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Sufficient $GL(2,\mathbb{R})$-invariant center conditions for some classes of two-dimensional cubic differential systems

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Abstract. The autonomous two-dimensional polynomial cubic systems of differential equations with pure imaginary eigenvalues of the Jacobian matrix at the singular point $(0,0)$ are considered in this paper. The center problem was studied for three classes of such systems: the class of cubic systems with zero divergence of the cubic homogeneities ($S_3 \equiv 0$), the class of cubic systems with zero divergence of the quadratic homogeneities ($S_2 \equiv 0$) and the class of cubic systems with nonzero divergence of the quadratic homogeneities ($S_2 \not\equiv 0$). For these systems, sufficient $GL(2,\mathbb{R})$-invariant center conditions for the origin of coordinates of the phase plane were established.

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1 Preliminaries

Let us consider the cubic system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= P_1(x,y) + P_2(x,y) + P_3(x,y) = P(x,y), \\ \frac{dy}{dt} &= Q_1(x,y) + Q_2(x,y) + Q_3(x,y) = Q(x,y), \end{aligned} \tag{1}$$

where $P_i(x,\ y),\ Q_i(x,\ y)$ are homogeneous polynomials of degree i in x and y with real coefficients. The system (1) can be written in the following coefficient form:

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \frac{dy}{dt} &= ex + fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3. \end{aligned} \tag{2}$$

Definition 1. [1] *Let $\varphi(x,y)$ and $\psi(x,y)$ be homogeneous polynomials in x and y with real coefficients of the degrees $\rho_1 \in \mathbb{N}^*$ and $\rho_2 \in \mathbb{N}^*$, respectively, and $j \in \mathbb{N}^*$. The polynomial*

$$(\varphi,\psi)^{(j)} = \frac{(\rho_1 - j)!(\rho_2 - j)!}{\rho_1!\rho_2!} \sum_{i=0}^{j} (-1)^i \binom{j}{i} \frac{\partial^j \varphi}{\partial x^{j-i} \partial y^i} \frac{\partial^j \psi}{\partial x^i \partial y^{j-i}} \tag{3}$$

is called the transvectant of index j of polynomials φ and ψ.

Using this formula we have the following remark.

Remark 1. [2] *If polynomials φ and ψ are $GL(2,\mathbb{R})$-comitants* [3–5] *of the degrees $\rho_1 \in \mathbb{N}^*$ and $\rho_2 \in \mathbb{N}^*$, respectively, for the system* (1), *then the transvectant of index $j \le \min\{\rho_\varphi, \rho_\psi\}$ is a $GL(2,\mathbb{R})$-comitant of the degree $\rho_\varphi + \rho_\psi - 2j$ for the system* (1). *If $j > \min\{\rho_\varphi, \rho_\psi\}$, then $(\varphi,\psi)^{(j)} = 0$.*

The $GL(2,\mathbb{R})$-comitants of the first degree with respect to the coefficients of the system (1) have the form

$$R_i = P_i(x,y)y - Q_i(x,y)x,\ S_i = \frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}\right),\ i = 1,2,3. \tag{4}$$

By using the comitants R_i and S_i, $i = 1,2,3$, the system (1) can be written [6] in the form

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2}\frac{\partial R_1}{\partial y} + \frac{1}{2}S_1 x + \frac{1}{3}\frac{\partial R_2}{\partial y} + \frac{2}{3}S_2 x + \frac{1}{4}\frac{\partial R_3}{\partial y} + \frac{3}{4}S_3 x,\\ \frac{dy}{dt} &= -\frac{1}{2}\frac{\partial R_1}{\partial x} + \frac{1}{2}S_1 y - \frac{1}{3}\frac{\partial R_2}{\partial x} + \frac{2}{3}S_2 y - \frac{1}{4}\frac{\partial R_3}{\partial x} + \frac{3}{4}S_3 y.\end{aligned} \tag{5}$$

