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Isohedral tilings by 14-, 16- and 18-gons for hyperbolic translation group of genus two

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Abstract. There are 2 types of isohedral tilings of the Euclidean plane with disks for the translation group $p1$. In the hyperbolic plane there exist countably many translation groups, each translation group is characterized by its genus. The present article continues work [7] and studies isohedral tilings of the hyperbolic plane with disks for the translation group of genus two. We use the technique of adjacency symbols, developed by B. N. Delone for the Euclidean plane. In [7] isohedral tilings of the hyperbolic plane with 8-, 10- and 12-gons were obtained. In the present article isohedral tilings of the hyperbolic plane with 14-, 16- and 18-gons are obtained, thus completing the enumeration.

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1 Introduction

Isohedral tilings of the Euclidean plane with disks are known. Moreover, there are different methods developed and applied by several authors which allowed to obtain all possible tilings of the Euclidean plane isohedral with respect to 17 crystallographic plane groups, i. e. discrete isometry groups with compact fundamental domains.

In the hyperbolic plane the number of such groups is infinite. All the two-dimensional hyperbolic discrete isometry groups with compact fundamental domains were classified and given the signature symbol by A. M. Macbeath [1]. J. Conway [2] proposed equivalent to Macbeath's but shorter orbifold symbol which we are going to use here.

Z. Lučić and E. Molnár [3, 4] developed a method of obtaining fundamental isohedral tilings for a given discrete isometry group with compact fundamental domain. The method was applied with success to some hyperbolic isometry groups, however it fails for hyperbolic translation groups. Another method of obtaining k-isohedral tilings (i. e. with k transitivity classes of tiles) of all three 2-dimensional spaces of constant curvature is based on Delauney–Dress symbols. In [5] some algorithms were developed that produce Delauney–Dress symbols corresponding to k-isohedral tilings for any given curvature of the symbol. In [6] some related problems, in particular, of computing orbifold symbol, were solved. Moreover, the algorithms were implemented on computer.

In the Euclidean plane the translation group $p1$ is one of 17 crystallographic plane groups. Its orbifold symbol by Conway is $\circ$ (a circle). In the hyperbolic plane there are a countable series of translation groups. Every hyperbolic translation group is characterized by its genus, which is the genus of the quotient of the hyperbolic plane by the group, and it has the orbifold symbol by Conway $\circ\circ\cdots\circ$ where the number of circles is equal to the genus. For a hyperbolic translation group the smallest genus is 2 and the hyperbolic group of genus 2 with the orbifold symbol $\circ\circ$ is the simplest hyperbolic group of translations. So the present article is a direct continuation of the work [7], and here we complete the study of isohedral tilings of the hyperbolic plane with disks for the hyperbolic translation group $\circ\circ$.

In the present paper we consider tilings with topological disks as it is done for the Euclidean plane in the classical monograph [8]. For obtaining isohedral tilings we use methods analogous to the methods developed by B. N. Delone [9] (see also [10]) in the Euclidean case and based on adjacency symbols and adjacency diagrams. The author described the method for the hyperbolic case in [11, 12] as well as in [7].

2 Basic notions, methods and some previous results

We recall basic concepts which will be used here.

A set W of closed topological disks in the plane is called a tiling of the plane with disks if every point of the plane belongs to at least one disk and no two disks have an inner point in common. The disks of a tiling are called tiles.

In a tiling a non-empty component of the intersection of two or more different tiles is called a vertex of the tiling if it is a single point and is called an edge of the tiling otherwise. The boundary of a tile is divided by vertices of the tiling into curves that are edges of the tiling, so any tile may be considered as a curvilinear polygon.

Definition 1. Let W be a tiling of the hyperbolic plane with disks, G be a discrete isometry group of the hyperbolic plane with a compact (bounded) fundamental domain. The tiling W is called isohedral with respect to the group G if the group G maps the tiling W onto itself and G acts transitively on the set of the tiles.

The enumeration of isohedral tilings is based on the concept of Delone class [9, 10] (homeomeric type in [8] is the same as well as equivariant type in [5]). Consider all possible pairs (W, G) where W is a tiling of the hyperbolic plane with disks which is isohedral relative to a discrete hyperbolic isometry group G with a bounded fundamental domain. Two pairs (W, G) and (W', G') belong to one Delone class if: 1) the tilings W and W' are combinatorially isomorphic; 2) the groups G and G' are isomorphic; 3) the groups G and G' act in the same way on the tilings W and W', respectively. Now give the precise definition of Delone class.

Definition 2. Consider all possible pairs (W, G) where W is a tiling of the hyperbolic plane with disks which is isohedral with respect to a discrete hyperbolic isometry group G with a bounded fundamental domain. Two pairs (W, G) and

(W', G') are said to belong to the same Delone class if there exists homeomorphic transformation φ of the plane which maps the tiling W onto the tiling W' and the relation $G = \varphi^{-1}G'\varphi$ holds.

A Delone class (W, G) is called fundamental if the group G acts one time transitively (simply transitively) on the set of tiles of W. Any translation group admits only fundamental Delone classes (or fundamental tilings for short).

