=m=p=q=r=s=0,\\ d=2f,\ n=f^2,\ f\neq 0; \end{array} \tag{60}$$

$$\begin{array}{l} a=(-1-Bf)/B,\ b=-B(2+Bf)l,\\ c=(1+Bf-B^5fl^2)/(B^3l),\\ d=(2B^4l^2+B^5fl^2-3-Bf)/B,\ g=B(3+Bf)l,\\ k=m=p=r=0,\ n=(1-2B^4l^2)/B^2,\\ q=l(B^4l^2-2),\ s=B^2l^2; \end{array} \tag{61}$$

$$\begin{array}{l} a=-(c^2+f^2)/f,\ l=-cf,\ n=-3c^2,\\ k=m=p=r=0,\ q=-3c^3/f,\ s=-c^4/f^2; \end{array} \tag{62}$$

$$\begin{array}{l} a=(1+AB^2l)^2/((1+A^2)B),\ k=m=p=r=0,\\ c=(3A+A^3-2B^2l+2A^2B^2l-2AB^4l^2)/((1+A^2)B),\\ n=-3Al,\ q=3A^2l,\ s=-A^3l,\ B(A-B^2l)\neq 0; \end{array} \tag{63}$$

$$\begin{array}{l} a=b=g=k=l=m=q=r=s=0,\\ d=2f,\ n=f^2,\ p\neq 0; \end{array} \tag{64}$$

$$\begin{array}{l} a=1/\gamma,\ b=-n/(p\gamma),\ c=-2p\gamma,\ d=-(1+n\gamma^2)/\gamma,\\ f=-2/\gamma,\ g=k=l=m=n=q=s=0,\ r=1/\gamma^2; \end{array} \tag{65}$$

$$\begin{aligned}
&a = k(-k^2s + k^2\gamma^2 + s^2\gamma^2)/(s(k^2+s^2)\gamma),\\
&b = (-s^5 + k^4\gamma^2 + k^2s^2\gamma^2)/(s^2(k^2+s^2)\gamma),\\
&c = (3k^2s^3 + s^5 + 2k^4\gamma^2 + 2k^2s^2\gamma^2)/(s^2(k^2+s^2)\gamma),\\
&d = k(k^2s + 3s^3 + 2k^2\gamma^2 + 2s^2\gamma^2)/(s(k^2+s^2)\gamma,\\
&f = k^3(s^3 + k^2\gamma^2 + s^2\gamma^2)/(s^3(k^2+s^2)\gamma),\\
&g = (s^3 + k^2\gamma^2 + s^2\gamma^2)/((k^2+s^2)\gamma),\\
&l = k^3/s^2,\ m = 3k^2/s,\ n = 3k^2/s,\ p = 3k^3/s^2,\\
&q = 3k,\ r = k^4/s^3,\ k \neq 0;
\end{aligned} \tag{66}$$

$$\begin{aligned}
&a = k(2\gamma^2 - s)/(s\gamma),\ b = (s - 2\gamma^2)/\gamma,\\
&c = (3s^3 + 2k^2\gamma^2 - 2s^2\gamma^2)/(s^2\gamma),\\
&d = (k^2s + 2s^3 + 2k^2\gamma^2 - 2s^2\gamma^2)/(ks\gamma),\\
&f = 2k(s - \gamma^2)/(s\gamma),\ g = 2\gamma,\ l = k(s-\gamma^2)/\gamma^2,\\
&m = (s^3 + 2k^2\gamma^2 - s^2\gamma^2)/(s\gamma^2),\\
&n = (2s^3 + k^2\gamma^2 - 2s^2\gamma^2)/(s\gamma^2),\\
&p = k(2s^3 + k^2\gamma^2 - 2s^2\gamma^2)/(s^2\gamma^2),\\
&q = (s^3 + 2k^2\gamma^2 - s^2\gamma^2)/(k\gamma^2),\\
&r = k^2(s-\gamma^2)/(s\gamma^2);
\end{aligned} \tag{67}$$