For every homogeneous $GL(2,\mathbb{R})$-comitant $\mathcal{K}(x,y)$ with degree $\rho \in \mathbb{N}^*$ of the system (1) from (5) we obtain the total derivative of $\mathcal{K}(x,y)$ with respect to t [7]:

$$\begin{gathered}\frac{d\mathcal{K}}{dt} = \frac{\partial\mathcal{K}}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial\mathcal{K}}{\partial y}\cdot\frac{dy}{dt} =\\ \frac{\partial\mathcal{K}}{\partial x}\left(\frac{1}{2}\frac{\partial R_1}{\partial y} + \frac{1}{2}S_1 x + \frac{1}{3}\frac{\partial R_2}{\partial y} + \frac{2}{3}S_2 x + \frac{1}{4}\frac{\partial R_3}{\partial y} + \frac{3}{4}S_3 x\right) +\\ + \frac{\partial\mathcal{K}}{\partial y}\left(-\frac{1}{2}\frac{\partial R_1}{\partial x} + \frac{1}{2}S_1 y - \frac{1}{3}\frac{\partial R_2}{\partial x} + \frac{2}{3}S_2 y - \frac{1}{4}\frac{\partial R_3}{\partial x} + \frac{3}{4}S_3 y\right) =\\ = \rho(\mathcal{K},R_1)^{(1)} + \frac{\rho}{2}\mathcal{K}S_1 + \rho(\mathcal{K},R_2)^{(1)} + \frac{2\rho}{3}\mathcal{K}S_2 + \rho(\mathcal{K},R_3)^{(1)} + \frac{3\rho}{4}\mathcal{K}S_3,\end{gathered} \tag{6}$$

where $(\mathcal{K},R_i)^{(1)}$ is a Jacobian (the transvectant of the first index) of $GL(2,\mathbb{R})$-comitants $\mathcal{K}$ and R_i. The representation (6) shows that the derivative with respect to t of every homogeneous $GL(2,\mathbb{R})$-comitant with the degree $\rho \ge 1$ of the system is a $GL(2,\mathbb{R})$-comitant too.

By using the comitants R_i and S_i ($i = 1,2,3$), and the transvectant (3) the following $GL(2,\mathbb{R})$-invariants [3–5] of the system (1) were constructed:

$$\begin{gathered}I_1 = S_1, \qquad I_2 = (R_1,R_1)^{(2)}, \qquad I_4 = (R_1,S_3)^{(2)},\\ I_{18} = ((R_3,R_1)^{(2)},S_3)^{(2)}, \quad I_{20} = ((R_2,R_2)^{(2)},S_3)^{(2)}, \quad I_{22} = ((S_3,S_2)^{(1)},S_2)^{(1)},\\ I_{38} = (((R_2,R_1)^{(2)},R_1)^{(1)},S_2)^{(1)}, \qquad I_{61} = (((R_3,S_3)^{(2)},R_1)^{(1)},S_3)^{(2)},\\ I_{111} = ((((R_3,R_1)^{(2)},R_1)^{(1)},S_2)^{(1)},S_2)^{(1)},\end{gathered}$$

$$I_{112} = (((R_2, R_1)^{(2)}, R_1)^{(1)}, (R_2, S_3)^{(2)})^{(1)},$$

$$I_{125} = ((((R_2, R_1)^{(1)}, S_2)^{(1)}, S_2)^{(1)}, S_2)^{(1)},$$

$$I_{174} = (((R_2, S_3)^{(2)}, S_3)^{(1)}, (R_2, S_3)^{(2)})^{(1)},$$

$$I_{278} = (((((R_3, R_1)^{(1)}, S_2)^{(1)}, S_2)^{(1)}, S_2)^{(1)}, S_2)^{(1)}.$$

We will consider the system (1) with the conditions $I_1 = 0, I_2 > 0$. These conditions mean that the eigenvalues of the Jacobian matrix at the singular point $(0,0)$ are pure imaginary, i.e., the system has the center or a weak focus at $(0,0)$. In these conditions the system (1) can be reduced, via a linear transformation and time rescaling, to the system (by preserving the same notations for the coefficients):

$$\begin{aligned} \frac{dx}{dt} &= y + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \frac{dy}{dt} &= -x + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3. \end{aligned} \tag{7}$$

