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# On limit cycles of polynomial systems of the first-order ODE's

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**Abstract.** Examples of four-dimensional Riemann metrics related with the ODE's of second order are constructed. Their properties and applications to the polynomial systems of ODE's of first order are considered.

## 1 Introduction

The second order ODE's

$$\frac{d^2y}{dx^2} + a_1(x,y)\left(\frac{dy}{dx}\right)^3 + 3a_2(x,y)\left(\frac{dy}{dx}\right)^2 + 3a_3(x,y)\frac{dy}{dx} + a_4(x,y) = 0 \tag{1}$$

are invariant with respect to non degenerate changes of the variables $x = f(u,v),\ y = g(u,v)$ and they have numerous applications to the theory of dynamical systems and differential geometry.

We will use this theory to study properties of polynomial systems of the first order ODE's

$$\frac{dy}{ds} = Q_n(x,y,a_i), \quad \frac{dx}{ds} = P_n(x,y,a_i)$$

containing a limit cycles at some value of parameters $a_i$.

## 2 Metrics of Riemann extensions in the theory of the second order ODE's

The equation (1) can be presented in the form of the system of equations

$$\frac{d^2}{ds^2}y(s)+a_4(x,y)\left(\frac{d}{ds}x(s)\right)^2+2\,a_3(x,y)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s)+a_2(x,y)\left(\frac{d}{ds}y(s)\right)^2=0,$$

$$\frac{d^2}{ds^2}x(s)-a_3(x,y)\left(\frac{d}{ds}x(s)\right)^2-2\,a_2(x,y)\left(\frac{d}{ds}x(s)\right)\frac{d}{ds}y(s)-a_1(x,y)\left(\frac{d}{ds}y(s)\right)^2=0, \tag{2}$$

which allows us to consider them as equations of geodesics of two-dimensional space in the coordinates $M(x,y)$ with components of affine connection $\Pi^i_{jk} = \Pi^i_{jk}(x,y)$

$$\frac{d^2x^i}{ds^2} + \Pi^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

To study solutions of system (2) we use the four-dimensional space equipped with of the Riemann metric

$$ds^2 = (2\,z a_3(x,y) - 2\,t a_4(x,y))\,dx^2 + 2\,\left(2\,z a_2(x,y) - 2\,t a_3(x,y)\right)dxdy + 2\,dxdz +$$
$$+ (2\,z a_1(x,y) - 2\,t a_2(x,y))\,dy^2 + 2\,dydt, \tag{3}$$

The full system of geodesics of the metric (3) decomposes into two parts.

The first part has the form of the linear system of equations for the coordinates $(\Psi = (\Psi_1 = z(s), \Psi_2 = t(s)))$

$$\frac{d^2\vec{\Psi}}{ds^2} + A(x,y)\frac{d\vec{\Psi}}{ds} + B(x,y))\vec{\Psi} = 0,$$

where $(\vec{\Psi} = (\Psi_1 = z(s),\ \Psi_2 = t(s))$ and $A(x,y)$, $B(x,y)$ are the $2\times 2$ matrix-functions. While the second part of geodesics for the local coordinates $x^i = (x,y)$ is defined by the system of equations (2).

The corresponding Ricci tensor of the metric is of the form

$$R_{11} = 2\,\frac{\partial}{\partial y} a_4(x,y) - 2\,\frac{\partial}{\partial x} a_3(x,y) - 4\,a_3(x,y)^2 + 4\,a_4(x,y) a_2(x,y),$$

$$R_{12} = -2\,\frac{\partial}{\partial x} a_2(x,y) + 2\,\frac{\partial}{\partial y} a_3(x,y) - 2\,a2(x,y) a_3(x,y) + 2\,a_4(x,y) a_1(x,y),$$

$$R_{22} = -2\,\frac{\partial}{\partial x} a_1(x,y) + 2\,\frac{\partial}{\partial y} a_2(x,y) + 4\,a_1(x,y) a_3(x,y) - 4\,a_2(x,y)^2.$$

The Weyl tensor of the metric (3) is $C_{1212} = tL_1 - zL_2$, where

$$L_1 = \frac{\partial^2}{\partial y^2} a_4(x,y) + 3\left(\frac{\partial}{\partial y} a_4(x,y)\right) a_2(x,y) + 3\,a_4(x,y)\frac{\partial}{\partial y} a_2(x,y) - 2\,\frac{\partial^2}{\partial x \partial y} a_3(x,y) +$$
$$+ \frac{\partial^2}{\partial x^2} a_2(x,y) - \left(\frac{\partial}{\partial x} a_4(x,y)\right) a_1(x,y) - 2\left(\frac{\partial}{\partial x} a_1(x,y)\right) a_4(x,y) -$$
$$-3\,a_3(x,y)\left(2\,\frac{\partial}{\partial y} a_3(x,y) - \frac{\partial}{\partial x} a_2(x,y)\right)$$

and

$$L_2 = \frac{\partial^2}{\partial x^2} a_1(x,y) - 3\left(\frac{\partial}{\partial x} a_1(x,y)\right) a_3(x,y) - 3\,a_1(x,y)\frac{\partial}{\partial x} a_3(x,y) + \frac{\partial^2}{\partial y^2} a_3(x,y) -$$
$$-2\,\frac{\partial^2}{\partial x \partial y} a_2(x,y) + 2\left(\frac{\partial}{\partial y} a_4(x,y)\right) a_1(x,y) + a_4(x,y)\frac{\partial}{\partial y} a_1(x,y) -$$
$$-3\,a_2(x,y)\left(\frac{\partial}{\partial y} a_3(x,y) - 2\,\frac{\partial}{\partial x} a_2(x,y)\right)$$

are the Liouville expressions which generate the invariants of the equation (1).