So our aim is to obtain all fundamental Delone classes of isohedral tilings in the hyperbolic plane with disks for the translation group of genus 2, which was begun in [7] and will be completed in the present article.

Following the scheme of work [9], in our previous work [7] we solved Diophantine equations and obtained possible sets of valencies from which ordered cycles of valencies $(\alpha_1, \alpha_2, \dots, \alpha_k)$ can be formed. For the number of vertices $k = 8$, 10, 12 the study was done in [7], so in the present paper we will work for $k = 14$ with two sets A : 33333333355555 and B : 33333344444444 as well as for $k = 16$ with the set 3333333333334444 and for $k = 18$ with the set 333333333333333333.

To ordered cycles of valencies we apply adjacency symbols and adjacency diagrams. For a given fundamental isohedral tiling (W, G) of the plane with disks, we choose a tile and label consecutively all its edges with the letters a, b,.... Applying the isometry group G yields a quite definite labelling of all the tiles in the tiling W. In a symbol the letter which labels the first chosen edge stands first, the letter which labels the adjacent edge of the neighbor tile stands next to it, then the lower index indicates the valency of the end vertex of the first edge, after that we pass to the second consecutive edge, and so on. If an isometry changes the orientation of a tile, this fact should be indicated with a bar over the second letter. In the present article we consider only translations, they preserve orientation, so no bars are needed. Such an adjacency symbol fully determines the Delone class of the pair (W, G).

The method works in the following way. We generate all possible adjacency symbols for each appropriate equivalence class of cycles. For each candidate in adjacency symbol we check if the condition of transition around a vertex is satisfied, for every vertex equivalence class.

Further we must choose one representative among equivalent adjacency simbols with the help of adjacency diagrams. An adjacency diagram is a polygonal tile where the vertices are labelled with their valencies and the paired edges are connected with arcs.

3 Tilings with 14-gons

For $k = 14$ there are 2 solutions of Diophantine equation. We begin with the first solution $14A$: 33333333355555.

First determine some rules to which we will adhere when generating adjacency symbols. Neighboring edges cannot be paired because all the isometries are translations. A 'parallel pairing' is not possible: for a tile labelling $(\dots ab \dots b'a' \dots)$ two pairs aa' and bb' cannot take place together. Any pair of edges separated by an edge

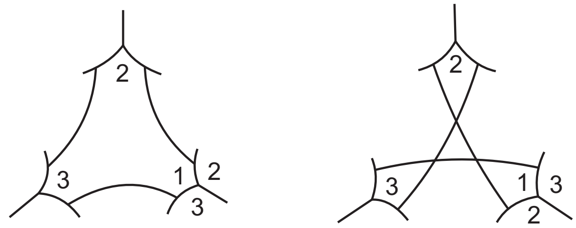


Figure 1. Two possible schemes as parts of adjacency diagrams for equivalent valencies 3

with both ends of valency 3 is not possible. For valencies 3 the two schemes possible as parts of adjacency diagrams are depicted in Fig. 1.

Now we form possible classes of ordered cycles for the set $14A$. We choose among equivalent cycles only the one corresponding to the smallest number. We enumerate only eligible ordered cycles of valencies for which the edges can fall into pairs, with edges in a pair having ends of the same valency, in appropriate order. To each eligible cycle of valencies we apply the technique of adjacency symbols and check the condition of transition around a vertex, for three vertices of valency 3 and a vertex 5.

We analyze which groups of vertices of the same valency are admissible in a cycle, first for valency 5. For cycle $14A1$: 33333333355555 the only Delone class $14A1$ of isohedral tilings of the hyperbolic plane exists and it is given by the adjacency symbol $(aj_3be_3cg_3dh_3eb_3fi_3gc_3hd_3if_3ja_5km_5ln_5mk_5nl_5)$. Divide vertices 5 into two groups. Falling into 4 consecutive and a separate valency 5: 5555 and 5 as well as falling into 3 consecutive and 2 consecutive valencies 5: 555 and 55 yield 3 edges 55 and both are not admissible.

We divide vertices 5 into 3 groups. Falling into 3 consecutive and 2 separate valencies 5: 555, 5 and 5 yields two neighboring edges 55 and is not admissible. Falling into 2 pairs of consecutive and a separate valency 5: 55, 55 and 5 yields two edges 55. We divide 9 vertices 3 into 3 groups, too. Falling into 7 consecutive and two separate valencies 3: 3333333, 3 and 3 suits us. By combining these groupings we obtain 2 classes of eligible ordered cycles of valencies: 33333335355355 and 33333335535355, however no Delone classes of isohedral tilings of the hyperbolic plane exist for them.

Falling into 6 consecutive, 2 consecutive and a separate valency 3: 333333, 33 and 3 suits us and can be combined with the above grouping of 5. So we obtain 3 classes of eligible ordered cycles of valencies: 33333353355355, 33333353553355 and 33333355335355, but there are no Delone classes for them.

