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On the number of topologies on countable fields

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Abstract. For any countable field R and any non-discrete metrizable field topology τ_0 of the field, the lattice of all field topologies of the field admits:
– Continuum of non-discrete metrizable field topologies of the field stronger than the topology τ_0 and such that $\sup\{\tau_1, \tau_2\}$ is the discrete topology for any different topologies;
– Continuum of non-discrete metrizable field topologies of the field stronger than τ_0 and such that any two of these topologies are comparable;
– Two to the power of continuum of field topologies of the field stronger than τ_0, each of them is a coatom in the lattice of all topologies of the field.

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1 Introduction

The study of possibility to set a non-discrete Hausdorff topology on infinite algebraic systems in which existing operations are continuous was begun in [1]. In this article, for any countable group, a method of constructing such group topologies is given.

For countable rings, the problem the possibility to set non-discrete Hausdorff ring topologies was studied in [2, 3].

For infinite fields the problem of the possibility to set of non-discrete field topologies was studied in [2].

The present article is a continuation of research in this direction. The main result of this paper is Theorem 3.1, in which for any countable field R and any non-discrete metrizable field topology τ_0, the number of topologies which have some properties in the lattice of all field topologies is specified.

For countable groups and rings, similar results were obtained in [4, 5, 6].

2 Notations and preliminaries

To present the main results we remind the following well-known result:

Theorem 2.1. *A set Ω of subsets of a field R is a basis of filter of neighborhoods of zero for some Hausdorff field topology on the field R if and only if the following conditions are satisfied:*

1) $\bigcap_{V\in\Omega} V = \{0\}$;

2) *For any subset V_1 and $V_2 \in \Omega$ there exists a subset $V_3 \in \Omega$ such that $V_3 \subseteq V_1 \cap V_2$;*

3) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $V_2 + V_2 \subseteq V_1$;*

4) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $-V_2 \subseteq V_1$;*

5) *For any subset $V_1 \in \Omega$ and any element $r \in R$ there exists a subset $V_2 \in \Omega$ such that $r \cdot V_2 \subseteq V_1$ and $V_2 \cdot r \subseteq V_1$;*

6) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $V_2 \cdot V_2 \subseteq V_1$.*

7) *For any subset $V_1 \in \Omega$ there exists a subset $V_2 \in \Omega$ such that $\frac{V_2}{(e+V_2)\setminus\{0\}} \subseteq V_1$.*

Proof. According to ([2], Proposition 1.2.2, Theorems 1.2.5 and 1.2.12) for the proof of the Theorem it suffices to verify that for any subset $V_1 \in \Omega$ there exists a subset $V_3 \in \Omega$ such that $\frac{e}{(e+V_3)\setminus\{0\}} \subseteq e + V_1$.

Let $V_1 \in \Omega$ and let $V_2 \in \Omega$ be a set such that $\frac{V_2}{(e+V_2)\setminus\{0\}} \subseteq V_1$. If $V_3 \in \Omega$ is a set such that $-V_3 + V_3 \subseteq V_2$ and $a \in V_3$ and $-e \neq b \in V_3$, then $\frac{e+a}{(e+b)} \in \frac{e+V_3}{(e+V_3)\setminus\{0\}} =$

$$\frac{e+b-b+a}{(e+b)} = e + \frac{-b+a}{(e+b)} \in e + \frac{-V_3+V_3}{(e+V_2)\setminus\{0\}} \subseteq e + \frac{e+V_2}{(e+V_2)\setminus\{0\}} \subseteq e + V_1.$$

From the arbitrariness of elements a and b it follows that $\frac{e}{(e+V_3)\setminus\{0\}} \subseteq \frac{e+V_3}{(e+V_3)\setminus\{0\}} \subseteq e + V_1$, and hence the theorem is completely proved. □

Definition 2.2. *A subset V of an Abelian group $R(+)$ is called symmetric* if $-V = V$.