In accordance with the Liouville theory the invariant

$$\nu_5 = L_2(L_1L_{2x} - L_2L_{1x}) + L_1(L_2L_{1y} - L_1L_{2y}) - a_1L_1^3 + 3a_2L_1^2L_2 - 3a_3L_1 + a_4L_2^3$$

is important to the theory of equation (1).

In case $\nu_5 \neq 0$ properties of the equation (1) are characterized by the absolute invariants

$$[5t_m - (m-2)t_7t_{m-2}]\nu_5^{2/5}] = 5(L_1\frac{\partial t_{m-2}}{\partial y} - L_2\frac{\partial t_{m-2}}{\partial x})$$

and by relations between them, where

$$t_m = \nu_m\nu_5^{-m/5}, \quad \nu_{m+2} = L_1\nu_{my} - L_2\nu_{mx} + m\nu_m(L_{2x} - L_{1y}).$$

## 3 Quadratic first order equation

**Theorem 1.** *In case $\nu_5 = 0$ the equation (1) admits invariant particular integral*

$$\frac{dy(x)}{dx} = -\frac{L_1(x,y)}{L_2(x,y)}$$

*if the following relation between coefficients $a_i(x,y)$ holds*

$$A_1\ a_1(x,y) + A_2\ a_2(x,y) + A_3\ a_3(x,y) + A_4\ a_4(x,y) + A_5 = 0. \tag{4}$$

*with some coefficients $A_i = A_i(x,y)$ that depend on the choice of the form of the equation $y' = h(x,y)$.*

As example in the case of the equation

$$\frac{d}{dx}y(x) = \frac{c_0 + c_1\,x + c_2\,y(x) + c_{11}\,x^2 + c_{12}\,xy(x) + c_{22}\,(y(x))^2}{e_0 + e_1\,x + e_2\,y(x) + e_{11}\,x^2 + e_{12}\,xy(x) + e_{22}\,(y(x))^2} \tag{5}$$

which determines properties of phase space of the quadratic system of equations

$$\dot{y} = c_0 + c_1x + c_2y + c_{11}x^2 + c_{12}xy + c_{22}y^2,$$

$$\dot{x} = e_0 + e_1x + e_2y + e_{11}x^2 + e_{12}xy + e_{22}y^2$$

with parameters $c_i,\ c_{jk}, e_i,\ e_{jk}$, the coefficients $A_i$ look cumbersome and we present their expressions for equations (5) with limit cycles.

1. Perko(1) system

$$\dot{x} = y + y^2, \quad \dot{y} = -x + \mu y - xy + (1+\mu)y^2. \tag{6}$$

The corresponding first order ODE looks as

$$\frac{d}{dx}y(x) = \frac{-x + \mu\,y(x) - xy(x) + (1+\mu)\,(y(x))^2}{y(x) + (y(x))^2} \tag{7}$$

and at the value of parameter $0 < \mu < 1/5$ system (6) has one limit cycle.

The second order ODE which has the particular integral defined by (7) has the form

$$\frac{d^2}{dx^2}y(x) + a1(x,y)\left(\frac{d}{dx}y(x)\right)^3 + 3\,a2(x,y)\left(\frac{d}{dx}y(x)\right)^2 + 3\,a3(x,y)\frac{d}{dx}y(x) +$$
$$+\frac{(y(x))^5 + (-\mu - x - x\mu + 2)(y(x))^4 + \left(-\mu + 3 - x - 3\,x\mu + x^2\right)(y(x))^3}{(y(x))^3(1+y(x))^3} +$$
$$+\frac{\left(3\,x^2 - 3\,x\mu + 1\right)(y(x))^2 + \left(3\,x^2 - x\mu\right)y(x) + x^2}{(y(x))^3(1+y(x))^3} = 0. \tag{8}$$

2. Perko(2)'s system

$$\dot{x} = y + y^2, \quad \dot{y} = -x/2 + \mu y - xy + (4/5 + mu)y^2$$

The first order ODE in this case is

$$\frac{d}{dx}y(x) = \frac{-1/2\,x + \mu\,y(x) - xy(x) + (4/5 + \mu)(y(x))^2}{y(x) + (y(x))^2}$$

and at the value of parameter $0 < \mu < 1/8$ the system has two limit cycles.