The center-focus problem for the system (1) was investigated in many papers. The necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions for distinguishing the center and focus for the system (1) (or (2)) were established in the case, when

$$P_3(x,y) \equiv Q_3(x,y) \equiv 0 \quad (R_3 \equiv S_3 \equiv 0)\,,$$

i.e. for the quadratic system of differential equations [3,4,8], also in the case, when

$$P_2(x,y) \equiv Q_2(x,y) \equiv 0 \quad (R_2 \equiv S_2 \equiv 0)\,,$$

i.e. for the system with nonlinearities of the third degree [4,9].

In the paper [10] the center-focus problem for the system (1) with $I_1 = 0$, $I_2 > 0$ and $R_3 \equiv 0$ was solved in terms of the coefficients of the normal forms (canonical forms) of this system. The necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions for the center-focus problem for the system (1) with $I_1 = 0$, $I_2 > 0$ and $R_3 \equiv 0$ were obtained in [11]. Also, the necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions for the center-focus problem for the system (1) with $I_1 = 0$, $I_2 > 0$, $R_2 \equiv 0$ and $S_3 \equiv 0$ were established in [12].

In the paper [13] the necessary and sufficient $GL(2,\mathbb{R})$-invariant conditions for the center-focus problem for a class of autonomous two-dimensional polynomial systems of differential equations with nonlinearities of the fourth degree were obtained.

In this paper we study the center-focus problem for the system (1) with $I_1 = 0$, $I_2 > 0$, $S_3 \equiv 0$, for the system (1) with $I_1 = 0$, $I_2 > 0$, $S_2 \equiv 0$, also for the system (1) with $I_1 = 0$, $I_2 > 0$, $S_2 \not\equiv 0$. For all these systems sufficient $GL(2,\mathbb{R})$-invariant center conditions for the origin of coordinates of the phase plane were established.

2 Sufficient $GL(2,\mathbb{R})$-invariant center conditions for system (1) with $I_1 = 0$, $I_2 > 0$ and $S_3 \equiv 0$

Theorem 1. *If the system* (1) *with* $I_1 = 0$, $I_2 > 0$, $S_3 \equiv 0$ *fulfills the conditions*

$$I_{38} = I_{111} = I_{125} = I_{278} = 0, \tag{8}$$

then the origin of coordinates of the phase plane of the system (1) *is a singular point of the center type.*

Proof. In this case $S_3 = (p+u)x^2 + 2(q+v)xy + (r+w)y^2 \equiv 0$ and via rotation we can obtain $g + m = 0$ in the system (7). So, the system (1) with $I_1 = 0$, $I_2 > 0$, $S_3 \equiv 0$, i.e. the system (7) with $S_3 \equiv 0$ can be reduced to the form (by preserving the same notations of the coefficients):

$$\begin{aligned}\frac{dx}{dt} &= y + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3,\\ \frac{dy}{dt} &= -x + lx^2 - 2gxy + ny^2 + tx^3 - 3px^2y - 3qxy^2 - ry^3.\end{aligned} \tag{9}$$

For the system (9) the comitant S_2 and the invariants I_{38}, I_{111}, I_{125}, I_{278} have the following values:

$$\begin{aligned}&S_2 = (h+n)y, &&\\ &I_{38} = -(g+k)(h+n), && I_{111} = -(p+r)(h+n)^2,\\ &I_{125} = -g(h+n)^3, && I_{278} = -p(h+n)^4.\end{aligned} \tag{10}$$

So, the conditions (8) imply the system of equalities:

$$\begin{aligned}I_{38} &= -(g+k)(h+n) = 0,\\ I_{111} &= -(p+r)(h+n)^2 = 0,\\ I_{125} &= -g(h+n)^3 = 0,\\ I_{278} &= -p(h+n)^4 = 0,\end{aligned}$$

which should can be verified in the following two cases:

Case I. $h + n = 0$;

Case II. $h + n \neq 0$, $g = k = p = r = 0$.