Falling into 5 consecutive, 3 consecutive and a separate valency 3: 33333, 333 and 3 suits us. Combining it with the above grouping of 5 we obtain 3 classes of eligible ordered cycles of valencies. For cycle $14A2$: 33333533355355 there exists the only Delone class $14A2$ of isohedral tilings which has the adjacency symbol $(am_3be_3ch_3di_3eb_3fl_5gj_3hc_3id_3jg_5kn_5lf_3ma_5nk_5)$. For cycle $14A3$: 33333553335355 there is also the only Delone class $14A3$ with the adjacency symbol

$(am_3be_3ci_3dj_3eb_3fl_5gn_5hk_3ic_3jd_3kh_5lf_5ma_5ng_5)$. For cycle 33333535533355 there is no Delone class.

Falling into 5 consecutive and 2 pairs of consecutive valencies 3: 33333, 33 and 33 gives 2 classes of eligible ordered cycles of valencies: 33333533553355 and 33333553353355, for them there are no Delone classes of isohedral tilings. Falling into 4 consecutive, 4 consecutive and a separate valency 3: 3333, 3333 and 3 gives 3 classes of eligible ordered cycles of valencies: 33335333355355, 33335355333355 and 33335533335355, there are no Delone classes of isohedral tilings for them.

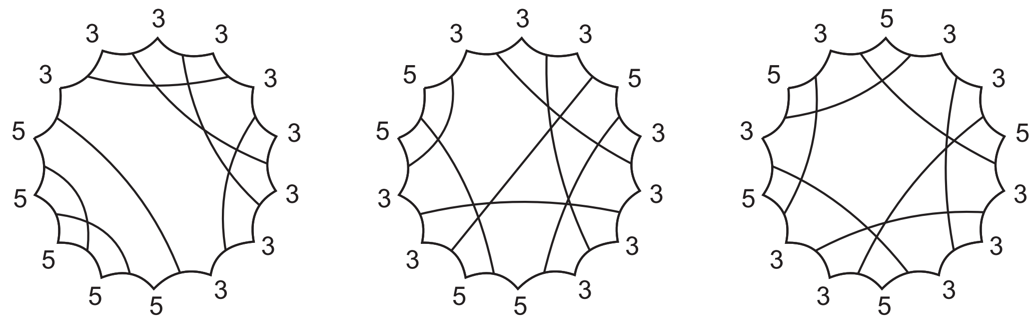
Falling into 4 consecutive, 3 consecutive and 2 consecutive valencies 3: 3333, 333 and 33 gives 3 classes of eligible ordered cycles of valencies. For cycle $14A4$: 33335333553355 the only Delone class $14A4$ of isohedral tilings has the adjacency symbol $(ae_3bg_3cl_3dh_3ea_5fm_3gb_3hd_3ik_5jn_5ki_3lc_3mf_5nj_5)$. For cycle $14A5$: 33335335533355 also the only Delone class $14A5$ with the adjacency symbol $(ae_3bk_3cg_3dl_3ea_5fm_3ge_3hj_5in_5jh_3kb_3ld_3mf_5ni_5)$ exists. For cycle 33335533353355 there is no Delone class of isohedral tilings. Falling into 3 triples of consecutive valencies 3: 333, 333 and 333 gives one class of eligible ordered cycles of valencies 33353335533355 for which there exists no Delone class.

Now examine the situation where no edge 55 exists, so all the vertices of valency 5 are separate. We divide 9 vertices 3 into 5 groups. Falling into 5 consecutive and 4 separate valencies 3: 33333, 3, 3, 3, and 3 gives one class of eligible ordered cycles of valencies: 33333535353535 for which no Delone class of isohedral tilings exists. Falling into 4 consecutive, 2 consecutive and 3 separate valencies 3: 3333, 33, 3, 3, and 3 gives 2 classes of eligible ordered cycles of valencies: 33335335353535 and 33335353353535, for them there are no Delone classes of isohedral tilings.

Falling into 2 triples of consecutive and 3 separate valencies 3: 333, 333, 3, 3, and 3 gives 2 classes of eligible ordered cycles of valencies. For cycle $14A6$: 33353335353535 there exists the only Delone class $14A6$ of isohedral tilings with the adjacency symbol $(ad_3bf_3cg_3da_5eh_3fb_3gc_3he_5il_3jm_5kn_3li_5mj_3nk_5)$. For cycle $14A7$: 33353533353535 also the only Delone class $14A7$ exists and it has the adjacency symbol $(ad_3bh_3ci_3da_5el_3fm_5gj_3hb_3ic_3jg_5kn_3le_5mf_3nk_5)$.

Falling into 3 consecutive, 2 pairs of consecutive and 2 separate valencies 3: 333, 33, 33, 3, and 3 gives 4 classes of eligible ordered cycles of valencies. For cycle $14A8$: 33353353353535 the only Delone class $14A8$ of isohedral tilings has the adjacency symbol $(al_3bf_3ci_3dm_5ej_3fb_3gk_5hn_3ic_3je_5kg_3la_5md_3nh_5)$. For cycle $14A9$: 33353353533535 also there is the only Delone class $14A9$ with the adjacency symbol $(ai_3bk_3cf_3dm_5en_3fc_3gj_5hl_3ia_5jg_3kb_3lh_5md_3ne_5)$. For cycle $14A10$: 33353353535335 there exist 3 Delone classes: $14A10,1$ with the adjacency symbol $(ai_3bm_3cf_3dj_5ek_3fc_3gl_5hn_3ia_5jd_3ke_5lg_3mb_3nh_5)$, $14A10,2$ with the adjacency symbol $(ak_3bf_3cm_3dh_5en_3fb_3gj_5hd_3il_5jg_3ka_5li_3mc_3ne_5)$ and $14A10,3$ with the adjacency symbol $(ak_3bm_3cf_3dh_5ei_3fc_3gl_5hd_3ie_5jn_3ka_5lg_3mb_3nj_5)$. For cycle $14A11$: 33353533533535 the only Delone class $14A11$ has the adjacency symbol $(af_3bh_3ck_3dm_5ei_3fa_5gl_3hb_3ie_5jn_3kc_3lg_5md_3nj_5)$.