The corresponding second order ODE depends on three arbitrary coefficients $a_i(x,y)$ and has the form

$$\frac{d^2}{dx^2}y(x) + \frac{1}{100}\,\frac{100\,(y(x))^5 - (80\,\mu + 80\,x - 186 + 100\,x\mu)(y(x))^4}{y^3(1+y)^3} +$$
$$+\frac{1}{100}\,\frac{-\left(200\,x\mu - 100\,x^2 - 200 + 80\,\mu\right)(y(x))^3 - \left(-50 - 150\,x^2 + 150\,x\mu\right)(y(x))^2}{y^3(1+y)^3} -$$
$$-\frac{1}{100}\,\frac{\left(-100\,x^2 + 50\,x\mu\right)y(x)}{y^3(1+y)^3} = 0$$

3. Cherkas system

$$\dot{x} = 1 + xy, \quad \dot{y} = c_0 + c_1x + c_2y + c_{11}x^2 + c_{12}xy + c_{22}y^2.$$

The first order ODE

$$\frac{d}{dx}y(x) = \frac{c_0 + c_1\,x + c_2\,y(x) + c_{11}\,x^2 + c_{12}\,xy(x) + c_{22}\,(y(x))^2}{1 + xy(x)},$$

and for this system the existence of limit cycles of normal size in an amount of one to four is proved.

The corresponding second order ODE is of the form

$$\frac{d^2}{dx^2}y(x) + \frac{\left(xc_{22} - xc_{22}{}^2\right)y^4 + \left(-x^2c_{12}\,c_{22} - 2\,c_{22}{}^2 + c_{22} + xc_2 - xc_2\,c_{22}\right)y^3}{(1+yx)^3} +$$

$$+\frac{\left(-3\,c_{12}\,xc_{22}+c_{2}-3\,c_{2}\,c_{22}-c_{12}\,x+xc_{0}-x^{3}c_{11}\right)y^{2}}{(1+yx)^{3}}+$$

$$+\frac{x^{4}yc_{11}\,c_{12}+(c_{1}\,c_{12}+c_{11}\,c_{2})\,yx^{3}+\left(-c_{12}{}^{2}+c_{1}\,c_{2}+c_{0}\,c_{12}-2\,c_{11}\,c_{22}-3\,c_{11}\right)yx^{2}}{(1+yx)^{3}}+$$

$$+\frac{(-c_{1}-2\,c_{1}\,c_{22}-2\,c_{2}\,c_{12}+c_{0}\,c_{2})\,yx+\left(-2\,c_{0}\,c_{22}-c_{2}{}^{2}+c_{0}-c_{12}\right)y}{(1+yx)^{3}}=0$$

## 4 The homogeneous quadratic first order systems of equations

Here we use the theory of second order ODE to study of $2D$-planar systems of the first order ODE's.

**Proposition 1.** The system of equations

$$\frac{dx}{ds}=a_0+a_1x+a_2y+a_{11}x^2+a_{12}xy+a_{22}y^2,$$

$$\frac{dy}{ds}=b_0+b_1x+b_2y+b_{11}x^2+b_{12}xy+b_{22}y^2 \tag{9}$$

with parameters $a_i, a_{ij}$ and $b_i, b_{ij}$ after the extension on the projective plane according to standard rule takes form of the Pfaff equation

$$\left(x\tilde{Q}-\tilde{P}y\right)dz-\ z\tilde{Q}dx+\ z\tilde{P}dy=0, \tag{10}$$

where the functions $\tilde{P}, \tilde{Q}$ are a homogeneous polynomials in the variables $(x, y, z)$.

In explicit form we get the expression

$$\left(xb_0\,z^2+b_1\,x^2z+xb_2\,yz+b_{11}\,x^3+b_{12}\,x^2y+xb_{22}\,y^2-ya_0\,z^2\right)dz-$$
$$-\left(ya_1\,xz-a_2\,y^2z-ya_{11}\,x^2-a_{12}\,xy^2-a_{22}\,y^3\right)dz+$$
$$+\left(z^2a_2\,y+za_{11}\,x^2+z^3a_0+z^2a_1\,x+za_{12}\,xy+za_{22}\,y^2\right)dy+$$
$$+\left(-z^3b_0-z^2b_1\,x-zb_{12}\,xy-zb_{22}\,y^2-z^2b_2\,y-zb_{11}\,x^2\right)dx=0. \tag{11}$$

**Theorem 2.** *The spatial homogeneous first order system of equations*

$$\frac{dx}{ds}=P(x,y,z),\quad \frac{dy}{ds}=Q(x,y,z),\quad \frac{dz}{ds}=R(x,y,z), \tag{12}$$

*connected with the Pfaff equation (11) has the following form*

$$\frac{dx}{ds}=Q_z-R_y,\quad \frac{dy}{ds}=R_x-P_z,\quad \frac{dz}{ds}=P_y-Q_x,$$

In the considered case it looks as follows

$$\frac{d}{ds}x(s) = 4\,a_0\,z^2 + (4\,a_2\,y + (3\,a_1 - b_2)\,x)\,z + 4\,a_{22}\,y^2 +$$
$$+ (3\,a_{12} - 2\,b_{22})\,xy + (2\,a_{11} - b_{12})\,x^2,$$
$$\frac{d}{ds}y(s) = 4\,b_0\,z^2 + ((3\,b_2 - a_1)\,y + 4\,b_1\,x)\,z + (2\,b_{22} - a_{12})\,y^2 +$$
$$+ (-2\,a_{11} + 3\,b_{12})\,xy + 4\,b_{11}\,x^2,$$
$$\frac{d}{ds}z(s) = (-b_2 - a_1)\,z^2 + ((-2\,b_{22} - a_{12})\,y - b_{12}\,x - 2\,a_{11}\,x)\,z, \tag{13}$$

and for it the condition on their right sides

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$$

is fulfilled.