In the case I the equality $h + n = 0$ implies $S_2 \equiv 0$. So, the divergence of the system (9) is $S \equiv S_1 + 2S_2 + 3S_3 \equiv 0$ and the origin of coordinates of the phase plane of the system (9) is a singular point of the center type. The system (1) with $S \equiv S_1 + 2S_2 + 3S_3 \equiv 0$ has the first integral

$$\frac{1}{2}R_1 + \frac{1}{3}R_2 + \frac{1}{4}R_3 = c,$$

where c is a real constant. The relation (6) directly implies this result:

$$\frac{d\left(\frac{1}{2}R_1+\frac{1}{3}R_2+\frac{1}{4}R_3\right)}{dt} = (R_1,R_1)^{(1)}+(R_1,R_2)^{(1)}+(R_1,R_3)^{(1)}+(R_2,R_1)^{(1)}+$$
$$+(R_2,R_2)^{(1)}+(R_2,R_3)^{(1)}+(R_3,R_1)^{(1)}+(R_3,R_2)^{(1)}+(R_3,R_3)^{(1)}=0,$$

because $(R_i,R_i)^{(1)}=0$, and $(R_i,R_j)^{(1)}=-(R_j,R_i)^{(1)}$, $i\neq j$, $i,j=1,2,3$.

In the case II the system (9) is reduced to the system:

$$\begin{aligned}\frac{dx}{dt}&=y+2hxy+3qx^2y+sy^3,\\ \frac{dy}{dt}&=-x+lx^2+ny^2+tx^3-3qxy^2.\end{aligned}\tag{11}$$

For the system (11), the condition

$$Q(x;-y)P(x;y)=-P(x;-y)Q(x;y)\tag{12}$$

is fulfilled, i.e. the straight line defined by the equation $y=0$ is a symmetry axis for the system (11). So, the point $(0;0)$ is a singular point of center type for the system (11), i.e. for the system (9) with $g=k=p=r=0$.

Thus, if for the system (9) the conditions (8) are fulfilled, then the origin of coordinates of the phase plane of the system (9) is a singular point of center type.

We remark that zero - equalities of the $GL(2,\mathbb{R})$-invariants and zero - identities of the $GL(2,\mathbb{R})$-comitants are preserved by non-degenerate linear transformations of the system, also by time rescaling.

Because the system (9) was obtained from the system (1) (or (2)) with $I_1=0$, $I_2>0$ and $S_3\equiv 0$ by some non-degenerate linear transformations and time rescaling, and the polynomials I_{38}, I_{111}, I_{125}, I_{278} are $GL(2,\mathbb{R})$-invariants, we are done.

Theorem 1 is proved.

3 Sufficient $GL(2,\mathbb{R})$-invariant center conditions for system (1) with $I_1=0$, $I_2>0$ and $S_2\equiv 0$

Theorem 2. *If the system* (1) *with* $I_1=0$, $I_2>0$, $S_2\equiv 0$ *fulfills the conditions*

$$I_4=I_{18}=I_{20}=I_{61}=I_{112}=I_{174}=0,\tag{13}$$

then the origin of coordinates of the phase plane of the system (1) *is a singular point of the center type.*