Falling into 4 pairs of consecutive and a separate valency 3: 33, 33, 33, 33, and 3 gives one class of eligible ordered cycles of valencies: 33533533533535 for which



14A1 14A4 14A10,2

Figure 2. Adjacency diagrams for the Delone classes of isohedral tilings $14A1$, $14A4$ and $14A10,2$

no Delone class of isohedral tilings exists. The adjacency diagrams for the Delone classes $14A1$, $14A4$ and $14A10,2$ of isohedral tilings are shown in Fig. 2.

Turn to the second solution of Diophantine equation $14B$: 33333344444444. From a rather large number of possible ordered cycles of valencies we must determine which are equivalent. For this purpose we always choose among eqivalent cycles the one which corresponds to the smallest number. We analyze which groups of vertices of the same valency are admissible in a cycle, first for valency 3. All the 6 consecutive valencies 3: 333333 yield 5 edges 33 which cannot be paired and it is not admissible. Falling into 5 consecutive and a separate valency 3: 33333 and 3 yields 4 neighboring edges 33 for which no pairing suits us. Falling into 4 consecutive and 2 consecutive valencies 3: 3333 and 33 yields 3 neighboring and a separate edge 33 for which no pairing suits us. Falling into 4 consecutive and 2 separate valencies 3: 3333, 3 and 3 yields 3 edges 33 and it is not admissible.

Falling into 2 triples of consecutive valencies 3: 333 and 333 suits us. Combining with different divisions of valencies 4 into two groups gives 4 classes of eligible ordered cycles of valencies. For each eligible cycle of valencies we apply the technique of adjacency symbols and check the condition of transition around a vertex, for two vertices of valency 3 and two vertices of valency 4.

For cycle $14B1$: 33343334444444 there exists the only Delone class $14B1$ of isohedral tilings with adjacency symbol $(ad_3bf_3cg_3da_4eh_3fb_3gc_3he_4ik_4jm_4ki_4ln_4mj_4nl_4)$. For cycle $14B2$: 33344333444444 also there is the only Delone class which has the adjacency symbol $(ad_3bg_3ch_3da_4el_4fi_3gb_3he_3if_4jm_4kn_4le_4mj_4nk_4)$. For cycle $14B3$: 33344443334444 there are 2 Delone classes of isohedral tilings: $14B3,1$ has the adjacency symbol $(ad_3bi_3cj_3da_4eg_4fm_4ge_4hk_3ib_3jc_3kh_4ln_4mf_4nl_4)$ and $14B3,2$ has the adjacency symbol $(ad_3bi_3cj_3da_4el_4fm_4gn_4hk_3ib_3jc_3kh_4le_4mf_4ng_4)$. For cycle 33344433344444 no Delone class exists.

Falling into 3 consecutive, 2 consecutive and a separate valency 3: 333, 33 and 3 yields 3 edges 33 and it is not admissible. Falling into 3 consecutive and 3 separate valencies 3: 333, 3, 3, and 3 yields 2 neighboring edges 33 and it is not admissible.

Falling into 3 pairs of consecutive valencies 3: 33, 33 and 33 yields 3 edges 33 and it is not admissible.

Falling into 2 pairs of consecutive and 2 separate valencies 3: 33, 33, 3, and 3 yields 2 edges 33 and it suits us. We divide 8 valencies 4 into 4 groups and combine with the above grouping of valencies 3. Falling into 5 consecutive and 3 separate valencies 4 yields 4 neighboring edges 44 and it does not allow a closed circle around a vertex 4, therefore it is not admissible.

Falling into 4 consecutive, 2 consecutive and 2 separate valencies 4: 4444, 44, 4, and 4 suits us. By combining with the above grouping of valencies 3 we obtain 10 classes of eligible ordered cycles of valencies. To each eligible cycle of valencies we apply the technique of adjacency symbols and check the condition of transition around a vertex, for two vertices of valency 3 and two vertices of valency 4.

For cycle $14B4$: 33434344334444 there exists the only Delone class $14B4$ with the adjacency symbol $(ae_3bj_3cf_4dk_3ea_4fc_3gi_4hm_4ig_3jb_3kd_4ln_4mh_4nl_4)$. For cycle $14B5$: 33433434444344 the only Delone class $14B5$ has the adjacency symbol $(am_3be_3cg_4dh_3eb_3fl_4gc_3hd_4ik_4jn_4ki_4lf_3ma_4nj_4)$. For cycles 33433434434444, 33433443434444, 33434334434444, 33434433434444, 33434434334444, 33434344443344, 33434433444434 and 33434444343344 there exist no Delone classes.