**Theorem 3.** *The solution of the homogeneous system (13) can be expressed through the solution of the algebraic ODE of the form*

$$\left(b_{22}\,t^2 + b_2\,t + a_0\,A + b_0 + a_{22}\,t^2 + a_2\,At\right)\left(\frac{d}{dt}A(t)\right)^2 +$$
$$+\left(-a_2\,A^2 + a_{12}\,At - 2\,b_{22}\,tA - 2\,a_{22}\,A(t)^2 t - b_2\,A + b_{12}\,t + a_1\,A + b_1\right)\frac{d}{dt}A(t) +$$
$$+b_{11} + a_{22}\,A^3 - b_{12}\,A + a_{11}\,A + b_{22}\,A^2 - a_{12}\,A^2 = 0. \tag{14}$$

*The equation (14) in the variables* $\frac{d}{dt}A(t) = u, A(t)$ *determines elliptic surface* $H(u,v,t) = 0$ *with genus* $(g = 1)$ *or rational surface* $(g = 0)$ *in a particular case.*

To investigate the properties of systems which have limit cycles we will use the representation of the system (13) in the form

$$\frac{d}{dx}z(x) = \frac{R(x,y,z)}{P(x,y,z)}, \quad \frac{d}{dx}y(x) = \frac{Q(x,y,z)}{P(x,y,z)}, \tag{15}$$

where

$$R(x,y,z) = -z(x)\left(a_1\,z(x) + a_{12}\,y(x) + 2\,a_{11}\,x + b12\,x + 2\,b_{22}\,y(x) + z(x)b_2\right),$$
$$Q(x,y,z) = 4\,b_0\,\left(z(x)\right)^2 + 4\,b_1\,xz(x) + 4\,b_{11}\,x^2 + 3\,b_{12}\,xy(x) + 2\,b_{22}\,\left(y(x)\right)^2 +$$
$$+3\,b_2\,y(x)z(x) - a_1\,z(x)y(x) - a_{12}\,\left(y(x)\right)^2 - 2\,a_{11}\,xy(x),$$
$$P(x,y,z) = 4\,a_0\,\left(z(x)\right)^2 + 3\,a_1\,xz(x) + 4\,a_2\,y(x)z(x) + 3\,a_{12}\,xy(x) -$$
$$-2\,xb_{22}\,y(x) - xb_2\,z(x) + 4\,a_{22}\,\left(y(x)\right)^2 - x^2 b_{12} + 2\,a_{11}\,x^2. \tag{16}$$

The system (15) after eliminating one of the variables $y(x)$ or $z(x)$ is reduced to one second-order differential equation of the type

$$F(x,y,y',y'')=0, \quad or \quad H(x,z,z',z'')=0.$$

Both equations can be used to study properties of integral curves of the system (15).

With this aim we apply the following theorem

**Theorem 4.** *As a result of the elimination of the function $y''$ from the system*

$$F(x,y,y',y'')=0, \quad \frac{\partial F}{\partial y''}=0$$

*the first order differential equation is obtained*

$$C(x,y,y')=0.$$

*This equation has the property that through each point $M$ of its integral curve $C=Q(x,y)$ passes integral curve of the equation $F(x,y,y',y'')=0$, for which the point $M$ is a return point of the second type.*

Further consideration of examples shows us that this property of the equation $F(x,y,y',y'')=0$ contains information about the existence of limit cycles or about their absence in the corresponding planar system (9).

## 5 An examples

1. The system of equations

$$\dot{x}=2+4x-4\mu x^2+12xy, \quad \dot{y}=8-3\mu-14\mu x-2\mu xy-8y^2, \tag{17}$$

has the algebraic curve

$$1/4+x-x^2+\mu\, x^3+xy+x^2y^2=0 \tag{18}$$

as limit cycle by the condition on the parameter $0<\mu<1/4$. The value $\mu=1/4$ is critical for the system (17) and it arises after additional computation.