Proof. In this case the comitant $S_2=(g+m)x+(h+n)y\equiv 0$, for the system (7) the invariant $I_4=p+r+u+w=0$ and after a rotation we can obtain $q+v=0$ in the system (7). So, we can consider in the system (7) $m=-g$, $n=-h$, $w=-p-r-u$, $v=-q$. Thus, the system (1) with $I_1=0$, $I_2>0$, $S_2\equiv 0$ and $I_4=0$, i.e. the system (7) with $S_2\equiv 0$ and $I_4=0$ can be reduced to the form (by preserving the same notations of the coefficients):

$$\begin{aligned}\frac{dx}{dt} &= y + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3,\\ \frac{dy}{dt} &= -x + lx^2 - 2gxy - hy^2 + tx^3 + 3ux^2y - 3qxy^2 - (p+r+u)y^3.\end{aligned} \tag{14}$$

For the system (14) the comitant S_3 and the invariants I_{18}, I_{20}, I_{61}, I_{112}, I_{174} have the following values:

$$\begin{aligned}&S_3 = (p+u)(x^2 - y^2), \quad I_{18} = (s+t)(p+u),\\ &I_{61} = 2(r+u)(p+u)^2, \quad I_{20} = 2(g^2 - h^2 + gk + hl)(p+u),\\ &I_{112} = (g^2 - h^2 - k^2 + l^2)(p+u),\\ &I_{174} = -(g+h-k+l)(g-h-k-l)(p+u)^3.\end{aligned} \tag{15}$$

Thus, the conditions (13) imply the system of equalities:

$$\begin{aligned}&I_{18} = (s+t)(p+u) = 0,\\ &I_{61} = 2(r+u)(p+u)^2 = 0,\\ &I_{20} = 2(g^2 - h^2 + gk + hl)(p+u) = 0,\\ &I_{112} = ((g+h)(g-h) - (k+l)(k-l))(p+u) = 0,\\ &I_{174} = -((g+h) - (k-l))((g-h) - (k+l))(p+u)^3 = 0.\end{aligned} \tag{16}$$

The invariant $I_{174} = 0$ in one of the following three cases:
1) $p + u = 0$; 2) $g + h = k - l$; 3) $g - h = k + l$.

If 1) $p + u = 0$, then the system of equalities (16) is fulfilled.

If 2) $g + h = k - l$, $g - h \neq k + l$ and $p + u \neq 0$, then from the equality $I_{112} = 0$ we obtain $(g + h)(g - h - k - l)(p + u) = 0$, i.e. $g + h = 0$, and then $k - l = 0$. The equalities $g + h = 0$ and $k - l = 0$ imply $I_{20} = 0$. So, in this case the system of equalities (16) is reduced to the conditions:

$$g + h = k - l = s + t = r + u = 0. \tag{17}$$

If 3) $g - h = k + l$, $g + h \neq k - l$ and $p + u \neq 0$, then from the equality $I_{112} = 0$ we obtain $(g - h)(g + h - k + l)(p + u) = 0$, i.e. $g - h = 0$, and then $k + l = 0$. The equalities $g - h = 0$ and $k + l = 0$ imply $I_{20} = 0$. So, in this case the system of equalities (16) is reduced to the conditions:

$$g - h = k + l = s + t = r + u = 0. \tag{18}$$

If 2) $g + h = k - l$, 3) $g - h = k + l$ and $p + u \neq 0$, then $g = k$ and $h = -l$. From the equality $I_{20} = 0$ we obtain $(g + h)(g - h) = 0$, i.e. $g + h = k - l = 0$ or $g - h = k + l = 0$. In the case $g + h = k - l = 0$, as well as in the case $g - h = k + l = 0$ the invariant $I_{112} = 0$. Thus, in this case the system of equalities (16) is reduced to one of the series of conditions (17) or (18).

So, the system of equalities (16) should be verified in the following cases:

Case I. $p + u = 0$;

Case II. $p+u \neq 0$, $g+h=k-l=s+t=r+u=0$ $(h=-g,\ l=k,\ t=-s,\ u=-r)$;

Case III. $p+u \neq 0$, $g-h=k+l=s+t=r+u=0$ $(h=g,\ l=-k,\ t=-s,\ u=-r)$.