$$\begin{aligned}
&a = -A(2+Bg)/B,\ b = (-1-Bg)/B,\\
&c = (-3 + 2A^2 - Bg + A^2Bg)/B,\ d = (1 + Bg - A^2Bg)/(AB),\\
&f = A(3+Bg)/B,\ l = (A - B^2k)/B^2,\ m = (1-2A^2)k/A,\\
&n = (-2+A^2)(A-B^2k)/(AB^2),\ p = (-2+A^2)k,\\
&q = (1-2A^2)(A-B^2k)/(A^2B^2),\ r = Ak,\\
&s = (A-B^2k)/(AB^2),\ kB(sA+k) \neq 0;
\end{aligned} \tag{68}$$

$$\begin{aligned}
&a = A(B^2s-1)/B,\ b = (1 + A^2 - A^2B^2s)/B,\\
&c = -(1 - 2A^2 + 2A^2B^2s)/B,\ d = A(1+2B^2s)/B,\\
&f = A^3(-2+B^2s)/B,\ g = -(1+B^2s)/B,\\
&k = A(1-B^2s)/B^2,\ l = -A^3s,\ n = 3A^2s,\\
&m = 3A^2(-1+B^2s)/B^2,\ p = 3A^3(1-B^2s)/B^2,\\
&q = -3As,\ r = A^4(-1+B^2s)/B^2,\ A \neq 0;
\end{aligned} \tag{69}$$

$$\begin{aligned}
&a = -(A^3 + B^2k - A^2B^2k + AB^4k^2 - 2A^3B^2s +\\
&\qquad 2A^2B^4ks + A^3B^4s^2)/((1+A^2)B),\\
&b = (1 + 2AB^2k + A^2B^4k^2 + A^2B^2s - A^4B^2s +\\
&\qquad 2A^3B^4ks + A^4B^4s^2)/((1+A^2)B),\\
&c = (-1 - 3A^2 + 4AB^2k + 2A^2B^4k^2 + 2A^2B^2s -\\
&\qquad 2A^4B^2s + 4A^3B^4ks + 2A^4B^4s^2)/((1+A^2)B),\\
&d = -(-3A - A^3 + 2B^2k - 2A^2B^2k + 2AB^4k^2 -\\
&\qquad 4A^3B^2s + 4A^2B^4ks + 2A^3B^4s^2)/((1+A^2)B),\\
&f = -A^2(-A + 3B^2k + A^2B^2k + AB^4k^2 + 2AB^2s +\\
&\qquad 2A^2B^4ks + A^3B^4s^2)/((1+A^2)B),
\end{aligned} \tag{70}$$

$$g = (-1 - 2AB^2k + B^4k^2 - B^2s - 3A^2B^2s + 2AB^4ks + A^2B^4s^2)/((1 + A^2)B),$$
$$l = -A^3s,\ m = -3Ak,\ n = 3A^2s,\ p = 3A^2k,$$
$$q = -3As,\ r = -A^3k,\ kB(sA + k) \neq 0.$$

5.1 Integrability of the cubic systems {(2),(57)}–{(2),(8)}

The *Systems* {(2),(57)}, {(2),(58)}, {(2),(62)}, {(2),(63)}, {(2),(66)}, {(2),(69)}, {(2),(8)} are integrable and they have the following integrating factors, respectively:

$$\mu(x, y) = 1/(1 + dy)^3,$$
$$\mu(x, y) = 1/(1 + cx)^3,$$
$$\mu(x, y) = 1/(1 + cx + fy)^3,$$
$$\mu(x, y) = 1/(-B - Ax + y)^3,$$
$$\mu(x, y) = 1/(sx + ky + \gamma)^3,$$
$$\mu(x, y) = 1/(-B + x - Ay)^3,$$
$$\mu(x, y) = 1/(-B + x - Ay)^3,$$

and therefore all of them have a center at $(0, 0)$.

For *System* {(2),(59)}, the first Lyapunov quantity is $L_1 = cd/8$ and L_1 vanishes if $d = 0$. The system {{(2),(59)}, $d = 0$} is equivalent with the system {(2),(58)}.