Falling into 2 triples of consecutive and 2 separate valencies 4: 444, 444, 4, and 4 suits us. By combining with the above grouping of valencies 3 we obtain 7 classes of eligible ordered cycles of valencies. For cycle $14B6$: 33434334443444 the only Delone class $14B6$ with adjacency symbol $(ae_3bg_3ck_4dh_3ea_4fl_3gb_3hd_4im_4jh_4kc_3lf_4mi_4nj_4)$ exists. For cycle $14B7$: 33434443344434 the only Delone class $14B7$ has the adjacency symbol $(ae_3bi_3cm_4dj_3ea_4fk_4gl_4hn_3ib_3jd_4kf_4lg_4mc_3nh_4)$. For cycles 33433434443444, 33434344433444, 33433444343444, 33434443343444 and 33434443433444 there are no Delone classes.

Falling into 3 consecutive, 2 pairs of consecutive and a separate valency 4: 444, 44, 44, and 4 suits us. By combining wih the above grouping of valencies 3 we obtain 10 classes of eligible ordered cycles of valencies. For cycle $14B8$: 33434434443344 there exists the only Delone class $14B8$ with the adjacency symbol $(ae_3bl_3cg_4dm_3ea_4fi_4gc_3hk_4in_4jf_4kh_3lb_3md_4ni_4)$. For cycle $14B9$: 33443344343444 also the only Delone class $14B9$ has the adjacency symbol $(al_3bf_3ci_4dn_4ej_3fb_3gk_4hm_4ic_3je_4kg_3la_4mh_4nd_4)$. For cycles 33433443443444, 33434433443444, 33434434433444, 33433443444344, 33434433444344, 33434443344344, 33434443443344 and 33443434433444 there are no Delone classes.

Falling into 4 pairs of consecutive valencies 4 suits us. By combining with the above grouping of valencies 3 we obtain 2 classes of eligible ordered cycles of valencies. For cycle $14B10$: 33443443344344 there exist 5 Delone classes of isohedral tilings: $14B10,1$ with the symbol $(af_3bi_3cl_4dg_4ej_3fa_4gd_4hm_3ib_3je_4kn_4lc_3mh_4nk_4)$, $14B10,2$ with adjacency symbol $(af_3bi_3cl_4dn_4ej_3fa_4gk_4hm_3ib_3je_4kg_4lc_3mh_4nd_4)$, $14B10,3$ with adjacency symbol $(am_3bi_3ce_4dg_4ec_3fh_4gd_4hf_3ib_3jl_4kn_4lj_3ma_4nk_4)$, $14B10,4$ with adjacency symbol $(am_3bi_3ce_4dk_4ec_3fh_4gn_4hf_3ib_3jl_4kd_4lj_3ma_4ng_4)$ and $14B10,5$ with the symbol $(am_3bi_3ce_4dn_4ec_3fh_4gk_4hf_3ib_3jl_4kg_4lj_3ma_4nd_4)$. For cycle 33443344344344 there is no Delone class.

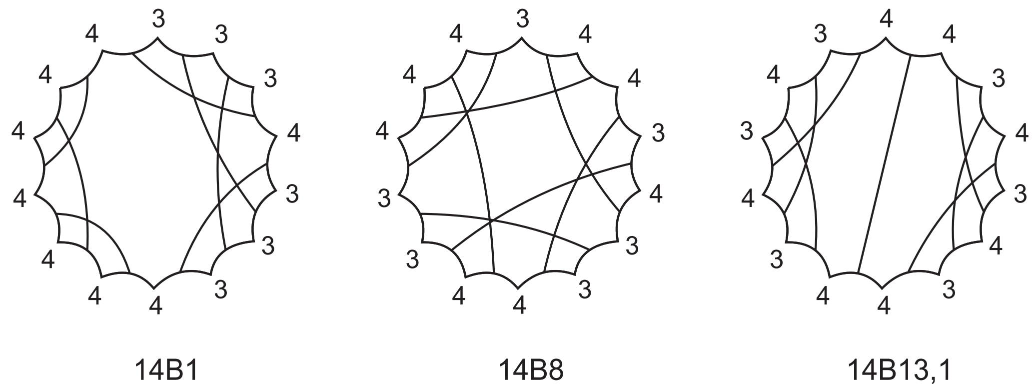


Figure 3. Adjacency diagrams for the Delone classes of isohedral tilings $14B1$, $14B8$ and $14B13,1$