In the case I the equality $p+u=0$ implies $S_3 \equiv 0$. So, the divergence of the system (14) $S \equiv S_1+2S_2+3S_3 \equiv 0$ and the origin of coordinates of the phase plane of the system (14) is a singular point of the center type.

In the case II the system (14) is reduced to the system:

$$\begin{aligned} \frac{dx}{dt} &= y + gx^2 - 2gxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \frac{dy}{dt} &= -x + kx^2 - 2gxy + gy^2 - sx^3 - 3rx^2y - 3qxy^2 - py^3. \end{aligned} \tag{19}$$

The trajectories of the system (19) are symmetric with respect to the straight line defined by the equation $x+y=0$. By the rotation of axes

$$x_1 = x\cos\alpha + y\sin\alpha, \quad y_1 = -x\sin\alpha + y\cos\alpha \tag{20}$$

with the angle $\alpha = \frac{\pi}{4}$, the system (19) can be reduced to the form:

$$\begin{aligned} \frac{dx_1}{dt} =& y_1 - \frac{1}{\sqrt{2}}(g-k){x_1}^2 + \frac{1}{\sqrt{2}}(3g+k){y_1}^2 + \\ &+ \frac{3}{2}(-p-q+r+s){x_1}^2y_1 + \frac{1}{2}(-p+3q-3r+s){y_1}^3, \\ \frac{dy_1}{dt} =& -x_1 + \frac{2}{\sqrt{2}}(g-k)x_1y_1 + \\ &+ \frac{1}{2}(-p-3q-3r-s){x_1}^3 + \frac{3}{2}(-p+q+r-s)x_1{y_1}^2. \end{aligned} \tag{21}$$

For the system (21) the condition

$$Q(-x_1;y_1)P(x_1;y_1) = -P(-x_1;y_1)Q(x_1;y_1) \tag{22}$$

is fulfilled, i.e. the straight line defined by the equation $x_1=0$ is a symmetry axis for the system (21), i.e. the straight line defined by the equation $x+y=0$ is the symmetry axis for the system (19). It results that the origin of coordinates of the phase plane of the system (19) is a singular point of the center type, i.e. for the system (14) with $g+h=k-l=s+t=r+u=0$.

In the case III the system (14) is reduced to the system:

$$\begin{aligned} \frac{dx}{dt} &= y + gx^2 + 2gxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \frac{dy}{dt} &= -x - kx^2 - 2gxy - gy^2 - sx^3 - 3rx^2y - 3qxy^2 - py^3. \end{aligned} \tag{23}$$

The trajectories of the system (23) are symmetric with respect to the straight line defined by the equation $x-y=0$. By the rotation of axes with the angle

$\alpha = -\dfrac{\pi}{4}$, the system (23) can be reduced to the form:

$$\begin{aligned}\frac{dx_1}{dt} =& y_1 - \frac{1}{\sqrt{2}}(g-k){x_1}^2 + \frac{1}{\sqrt{2}}(3g+k){y_1}^2 + \\ &+ \frac{3}{2}(p-q-r+s){x_1}^2 y_1 + \frac{1}{2}(p+3q+3r+s){y_1}^3, \\ \frac{dy_1}{dt} =& - x_1 + \frac{2}{\sqrt{2}}(g-k)x_1 y_1 + \\ &+ \frac{1}{2}(p-3q+3r-s){x_1}^3 + \frac{3}{2}(p+q-r-s)x_1{y_1}^2.\end{aligned} \tag{24}$$

For the system (24) the condition (22) is fulfilled, i.e. the straight line defined by the equation $x_1 = 0$ is a symmetry axis for the system (24), i.e. the straight line defined by the equation $x - y = 0$ is the symmetry axis for the system (23). It results that the origin of coordinates of the phase plane of the system (23) is a singular point of the center type, i.e. for the system (23) with $g - h = k + l = s + t = r + u = 0$.

Thus, if for the system (14) the conditions

$$I_{18} = I_{20} = I_{61} = I_{112} = I_{174} = 0,$$

are fulfilled, then the origin of coordinates of the phase plane of the system (14) is a singular point of the center type.