We show here that the given value of the parameter $\mu$ is derived from the study of the system (17) in the projective coordinates. Really, the system (17) after projective extension takes the form

$$\frac{d}{dt}x(t)=-8\,z^2-12\,xz-52\,xy+6\,\mu\,x^2,$$

$$\frac{d}{dt}y(t)=-4\,(8-3\,\mu)\,z^2+56\,\mu\,xz-2\,\mu\,xy+28\,y^2+4\,zy,$$

$$\frac{d}{dt}z(t)=4\,z^2-4\,zy-10\,\mu\,xz. \tag{19}$$

A homogeneous form of algebraic curve (18) looks as follows

$$z^4 + 4\,xz^3 - 4\,x^2z^2 + 4\,\mu\,x^3z + 4\,xyz^2 + 4\,x^2y^2 = 0. \tag{20}$$

With the help of the system (26) after differentiation of the condition (20) with respect to the parameter $t$ we obtain the relation

$$8\,\left(z^4 + 4\,xz^3 - 4\,x^2z^2 + 4\,\mu\,x^3z + 4\,xyz^2 + 4\,x^2y^2\right)(x\mu - 2\,z - 6\,y) = 0. \tag{21}$$

**Proposition 2.** Substitution of the expression

$$z = 1/2\,x\mu - 3\,y$$

from the condition (21) into the (20) gives us the equation of straight line $y = Kx$ with the coefficient $K$ which is determined from the algebraic equation

$$(1296\,K^4 + 8\,\mu^3 + \mu^4 + 16\,\mu^2 + \left(-24\,\mu^3 - 128\,\mu^2\right)K + \left(216\,\mu^2 + 672\,\mu - 512\right)K^2 +$$

$$+ (-864\,\mu - 1152)\,K^3 = 0. \tag{22}$$

**Proposition 3.** The equation (22) has double roots when the parameter $\mu$ satisfies the conditions

$$\mu^2\,(4\mu - 1)\left(9\mu^2 - 384\mu - 2048\right) = 0. \tag{23}$$

The value of parameter $\mu = \frac{1}{4}$ is critical for the system (17).

The study of behavior of the system at another value of parameters

$$\mu = 0, \quad \mu = \frac{64}{3} + \frac{32}{3}\sqrt{6}, \quad \mu = \frac{64}{3} - \frac{32}{3}\sqrt{6}$$

requires additional consideration.

2. The system of equations

$$\dot{x} = 5x + 6x^2 + 4(1+\mu)xy + \mu y^2, \quad \dot{y} = x + 3y + 4xy + (2+3\mu)y^2 \tag{24}$$

has the algebraic curve

$$x^2 + x^3 + x^2y + 2\mu xy^2 + 2\mu xy^3 + \mu^2y^4 = 0 \tag{25}$$

as limit cycle by the condition on the parameter $-\frac{71}{32} + \frac{17}{32}\sqrt{17} < \mu < 0$.

The value of parameter $\mu = -\frac{71}{32} + \frac{17}{32}\sqrt{17}$ is critical for the system (26) and it arises after additional computation.

We show here that the given value of the parameter $\mu$ is derived from the study of the system (26) in the projective coordinates.

Really, the system (26) after projective extension takes the form

$$\frac{d}{dt}x(t) = -13\,xz - 3\,(4 + 4\,\mu)\,xy + 2\,x\,(2 + 3\,\mu)\,y - 4\,\mu\,y^2 - 8\,x^2,$$

$$\frac{d}{dt}y(t) = -4\,xz - 2\,\left(2+3\,\mu\right)y^2 - zy + \left(4+4\,\mu\right)y^2,$$
$$\frac{d}{dt}z(t) = 7\,z^2 + \left(4+4\,\mu\right)yz + 16\,xz + 2\,z\left(2+3\,\mu\right)y. \tag{26}$$

A homogeneous form of algebraic curve (25) looks as follows

$$x^2z^2 + x^3z + x^2yz + 2\,\mu\,xy^2z + 2\,\mu\,xy^3 + \mu^2y^4 = 0. \tag{27}$$

With the help of the system (26) after differentiation of the condition (27) with respect to the parameter $t$ we obtain the relation

$$-4\,\left(2\,y\mu + 3\,z + 2\,x + 2\,y\right)\left(x^2z^2 + x^3z + x^2yz + 2\,\mu\,xy^2z + 2\,\mu\,xy^3 + \mu^2y^4\right) = 0. \tag{28}$$

**Proposition 4.** Substitution of the expression

$$z = -2/3\,y\mu - 2/3\,x - 2/3\,y$$

from the condition (28) into the (27) give us the equation of straight line $y = Kx$ with the coefficient $K$ which is determined from the algebraic equation

$$34992\,\mu^5\left(16\,\mu^2 + 71\,\mu + 2\right)\left(\mu^4 - 2\,\mu^3 - 24\,\mu^2 + 16\,\mu - 8\right) = 0. \tag{29}$$

**Proposition 5.** The equation (29) has double roots when the parameter $\mu$ satisfies the conditions

$$34992\,\mu^5\left(16\,\mu^2 + 71\,\mu + 2\right)\left(\mu^4 - 2\,\mu^3 - 24\,\mu^2 + 16\,\mu - 8\right) = 0. \tag{30}$$

The value of parameter

$$\mu = -\frac{71}{32} + \frac{17}{32}\sqrt{17}$$

is a critical for the system (26).

Behavior of the system at the another values of parameters requires additional consideration.

## 6 Supplement1

In this section, we will consider an example of equation (1) for which associated metric has only one component of the Ricci tensor.