Now take all the valencies 3 separate and divide 8 valencies 4 into 6 groups. Falling into 3 consecutive and 5 separate valencies 4: 444, 4, 4 , 4, 4, and 4 yields 2 neighboring edges 44 and it is not admissible. So the only possible division is falling into 2 pairs of consecutive and 4 separate valencies 4: 44, 44, 4, 4, 4, and 4. We obtain 3 classes of eligible ordered cycles of valencies. For cycle $14B11 : 34343434344344$ there exist 4 Delone classes of isohedral tilings: $14B11,1$ with the adjacency symbol $(af_3bg_4cj_3dl_4eh_3fa_4gb_3he_4im_3jc_4kn_4ld_3mi_4nk_4)$, $14B11,2$ with the adjacency symbol $(af_3bi_4cm_3dg_4ej_3fa_4gd_3hl_4ib_3je_4kn_4lh_3mc_4nk_4)$, $14B11,3$ with the adjacency symbol $(ah_3bl_4cf_3di_4ej_3fc_4gm_3ha_4id_3je_4kn_4lb_3mg_4nk_4)$ and $14B11,4$ with the adjacency symbol $(am_3bi_4cf_3dg_4eh_3fc_4gd_3he_4ib_3jl_4kn_4lj_3ma_4nk_4)$. For cycle $14B12 : 34343434434344$ the only Delone class $14B12$ of isohedral tilings has the adjacency symbol $(am_3be_4ck_3dg_4eb_3fl_4gd_3hj_4in_4jh_3kc_4lf_3ma_4ni_4)$. For cycle $14B13 : 34343443434344$ there are 6 Delone classes: $14B13,1$ with the adjacency symbol $(ad_3be_4cf_3da_4eb_3fc_4gn_4hk_3il_4jm_3kh_4li_3mj_4ng_4)$, $14B13,2$ with the adjacency symbol $(ad_3bh_4ci_3da_4ek_3fl_4gn_4hb_3ic_4jm_3ke_4lf_3mj_4ng_4)$, $14B13,3$ with the adjacency symbol $(ad_3bl_4cm_3da_4ei_3fj_4gn_4hk_3ie_4jf_3kh_4lb_3mc_4ng_4)$, $14B13,4$ with the adjacency symbol $(ai_3bj_4cf_3dl_4em_3fc_4gn_4hk_3ia_4jb_3kh_4ld_3me_4ng_4)$, $14B13,5$ with the adjacency symbol $(ak_3be_4cm_3dh_4eb_3fj_4gn_4hd_3il_4jf_3ka_4li_3mc_4ng_4)$ and $14B13,6$ with the adjacency symbol $(ak_3bl_4cf_3dh_4ei_3fc_4gn_4hd_3ie_4jm_3ka_4lb_3mj_4ng_4)$. The adjacency diagrams for the Delone classes $14B1$, $14B8$ and $14B13,1$ of isohedral tilings are shown in Fig. 3.

4 Tilings with 16-gons

For $k = 16$ there is the only solution of the Diophantine equation $16A$: 3333333333334444 (here the letter A is added for further convenience).

We form possible classes of ordered cycles for the set $16A$ and enumerate only eligible ordered cycles of valencies. We analyze which groups of vertices of the same valency are admissible in a cycle, first for valency 4. If all the valencies 4 are consecutive, it yields 3 edges 44 which cannot be paired, so it is not admissible. Falling into 3 consecutive and a separate valency 4: 444 and 4 yields 2 neighboring edges 44 which cannot be paired and it is not admissible.

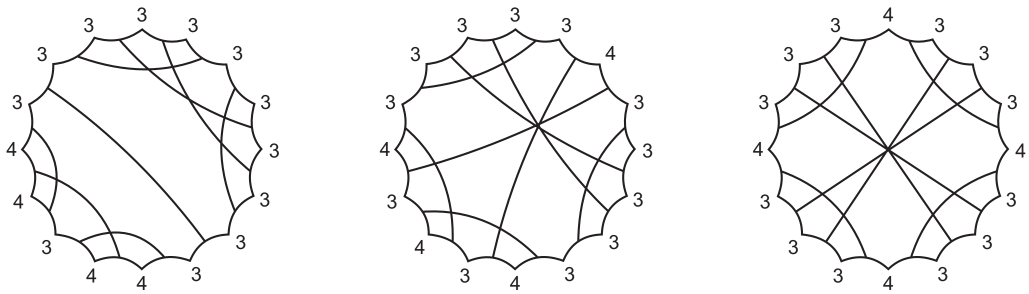
Falling into 2 pairs of consecutive valencies 4: 44 and 44 suits us. We divide 12 valencies 3 into 2 groups. By combining groupings of 3 with the above groups of 4 we obtain 6 classes of eligible ordered cycles of valencies. To each eligible cycle of valencies we apply the technique of adjacency symbols and check the condition of transition around a vertex, for four vertices of valency 3 and a vertex of valency 4.

For cycle $16A1$: 3333333333344344 there exists the only Delone class $16A1$ of isohedral tilings with symbol $(ao_3bk_3cf_3dh_3ei_3fc_3gj_3hd_3ie_3jg_3kb_3ln_4mp_4nl_3oa_4pm_4)$. For cycle $16A2$: 3333333333443344 there is also the only Delone class $16A2$ with the adjacency symbol $(ao_3be_3ch_3di_3eb_3fn_3gj_3hc_3id_3jg_3km_4lp_4mk_3nf_3oa_4pl_4)$. For cycle $16A3$: 3333333334433344 the only Delone class $16A3$ has the adjacency symbol $(ao_3bf_3ch_3dm_3ei_3fb_3gn_3hc_3ie_3jl_4kp_4lj_3md_3ng_3oa_4pk_4)$. For cycle $16A4$: 3333333344333344 the Delone class $16A4$ with the adjacency symbol $(ao_3bf_3cl_3dh_3em_3fb_3gn_3hd_3ik_4jp_4ki_3lc_3me_3ng_3oa_4pj_4)$ is the only possible, too. For cycle $16A5$: 3333334433333344 also the only Delone class $16A5$ with the adjacency symbol $(ao_3be_3ck_3dl_3eb_3fn_3gi_4hp_4ig_3jm_3kc_3ld_3mj_3nf_3oa_4ph_4)$ exists. For cycle 3333333443333344 there is no Delone class.