For *Systems* {(2),(60)}, {(2),(61)}, {(2),(64)}, {(2),(68)}, the first Lyapunov quantity has the form, respectively:

$$L_1 = -cf/4,$$
$$L_1 = (1 + Bf)(1 + B^4l^2)^2/(4B^4l),$$
$$L_1 = (p - cf)/4,$$
$$L_1 = (1 + A^2)^2(-2A - ABg + B^2k)/(4A^2B^2).$$

If the quantity L_1 vanishes, then all of these systems are degenerate.

For *System* {(2),(65)} the first Lyapunov quantity $L_1 = -p/4$ is not equal to zero and therefore {(2),(65)} has a focus at $(0, 0)$.

The first Lyapunov quantity L_1 for *System* {(2),(67)} has the form $L_1 = (k^2 + s^2)(s^3 - k^2\gamma^2 - s^2\gamma^2)/(4ks^2\gamma^2)$. If $L_1 = 0$, then the system is integrable and has the following integrating factor

$$\mu(x, y) = 1/(sx + ky + \gamma)^3.$$

We have proved the following theorem:

Theorem 3. *Cubic differential system* (2) *with the line at infinity of algebraic multiplicity at least two and an invariant straight line* l_1 *of the multiplicity three has a center at origin* $(0, 0)$ *if and only if it has an integrating factor of the form* $\mu(x, y) = 1/l_1^3$.

6 The problem of the center for cubic systems with a real affine invariant line of multiplicity four

Without loss of generality we consider that the real invariant straight line is described by equation $x-1=0$. Then there are only three families of cubic systems (2) for which the line $x-1=0$ has multiplicity four [29]:

$$\dot{x}=y(x-1)(x-fy-1),\ \dot{y}=-(x-2x^2+y^2+x^3-xy^2+fy^3),\ f\neq 0; \tag{71}$$

$$\dot{x}=y(x-1)^2,\ \dot{y}=-(x-2x^2+dxy+2y^2+x^3+qx^2y+2xy^2),\ d\neq 0; \tag{72}$$

$$\begin{array}{l}\dot{x}=-(x-1)((b-1)^2x^2+(2b-3)fxy+fy(1+fy))/f,\\ \dot{y}=-(x-1)(x^2-x-y^2)-fy^3-(b-1)y^2(1+2x)\\ \qquad -x(b-1)^2(x(x-1)+fy(x+2)+x(b-1))/f^2,\ f(b-1)\neq 0.\end{array} \tag{73}$$

The system (71) has a first integral

$$F(x,y)=(x-1)^6\exp\left[\frac{4-6x+3y^2+2x^3-3xy^2+2fy^3}{(1-x)^3}\right].$$

For system (72) the first Lyapunov quantity looks as $L_1=-q/4$. If $q=0$, then (72) has an integrating factor

$$\mu(x,y)=\frac{1}{(x-1)^6}\exp\left[\frac{d(-d+3dx+6y-6xy)}{6(x-1)^3}\right].$$

System (73) has a first integral of the form $F(x,y)=f_1^{\alpha_1}f_2^{\alpha_2}f_3^{\alpha_3}f_4^{\alpha_4}$, where

$$\begin{array}{l}f_1=x-1,\ \ f_2=\exp\left[\frac{b-1+fy}{f(x-1)}\right],\ \ f_3=\exp\left[\frac{1-2b+b^2+2bfy-2fxy+f^2y^2}{(x-1)^2}\right],\\ f_4=\exp[\frac{1}{(x-1)^3}\cdot(-7+12b-6b^2+b^3+3f^2+3(3-4b+b^2-2f^2)x\\ \qquad +3f(b-2)(b-1)y+3(b-1+f^2)x^2+3f(b-1)xy+3f^2(b-1)y^2+f^3y^3)],\\ \alpha_1=6((b-1)^2+f^2),\ \alpha_2=6f(2-b^2),\ \alpha_3=3(3-2b),\ \alpha_4=-2.\end{array}$$

The Main Theorem results from Sections 2–6.

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