Notation 2.3. Let $V_1, V_2, \ldots$ and $S_1, S_2, \ldots$ be non-empty symmetric subsets of a field R, and e is the unit of the field R. If $e \in S_k$, $e \notin V_k$ and $0 \in V_k$ for any natural number k then we define by induction the subsets $F_k\big(S_1, \ldots, S_k; V_1, \ldots, V_k\big)$ of the field R:

We take $F_1(S_1; V_1) = V_1 + V_1 + V_1 \cdot V_1 + V_1 \cdot V_1 + S_1 \cdot V_1$, and

$$F_{t+1}(S_1, S_2, \ldots, S_{t+1}; V_1, V_2, \ldots, V_{t+1}) = F_1\big(S_1; V_1 \cup F_t(S_2, \ldots, S_{t+1}; V_2, \ldots, V_{t+1})\big)$$

and we take

$$\widetilde{F}_k(S_1, \ldots, S_k; V_1, \ldots, V_k) = \frac{F_k(S_1, \ldots, S_k; V_1, \ldots, V_k)}{e + F_k(S_1, \ldots, S_k; V_1, \ldots, V_k) \setminus \{0\}}$$

for any natural number k.

Proposition 2.4. *Let $V_1, V_2, \ldots$ and $S_1, S_2, \ldots$ be some sequences of non-empty finite symmetric subsets of a field R. If $e \in S_1 \subseteq S_2 \subseteq \ldots$ and $e \notin V_i$ and $0 \in V_i$ for any natural number i, then the following statements are true:*

Statement 1. $F_{k-1}(S_2, \ldots, S_k; V_2, \ldots, V_k) + F_{k-1}(S_2, \ldots, S_k; V_2, \ldots, V_k)+$

$$F_{k-1}(S_2, \ldots, S_k; V_2, \ldots, V_k) \cdot F_{k-1}(S_2, \ldots, S_k; V_2, \ldots, V_k)+$$

$$F_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)\cdot F_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)+$$
$$S_1\cdot F_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)\subseteq F_k(S_1,\dots,S_k;V_1,\dots,V_k)$$

for any natural number $k>1$, and hence,

1.1. $\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)+\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)\subseteq \widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k)$ *for any natural number* $k>1$,

1.2. $\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)\cdot\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)\subseteq \widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k)$ *for any natural number* $k>1$,

1.3. $\frac{\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)}{e+\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)\setminus\{0\}}\subseteq \widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k)$ *for any natural number* $k>1$,

1.4. $S_1\cdot\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)\subseteq \widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k)$ *for any natural number* $k>1$;

Statement 2. $F_k(S_1,\dots,S_k;V_1,\dots,V_k)$ and $\widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k)$ *are finite symmetric sets for any natural number* k*;*

Statement 3. $\widetilde{F}_k(S_1,\dots,S_k;\{0\},\dots,\{0\})=\widetilde{F}_k(S_1,\dots,S_k;\{0\},\dots,\{0\})=\{0\}$ *for any natural number* k*;*

Statement 4. *If* k *is a natural number and* $U_i\subseteq V_i\subseteq R$ *and* $T_i\subseteq S_i\subseteq R$ *for any natural number* $1\le i\le k$*, then*

$$F_k(T_1,\dots,T_k;U_1,\dots,U_k)\subseteq F_k(S_1,\dots,S_k;V_1,\dots,V_k)$$

and

$$\widetilde{F}_k(T_1,\dots,T_k;U_1,\dots,U_k)\subseteq \tilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k);$$

Statement 5. *If* k *and* p *are natural numbers and* $V_{k+j}=\{0\}$ *for any natural number* $1\le j\le p$*, then*

$$F_k(S_1,\dots,S_k;V_1,\dots,V_k)=F_{k+p}(S_1,\dots,S_{k+p};V_1,\dots,V_{k+p})$$

and

$$\widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k)=\widetilde{F}_{k+p}(S_1,\dots,S_{k+p};V_1,\dots,V_{k+p});$$

Statement 6. *For any natural number* $k\ge 2$ *the following equalities are true:*

$$F_k(S_1,\dots,S_k;V_1,\dots,V_k)=$$

$$F_k(S_1,\dots,S_k;V_1\cup F_{k-1}\ (S_2,\dots,S_k;V_2,\dots,V_k),\dots,V_{k-1}\cup F_1(S_k;V_k),V_k)$$

and

$$\widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k)=$$
$$\widetilde{F}_k\big(S_1,\dots,S_k;V_1\cup F_{k-1}\ (S_2,\dots,S_k;V_2,\dots,V_k),\dots,V_{k-1}\cup F_1(S_k;V_k),V_k\big);$$