Falling into 2 consecutive and 2 separate valencies 4: 44, 4 and 4 yields one edge 44 and it is not admissible. So we examine the situation where all valencies 4 are separate. We divide 12 valencies 3 into 4 groups. Here there are many possibilities. We begin with the division where one group contains 9 consecutive valencies 3, then diminish this number to 8 and so on. In such a way we obtain 29 classes of eligible ordered cycles of valencies.

For cycle $16A6$: 3333333334343434 there exists the only Delone class $16A6$ with the adjacency symbol $(aj_3be_3cg_3dh_3eb_3fi_3gc_3hd_3if_3ja_4kn_3lo_4mp_3nk_4ol_3pm_4)$. For cycle $16A7$: 3333334333433434 the only Delone class $16A7$ has the adjacency symbol $(an_3be_3ci_3dj_3eb_3fm_3go_4hk_3ic_3jd_3kh_4lp_3mf_3na_4og_3pl_4)$. For cycle $16A8$: 3333334333434334 there are 2 Delone classes: $16A8,1$ with the adjacency symbol $(am_3bf_3ci_3do_3ej_3fb_3gl_4hp_3ic_3je_3kn_4lg_3ma_4nk_3od_3ph_4)$ and $16A8,2$ with the adjacency symbol $(am_3bo_3cf_3di_3ej_3fc_3gn_4hk_3id_3je_3kh_4lp_3ma_4ng_3ob_3pl_4)$. For cycle $16A9$: 3333334334333434 the only Delone class $16A9$ has the adjacency symbol $(aj_3be_3cl_3dm_3eb_3fi_3go_4hp_3if_3ja_4kn_3lc_3md_3nk_4og_3ph_4)$. For cycles 3333333343343434, 3333333343433434, 3333333433343434, 3333333434333434, 3333333433433434, 3333333433434334, 3333334333343434, 3333334343333434 and 3333334334334334 there are no Delone classes.

For cycle $16A10$: 3333343333343434 there exists the only Delone class $16A10$ with the adjacency symbol $(an_3be_3ci_3dj_3eb_3fm_4gp_3hk_3ic_3jd_3kh_3lo_4mf_3na_4ol_3pg_4)$. For cycle $16A11$: 3333343334334334 there is also the only Delone class $16A11$ with the adjacency symbol $(af_3bh_3cl_3do_3ei_3fa_4gm_3hb_3ie_3jn_4kp_3lc_3mg_4nj_3od_3pk_4)$. For cycle $16A12$: 3333343343334334 the only Delone class $16A12$ has the adjacency symbol $(af_3bk_3co_3dh_3el_3fa_4gm_3hd_3in_4jp_3kb_3le_3mg_4ni_3oc_3pj_4)$. For cycles 3333343433333434, 3333343333433434, 3333343333434334, 3333343343333434,



16A1 16A10 16A16,2

Figure 4. Adjacency diagrams for the Delone classes of isohedral tilings $16A1$, $16A10$ and $16A16, 2$

3333343334333434 and 3333343334343334 there are no Delone classes.

For cycle $16A13$: 3333433334333434 there exist 2 Delone classes: $16A13, 1$ with the adjacency symbol $(ae_3bg_3cm_3dh_3ea_4fn_3gb_3hd_3il_3jo_4kp_3li_3mc_3nf_4oj_3pk_4)$ and $16A13, 2$ with symbol $(an_3bg_3ci_3dl_3eo_4fj_3gb_3hm_3ic_3jf_4kp_3ld_3mh_3na_4oe_3pk_4)$. For cycle $16A14$: 3333433343333434 there are also 2 Delone classes: $16A14, 1$ has the adjacency symbol $(ae_3bl_3cg_3dm_3ea_4fn_3gc_3hk_3io_4jp_3kh_3lb_3md_3nf_4oi_3pj_4)$ and $16A14, 2$ has the symbol $(ai_3bk_3cm_3dg_3eo_4fp_3gd_3hl_3ia_4jn_3kb_3lh_3mc_3nj_4oe_3pf_4)$. For cycle $16A15$: 3333433334334334 the Delone class $16A15$ with the adjacency symbol $(am_3bg_3co_3di_3ek_4fp_3gb_3hl_3jn_4ke_3lh_3ma_4nj_3oc_3pf_4)$ is the only possible. For cycle $16A16$: 3334333433343334 there exist 3 Delone classes of isohedral tilings: $16A16, 1$ with the symbol $(ad_3bf_3cg_3da_4eh_3fb_3gc_3he_4il_3jn_3ko_3li_4mp_3nj_3ok_3pm_4)$, $16A16, 2$ with the symbol $(ad_3bj_3ck_3da_4eh_3fn_3go_3he_4il_3jb_3kc_3li_4mp_3nf_3og_3pm_4)$ and $16A16, 3$ with symbol $(al_3bf_3co_3di_4ep_3fb_3gk_3hm_4id_3jn_3kg_3la_4mh_3nj_3oc_3pe_4)$. For cycles 3333433433334334, 3333433343334334 and 3333433343343334 there are no Delone classes. The adjacency diagrams for the Delone classes $16A1$, $16A10$ and $16A16, 2$ of isohedral tilings are shown in Fig. 4.