**Theorem 5.** *Tensor Ricci of the metric (3) associated with the equation*

$$\frac{d^2}{dx^2}y(x) + \left(\frac{\partial}{\partial y}h(x,y)\right)\left(\frac{d}{dx}y(x)\right)^3 + 3\left(\frac{\partial}{\partial x}h(x,y)\right)\left(\frac{d}{dx}y(x)\right)^2 -$$

$$-\frac{-\frac{\partial^2}{\partial x^2}h(x,y) - 3\left(\frac{\partial}{\partial x}h(x,y)\right)^2\frac{d}{dx}y(x)}{\frac{\partial}{\partial y}h(x,y)} - \frac{-\left(\frac{\partial}{\partial x}h(x,y)\right)^3 + 2\left(\frac{\partial^2}{\partial x\partial y}h(x,y)\right)\frac{\partial}{\partial x}h(x,y)}{\left(\frac{\partial}{\partial y}h(x,y)\right)^2} +$$

$$+\frac{\left(\frac{\partial}{\partial x}h(x,y)\right)^2 \frac{\partial^2}{\partial y^2}h(x,y)}{\left(\frac{\partial}{\partial y}h(x,y)\right)^3} = 0 \tag{31}$$

*has only one component*

$$R_{xx} = 2\,\frac{\frac{\partial^3}{\partial x^2 \partial y}h(x,y)}{\frac{\partial}{\partial y}h(x,y)} +$$

$$+\frac{-2\,\left(\frac{\partial^2}{\partial x^2}h(x,y)\right)\frac{\partial^2}{\partial y^2}h(x,y) - 4\,\left(\frac{\partial^2}{\partial x \partial y}h(x,y)\right)^2 - 4\,\left(\frac{\partial}{\partial x}h(x,y)\right)\frac{\partial^3}{\partial y \partial x \partial y}h(x,y)}{\left(\frac{\partial}{\partial y}h(x,y)\right)^2} +$$

$$+\frac{12\,\left(\frac{\partial}{\partial x}h(x,y)\right)\left(\frac{\partial^2}{\partial x \partial y}h(x,y)\right)\frac{\partial^2}{\partial y^2}h(x,y) + 2\,\left(\frac{\partial}{\partial x}h(x,y)\right)^2 \frac{\partial^3}{\partial y^3}h(x,y)}{\left(\frac{\partial}{\partial y}h(x,y)\right)^3} -$$

$$-6\,\frac{\left(\frac{\partial}{\partial x}h(x,y)\right)^2\left(\frac{\partial^2}{\partial y^2}h(x,y)\right)^2}{\left(\frac{\partial}{\partial y}h(x,y)\right)^4} \tag{32}$$

*and from the condition $R_{xx} = 0$ it follows that the metric is a flat $R_{ijkl} = 0$ on the solutions of the corresponding pde.*

The function

$$h(x,y) = e^{-\left(LambertW(-\_C3\;\_C1\;e^{-\frac{x-\_C1\;y+\_C2\;x\_C1}{x}})x + x - \_C1\;y + \_C2\;x\_C1\right)x^{-1}}x \tag{33}$$

is an example of such solution. From here it follows that all equations with such condition are point-equivalent to the equation $y'' = 0$.

**Theorem 6.** *To the equation (31) with the function $h(x,y) = f(x) + \mu\sqrt(y)$ invariant $\nu_5 = 0$ the equation takes the form*

$$\frac{d^2}{dx^2}y(x) + 1/2\,\frac{\mu\,\left(\frac{d}{dx}y(x)\right)^3}{\sqrt{y(x)}} + 3\,\left(\frac{d}{dx}f(x)\right)\left(\frac{d}{dx}y(x)\right)^2 + 6\,\frac{\left(\frac{d}{dx}f(x)\right)^2\sqrt{y(x)}\,\frac{d}{dx}y(x)}{\mu} +$$

$$+1/2\,\frac{-4\,\left(\frac{d}{dx}f(x)\right)^2 y(x) + 8\,\left(\frac{d}{dx}f(x)\right)^3 (y(x))^2 + 4\,\mu\,\left(\frac{d^2}{dx^2}f(x)\right)(y(x))^{3/2}}{y(x)\mu^2} = 0, \tag{34}$$

*and the Ricci tensor of corresponding Riemann space $R_{xx} \neq 0$.*

In this situation both functions of the equation (34) $L_1 \neq 0$ and $L_2 \neq 0$ and for the further study of the equation it is necessary to use the invariant

$$W_1 := \frac{[L_2{}^3\,(\alpha_1\,L_2 - \alpha\,L_1) - 2\,R_2\,L_2\,L_{2y} + L_2{}^2 R_{2y} - L_2\,R_2\,(a_1\,L_1 - a_2\,L_2)]}{L_2{}^4} =$$

$$= \frac{[L_1{}^3\,(\alpha_1\,L_1 - \alpha_2\,L_2) + R_1\,L_1\,L_{1x} - L_1{}^2 R_{1x} + L1\,R_1\,\,(a_3\,L_1 - a_4\,L_2)]}{L_1{}^4}, \quad (35)$$

where

$$\alpha = a_{2y} - a_{1x} + 2(a_1a_3 - a_2^2),\;\; \alpha_1 = a_{3y} - a_{2x} + a_1a_4 - a_2a_3,$$
$$\alpha_2 = a_{4y} - a_{3x} + 2(a_2a_4 - a_3^2),$$

and

$$R_1 = L_1L_{2x} - L2L_{1x} + a_2^2L_1^2 - 2a_3L_1L_2 + a_4L_2^2,$$
$$R_2 = L_1L_{2y} - L_2L_{1y} + a_1^2L_1^2 - 2a_2L_1L_2 + a_3L_2^2.$$