Statement 7. $V_t\subseteq F_k(S_1,\dots,S_k;V_1,\dots,V_k)\subseteq\widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k)$ *for any natural numbers* k *and* $1\le t\le k$*;*

Statement 8. *f For any natural numbers k, s, t and $t \le s$ the following inclusions are true:*

$$F_{k+1}(S_s, \dots, S_{k+s}; V_s, \dots, V_{k+s}) \subseteq F_{k+s-t+1}(S_t, \dots, S_{k+s}; V_t, \dots, V_{k+s})$$

and

$$\widetilde{F}_{k+1}(S_s, \dots, S_{k+s}; V_s, \dots, V_{k+s}) \subseteq \widetilde{F}_{k+s-t+1}(S_t, \dots, S_{k+s}; V_t, \dots, V_{k+s})$$

Proof. Statement 1 for sets $F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ follows from the definition of the set $F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ for $k > 1$.

Statements 2 – 7 for sets $F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ are proved easy by induction on the number k and from the definition of sets $F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ (see [2], Proposition 5.3.2, or [6], Proposition 2.4).

Statement 8 for sets $F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ is proved easy by induction on the number $s - t$ (see [6], Proposition 2.4).

We proceed to the proof of these statements for sets $\widetilde{F}_k(S_1, \dots, S_k; V_1, \dots, V_k)$.

If $\tilde{a}, \tilde{b} \in \widetilde{F}_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k)$, and $c \in S_2$ then there exist $a_1, a_2, b_1, b_2 \in F_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k)$ such that $\tilde{a} = \frac{a_1}{e+a_2}$ and $\tilde{b} = \frac{b_1}{e+b_2}$. As $0 \in V_i$ for any i then from Statement 1 for the set $F_k(S_1, \dots, S_k; V_1, \dots, V_k)$ it follows that:

$$\tilde{a} + \tilde{b} = \frac{a_1}{e+a_2} + \frac{b_1}{e+b_2} = \frac{a_1 + a_1 \cdot b_2 + b_1 + a_2 \cdot b_1}{e + b_2 + b_1 + b_1 \cdot b_2} \in$$

$$\frac{F_k(S_1, \dots, S_k; V_1, \dots, V_k)}{e + F_k(S_1, \dots, S_k; V_1, \dots, V_k) \setminus \{0\}} = \widetilde{F}_k(S_1, \dots, S_k; V_1, \dots, V_k)$$

and

$$\tilde{a} \cdot \tilde{b} = \frac{a_1}{e+a_2} \cdot \frac{b_1}{e+b_2} = \frac{a_1 \cdot b_1}{e + b_2 + b_1 + b_1 \cdot b_2} \in$$

$$\frac{F_k(S_1, \dots, S_k; V_1, \dots, V_k)}{e + F_k(S_1, \dots, S_k; V_1, \dots, V_k) \setminus \{0\}} = \widetilde{F}_k(S_1, \dots, S_k; V_1, \dots, V_k)$$

and

$$\frac{\tilde{a}}{e+\tilde{b}} = \Big(\frac{a_1}{e+a_2}\Big) \cdot \Big(e + \frac{b_1}{e+b_2}\Big)^{-1} = \Big(\frac{a_1}{e+a_2}\Big) \cdot \Big(\frac{e \cdot (e+b_2) + b_1}{e+b_2}\Big)^{-1} =$$

$$\frac{a_1 \cdot (e+b_2)}{e + b_2 + (e+a_2) \cdot b_1} = \frac{a_1 + a_1 \cdot b_2}{e + b_2 + b_1 + a_2 \cdot b_1} \in \frac{F_k(S_1, \dots, S_k; V_1, \dots, V_k)}{e + F_k(S_1, \dots, S_k; V_1, \dots, V_k) \setminus \{0\}} =$$

$\widetilde{F}_k(S_1, \dots, S_k; V_1, \dots, V_k)$ and

$$c \cdot \tilde{a} = c \cdot \frac{a_1}{e+a_2} = \frac{c \cdot a_1}{e+a_2} \in \frac{c \cdot F_{k-1}(S_1, \dots, S_k; V_1, \dots, V_k)}{e + F_{k-1}(S_1, \dots, S_k; V_1, \dots, V_k) \setminus \{0\}} \subseteq$$

$\widetilde{F}_k(S_1, \dots, S_k; V_1, \dots, V_k)$.