5 Tilings with 18-gons and summarized results

For $k = 18$ the only solution of Diophantine equaton as well as the unique ordered cycle of valencies is 18 : 333333333333333333.

Here any edge is 33. We use the technique of adjacency symbols and draw adjacency diagrams taking in consideration some restrictions. Any neighboring edges cannot be paired and any two edges separated by an edge cannot be paired. We consult Fig. 1 for possible schemes of equivalent vertices of valency 3. We begin with the situation where the first two eqivalent edges are separated by an edge and work out all the possibilities, then we examine the situation where the first two equivalent edges are separated by two edges with no equivalent edges being separated by an edge, and so on. The thorough examination yields the following results.

There exist 6 Delone classes of isohedral tilings: 18, 1 with the adjacency sym-

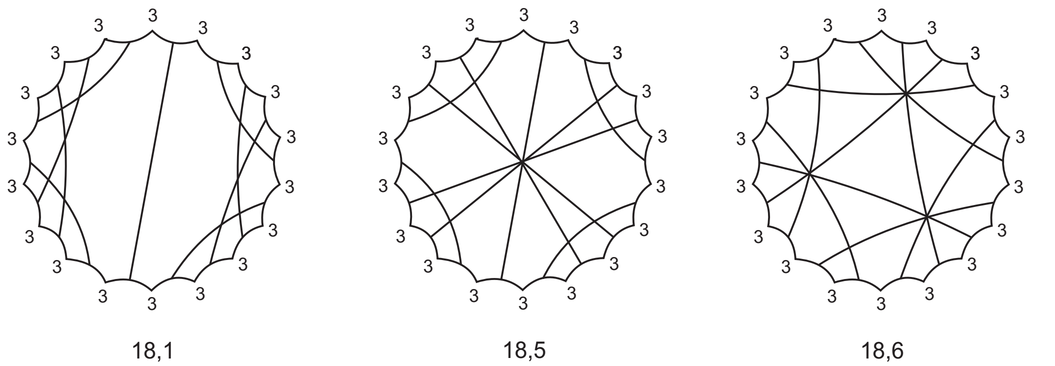


Figure 5. Adjacency diagrams for the Delone classes of isohedral tilings $18,1$, $18,5$ and $18,6$

bol $(ad_3bf_3cg_3da_3eh_3fb_3gc_3he_3ir_3jm_3ko_3lp_3mj_3nq_3ok_3pl_3qn_3ri_3)$, $18,2$ with the adjacency symbol $(ad_3bg_3ch_3da_3en_3fi_3gb_3hc_3if_3jm_3kp_3lq_3mj_3ne_3or_3pk_3ql_3ro_3)$, $18,3$ with the symbol $(ad_3bh_3ci_3da_3em_3fp_3gj_3hb_3ic_3jg_3ko_3lq_3me_3nr_3ok_3pf_3ql_3rn_3)$, $18,4$ with the symbol $(ad_3bi_3cj_3da_3em_3fo_3gq_3hk_3ib_3jc_3kh_3lp_3me_3nr_3of_3pl_3qg_3rn_3)$, $18,5$ with the symbol $(ad_3bk_3cl_3da_3eh_3fo_3gp_3he_3ir_3jm_3kb_3lc_3mj_3nq_3of_3pg_3qn_3ri_3)$ and $18,6$ with the symbol $(af_3bi_3cn_3dq_3ej_3fa_3gl_3ho_3ib_3je_3kp_3lg_3mr_3nc_3oh_3pk_3qd_3rm_3)$. The adjacency diagrams for the Delone classes $18,1$, $18,5$ and $18,6$ of isohedral tilings are shown in Fig. 5.

Now summarizing results obtained in previous work [7] and in the present article we conclude that we have proved the following

Theorem 1. *For translation group of genus two there exist 119 Delone classes of isohedral tilings of the hyperbolic plane with disks, and namely 4 classes with 8-gons, 18 classes with 10-gons, 31 classes with 12-gons, 39 classes with 14-gons, 21 classes with 16-gons, and 6 classes with 18-gons.*

The results for 8-, 10- and 12-gons have been described in [7], the results for 14-, 16- and 18-gons are described in the present article. The adjacency symbols have been given for all 119 Delone classes, the adjacency diagrams are depicted for 28 of them. An adjacency symbol can be obtained from the corresponding adjacency diagram if the letter a labels the right bottom edge, going round counter-clockwise.

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