Starting from the invariant (35) the sequence of invariants

$$W_{m+2} = L_1\frac{\partial W_m}{\partial y} - L_2\frac{\partial W_m}{\partial x} + mW_m\left(\frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial y}\right)$$

can be constructed.

## 7 Supplement2

It is known that with partial first order equation

$$F(x,y,z,p,q) = F(x,y,z,z_x,z_y) = 0 \quad (36)$$

can be associated the Monge equation

$$\Phi\left(x,y,z,\frac{dy}{dx},\frac{dz}{dx}\right) = 0, \quad (37)$$

homogeneous with respect to the differentials.

Such type of equation determines a set of the Monge curve lines which of is the rib return of corresponding surface composed from characteristics of the equation (36).

We shall apply theory of Monge equation (37) for study of planar systems of ODE's with the limit cycles.

**Proposition 6.** For the system of equation which has limit cycle at the condition $0 < \mu < 1/4$

$$\frac{dx}{dt} = y + y^2,\;\; \frac{dy}{dt} = -x + \mu y - xy + (1+\mu)y^2 \quad (38)$$

corresponding the p.d.e. (36) looks as follows

$$\left(\frac{\partial}{\partial x}f(x,y)\right)\left(y+y^2\right) + \left(\frac{\partial}{\partial y}f(x,y)\right)\left(-x + \mu\,y - xy + (1+\mu)\,y^2\right) = 0. \quad (39)$$

After change of the variables

$$f(x,y) = u(x,t),\;\; \frac{\partial}{\partial x}f(x,y) = \frac{\partial}{\partial x}u(x,t) - \frac{\left(\frac{\partial}{\partial x}v(x,t)\right)\frac{\partial}{\partial t}u(x,t)}{\frac{\partial}{\partial t}v(x,t)},$$

$$\frac{\partial}{\partial y}f(x,y)=\frac{\frac{\partial}{\partial t}u(x,t)}{\frac{\partial}{\partial t}v(x,t)},$$

where

$$u(x,t)=t\frac{\partial}{\partial t}z(x,t)-z(x,t),\quad v(x,t)=\frac{\partial}{\partial t}z(x,t)$$

the equation (39) takes the form

$$\left(-\frac{\partial}{\partial x}z(x,t)+t+\mu\,t\right)\left(\frac{\partial}{\partial t}z(x,t)\right)^2+\left(-\frac{\partial}{\partial x}z(x,t)+\mu\,t-tx\right)\frac{\partial}{\partial t}z(x,t)-tx=0. \tag{40}$$

**Theorem 7.** *Homogeneous equation associated with the equation (40) is*

$$512\,dx^4t^4x+320\,dx^4t^4x^2-1000\,dz\,dx^3\mu\,t^3x^2+800\,dx^2t^2dz^2x-800\,dx^3t^3dz\,x^2+$$
$$+500\,dx^4\mu^2t^4x^2+500\,dz^2dx^2t^2x^2-1600\,dx^3t^3\mu\,x\,dz+800\,dx^4\mu^2t^4x-$$
$$-1280\,dx^3t^3x\,dz+800\,dx^4t^4\mu\,x^2+1280\,dx^4t^4\mu\,x+500\,dx^2x^2t^2\,dt\,dz-$$
$$-500\,dx^3x^2t^3\mu\,dt+4400\,dx^3x^2\,dt\,t^3+2000\,dx^3\,dt\,t^3x^3-5000\,dt\,dx^2\mu\,t^2dz\,x+$$
$$+3000\,dt\,dx^2\mu^2t^2dz-3000\,dt\,dx\,\mu\,t\,dz^2-1600\,dt\,dx^2\mu\,t^2dz-2500\,dt^2dx^2\mu\,t^2x+$$
$$+2500\,dt\,dx^3\mu^2t^3x+800\,dt\,dx^3\mu^2t^3-1000\,dt\,dx^3\mu^3t^3+1625\,dt^2dx^2t^2x^2+$$
$$+1920\,dt\,dx^3t^3x+2400\,dt^2dx^2t^2x-800\,dz^3dx\,t+500\,dx^4\mu^4t^4+800\,dx^4t^4\mu^3+$$
$$+500\,dz^4+2500\,dt\,dx\,tx\,dz^2+2500\,dt^2dx\,tx\,dz+400\,dt\,dx^2t^2x\,dz+1000\,dz^3dt-$$
$$-400\,dt\,dx^3\mu\,t^3x+3000\,dz^2dx^2\mu^2t^2-2000\,dz^3dx\,\mu\,t+2400\,dz^2dx^2\mu\,t^2-$$
$$-2400\,dz\,dx^3\mu^2t^3-2000\,dz\,dx^3\mu^3t^3+800\,dz^2dx\,t\,dt+320\,dx^2t^2dz^2+$$
$$+320\,dx^4t^4\mu^2-1000\,dt^2dx\,\mu\,t\,dz+500\,dt^2dz^2+1000\,dt^3dx\,tx+$$
$$+500\,dt^2dx^2\mu^2t^2-640\,dx^3t^3dz\,\mu=0,$$