From the arbitrariness of elements $\tilde{a}$, $\tilde{b}$ and c it follows that

$$\widetilde{F}_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) + \widetilde{F}_{k-1}(S_2, \dots, S_k; V_2, \dots, V_k) \subseteq \widetilde{F}_k(S_1, \dots, S_k; V_1, \dots, V_k),$$

and hence the inclusion 1.1 is proved and

$$\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)\cdot\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k) \subseteq \widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k),$$

and hence the inclusion 1.2 is proved and

$$\frac{\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)}{e+\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k)\setminus\{0\}} \subseteq \widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k),$$

and hence the inclusion 1.3 is proved and

$$S_2\cdot\widetilde{F}_{k-1}(S_2,\dots,S_k;V_2,\dots,V_k) \subseteq \widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k),$$

and hence the inclusion 1.4 is proved.

Hence we have proved Statement 1 also for the set $\widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k)$.

For any set $\widetilde{F}_k(S_1,\dots,S_k;V_1,\dots,V_k)$ each of Statements 2 – 8 follows from the definition the set
$widetilde F_k(S_1,\dots,S_k;V_1,\dots,V_k)$ and of the corresponding statement for the set $F_k(S_1,\dots,S_k;V_1,\dots,V_k)$.

Hence, Proposition 2.4 is proved. □

Definition 2.5. If R is a field and x is some variable, then we denote by:
– $R[x]$ the polynomial ring on the field R;
– $\tilde{R}[x]$ the field of fractions of the ring $R[x]$.

We call elements of the field $\tilde{R}[x]$ *a rational function of x over the field R.*

Definition 2.6. As usual, an element $a \in R$ is called a root of a rational function $f(x) \in \tilde{R}[x]$ if $f(a) = 0$.

Notation 2.7. If $R = \{0, \pm 1, \pm r_1, \pm r_2, \dots\}$ is a countable field, then for any natural number k we put $S_k = \{\pm 1, \pm r_1, \pm r_2, \dots, \pm r_k\}$.

Theorem 2.8. *Let (R,τ) be a topological field. If τ is a non-discrete Hausdorff topology then for any rational function $\tilde{f}(x) = \frac{f_1(x)}{e+f_2(x)}$ of x over the field R such that $\tilde{f}(0) \neq 0$ and $e + f_2(0) \neq 0$ there exists a neighborhood W of zero such that each element $r \in W$ is not a root of the rational function $\tilde{f}(x)$.*

Proof. As $\tilde{f}(0) = \frac{f_1(0)}{e+f_2(0)} \neq 0$ then $f_1(0) \neq 0$ and since (R,τ_0) is a Hausdorff space, then there exists a neighborhood W_0 of the element $\tilde{f}(0)$ such that $0 \notin W_0$.

As (R,τ_0) is a topological field then there exist neighborhoods W_1 and W_2 of elements $f_1(0)$ and $f_2(0)$ such that $W_1 \cdot (e+W_2)^{-1} \subseteq W_0$.

Since any polynomial over a topological field is a continuous function, then there exists a neighborhood W of zero in (R,τ_0) such that $f_1(r) \in W_1$ and $f_2(r) \in W_2$ for any element $r \in W$. Then $\tilde{f}(r) = \frac{f_1(r)}{e+f_2(r)} \in W_1 \cdot (e+W_2)^{-1} \subseteq W_0$, and hence $\tilde{f}(r) \neq 0$ for any element $r \in W$.