Because the system (14) was obtained from the system (1) (or (2)) with $I_1 = 0$, $I_2 > 0$, $S_3 \equiv 0$ and $I_4 = 0$ by some non-degenerate linear transformations, and the polynomials I_4, I_{18}, I_{20}, I_{61}, I_{112}, I_{174} are $GL(2,\mathbb{R})$-invariants, we are done.

Theorem 2 is proved.

4 Sufficient $GL(2,\mathbb{R})$-invariant center conditions for system (1) with $I_1 = 0$, $I_2 > 0$ and $S_2 \not\equiv 0$

Theorem 3. *If the system* (1) *with* $I_1 = 0$, $I_2 > 0$ *and* $S_2 \not\equiv 0$ *fulfills the conditions*

$$I_4 = I_{22} = I_{38} = I_{111} = I_{125} = I_{278} = 0, \tag{25}$$

then the origin of coordinates of the phase plane of the system (1) *is a singular point of the center type.*

Proof. In this case after a rotation we can obtain $g + m = 0$ in the system (7). So, the system (1) with $I_1 = 0$, $I_2 > 0$, i.e. the system (7), can be reduced to the form (by preserving the same notations of the coefficients):

$$\begin{aligned}\frac{dx}{dt} &= y + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \frac{dy}{dt} &= -x + lx^2 - 2gxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3.\end{aligned} \tag{26}$$

For the system (26) the comitant S_2 and the invariants I_4, I_{22},I_{38}, I_{111}, I_{125}, I_{278} have the following values:

$$\begin{aligned}
&S_2 = (h+n)y, & &\\
&I_4 = p + r + u + w, & &I_{22} = (p+u)(h+n)^2,\\
&I_{38} = -(g+k)(h+n), & &I_{111} = -\frac{1}{4}(p+3r-3u-w)(h+n)^2,\\
&I_{125} = -g(h+n)^3, & &I_{278} = -\frac{1}{4}(p-3u)(h+n)^4.
\end{aligned} \tag{27}$$

Thus, the conditions (25) imply the system of equalities:

$$\begin{aligned}
&I_4 = p + r + u + w = 0,\\
&I_{22} = (p+u)(h+n)^2 = 0,\\
&I_{38} = -(g+k)(h+n),\\
&I_{111} = -\frac{1}{4}(p+3r-3u-w)(h+n)^2 = 0,\\
&I_{125} = -g(h+n)^3 = 0,\\
&I_{278} = -\frac{1}{4}(p-3u)(h+n)^4 = 0.
\end{aligned} \tag{28}$$

If $S_2 \not\equiv 0$, then the system (28) is fulfilled if the following conditions are fulfilled:

$$g = k = p = r = u = w = 0.$$

In this case the system (26) is reduced to the system:

$$\begin{aligned}
\frac{dx}{dt} &= y + 2hxy + 3qx^2y + sy^3,\\
\frac{dy}{dt} &= -x + lx^2 + ny^2 + tx^3 + 3vxy^2.
\end{aligned} \tag{29}$$

For the system (29), the condition (12) is fulfilled, i.e. the straight line defined by the equation $y = 0$ is a symmetry axis for the system (29). So, the point $(0;0)$ is a singular point of center type for the system (29), i.e. for the system (26) with $g = k = p = r = u = w = 0$.

So, if for the system (26) the conditions (25) are fulfilled, then the origin of coordinates of the phase plane of the system (26) is a singular point of the center type.

Because the system (26) was obtained from the system (1) (or (2)) with $I_1 = 0$, $I_2 > 0$ and $S_2 \not\equiv 0$ by some non-degenerate linear transformations, and the polynomials I_4, I_{22}, I_{38}, I_{111}, I_{125}, I_{278} are $GL(2,\mathbb{R})$-invariants, we are done.

Theorem 3 is proved.

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