*and from here we obtain the Monge equation*

$$\left(\frac{d}{dt}z(t)\right)^4-2\,t\,(1+x+2\,\mu)\left(\frac{d}{dt}x(t)\right)\left(\frac{d}{dt}z(t)\right)^3+$$
$$+t^2\left((x)^2+6\,\mu^2+6\,\mu\,x+6\,\mu+8\,x+1\right)\left(\frac{d}{dt}x(t)\right)^2\left(\frac{d}{dt}z(t)\right)^2-$$
$$-2\,t^3\,(1+\mu)\left(2\,\mu^2+\mu+3\,\mu\,x+5\,x+(x)^2\right)\left(\frac{d}{dt}x(t)\right)^3\frac{d}{dt}z(t)+$$
$$+t^4\,(1+\mu)^2\left(\mu^2+2\,\mu\,x+4\,x+(x)^2\right)\left(\frac{d}{dt}x(t)\right)^4+4\,\left(\frac{d}{dt}x(t)\right)tx-$$

$$-t^2\left(8\,\mu\,x-\mu^2+8\,(x)^2-12\,x\right)\left(\frac{d}{dt}x(t)\right)^2-2\,t\,(\mu-4\,x)\left(\frac{d}{dt}x(t)\right)\frac{d}{dt}z(t)+$$
$$+\left(\frac{d}{dt}z(t)\right)^2+2\left(\frac{d}{dt}z(t)\right)^3-2\,t\,(-x-1+3\,\mu)\left(\frac{d}{dt}x(t)\right)\left(\frac{d}{dt}z(t)\right)^2+$$
$$+2\,t^2\left(-x-2\,\mu\,x-2\,\mu+3\,\mu^2-4\,(x)^2\right)\left(\frac{d}{dt}x(t)\right)^2\frac{d}{dt}z(t)+$$
$$+2\,t^3\left(2\,(x)^3+\mu^2+6\,x-\mu^3+10\,(x)^2+\mu^2x+4\,\mu\,(x)^2+\mu\,x\right)\left(\frac{d}{dt}x(t)\right)^3=0. \quad (41)$$

The solutions of equation (41) depend on the value of parameter $\mu$ and contains information about properties of integral surface formed from characteristics of the equation (40). In particular a following proposition is valid

**Proposition 7.** Under substitution

$$z(t)=t^2\frac{\partial}{\partial t}\omega(x,t)-2\,t\frac{\partial}{\partial t}\omega(x,t)+2\,\omega(x,t),\quad x(t)=\frac{\partial}{\partial t}\omega(x,t)$$

undetermined equation (41) is reduced to algebraic-differential equation with respect to the function $\omega(x,t)$

$$A_4\left(\frac{\partial^2}{\partial t^2}\omega(x,t)\right)^4+A_3\left(\frac{\partial^2}{\partial t^2}\omega(x,t)\right)^3+A_2\left(\frac{\partial^2}{\partial t^2}\omega(x,t)\right)^2+A_1\frac{\partial^2}{\partial t^2}\omega(x,t)+A_0=0, \quad (42)$$

where $A_i=A_i(\mu,t,\frac{d\omega}{dt})$ are some polynomial functions with respect to the variables $t$ and $\omega_t$.

Study of return points of integral curves of the equation(42) in accordance with the $Theorem 4$ alow us to formulate a following result

**Theorem 8.** *From the set of real values of the parameter $\mu$, which satisfy to the equation*

$$-22265110462464\,\mu^{11}-485606182354944\,\mu^{10}-3541595672543232\,\mu^9-$$
$$-7724112134799360\,\mu^8+24161394358222848\,\mu^7+131429131899371520\,\mu^6+$$
$$+52585845479178240\,\mu^5-573323004352659456\,\mu^4-661065167731163136\,\mu^3+$$
$$+894761666057601024\,\mu^2+1196951371505467392\,\mu-$$
$$-365080998536282112=0,$$

*or*

$$\mu=1.974873734,\ \ \mu=-2.974873734,\ \ \mu=0.2650452857$$

*only the value*

$$\mu=0.2650452857$$

*satisfies the condition of existence of limit cycle in the system (38).*

*Analogous consideration for the system*

$$\frac{dx}{dt}x = y + y^2, \frac{dy}{dt} = -1/2x + \mu y - xy + (4/5 + \mu)y^2,$$

*which has two limit cycles at the condition* $0 < \mu < \frac{4}{5}$, *give us two values*

$$\mu = 0.3194424124, \ \ \textit{and} \ \ \mu = 0.3194444444.$$

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