The theorem is proved. □

3 Basic results

Theorem 3.1. *If* $R = \{0, \pm r_{1,}, \pm r_2, \dots\}$ *is a countable field and* τ_0 *is a non-discrete, Hausdorff, field topology such that the topological field* (R, τ_0) *has a countable basis of the filter of neighborhoods of zero, then the following statements are true:*

1. *For any infinite set* A *of natural numbers there exists a field topology* $\tau(A)$ *such that* $\tau_0 \leq \tau(A)$ *and the topological field* $(R, \tau(A))$ *has a countable basis of the filter of neighborhoods of zero*;

2. $\sup\{\tau(A), \tau(B)\}$ *is the discrete topology for any infinite sets* A *and* B *of natural numbers such that* $A \cap B$ *is a finite set;*

3. *There are continuum of field topologies stronger than* τ_0 *and such that any two of them are comparable to each other;*

4. *There exist two to the power of continuum of field topologies such that* $\sup\{\tau_1, \tau_2\}$ *is the discrete topology for any two different topologies* τ_1 *and* τ_2*;*

5. *There exist two to the power of continuum of coatoms in the lattice of all field topologies of the field* R.

Proof. Since (R, τ_0) is a topological field and it is a Hausdorff space, then there exists a countable basis $\{V_1, V_2, \dots\}$ of the filter of neighborhoods of zero such that $-V_k = V_k$, $V_k \cap S_k = \emptyset$ and

$$\widetilde{F}_1(S_{k+1}; V_{k+1}) = \frac{V_{k+1} + V_{k+1} + V_{k+1} \cdot V_{k+1} + V_{k+1} \cdot V_{k+1} + S_{k+1} \cdot V_{k+1}}{e + V_{k+1} + V_{k+1} + V_{k+1} \cdot V_{k+1} + V_{k+1} \cdot V_{k+1} + S_{k+1} \cdot V_{k+1}} \subseteq V_k$$

for any natural number k.

Then for any natural numbers i and n by induction on the number n it is easy to prove that $\widetilde{F}_n(S_{i+1}, \dots, S_{i+n}; V_{i+1} \dots, V_{i+n}) \subseteq V_i$.

Further the proof of Statement 1 will be realized in several steps.

Step I. By induction we construct a sequence $k_1, k_2, \dots$ of natural numbers such that $k_i \geq i$, for any natural number i and we construct a sequence $h_1, h_2, \dots$ of nonzero elements of the field R such that $\{-h_i, h_i\} \subseteq V_{k_i}$ and

$$\widetilde{F}_n(S_1, \dots, S_k; U_{A,1}, \dots, U_{A,n}) \bigcap \widetilde{F}_n(S_1, \dots, S_k; U_{B,1}, \dots, U_{B,n}) = \{0\}$$

for all subsets A and B of the set of natural numbers such that $A \cap B = \emptyset$, where $U_{C,i} = \{h_i, 0, -h_i\}$ if $i \in C$ and $U_{C,i} = \{0\}$ if $i \notin C$, for any set C of natural numbers.

We take $k_1 = 2$, and as h_1 we take an arbitrary element of the set $V_2 \backslash \{0\}$.

If A and B are some sets of natural numbers such that $A \bigcap B = \emptyset$, then $k_1 \notin A$ or $k_1 \notin B$, and hence, $U_{A,1} = \{0\}$ or $U_{B,1} = \{0\}$. Then $\tilde{F}_1(S_1; U_{A,1}) \cap \tilde{F}_1(S_1; U_{B,1}) = \{0\}$ for any sets A and B of natural number such that $A \cap B = \emptyset$.

Suppose that we defined natural numbers $k_1 < k_2 < \dots < k_n$ such that $k_i \geq i$ and we defined nonzero elements $h_1, h_2, \dots, h_n$ of the field R such that $\{h_i, -h_i\} \subseteq V_{k_i}$ and

$$\widetilde{F}_n(S_1, \dots, S_k; U_{A,1}, \dots, U_{A,n}) \cap \widetilde{F}_n(S_1, \dots, S_k; U_{B,1}, \dots, U_{B,n}) = \{0\}$$

for any sets A and B of natural numbers such that $A \cap B = \emptyset$.

For any sets $A' \subseteq \{1, \ldots, n\}$ and $B' \subseteq \{1, \ldots, n\}$ of natural numbers such that $A' \cap B' = \emptyset$ we consider a finite set

$$\Omega_{(A',B')} = \widetilde{F}_{n+1}(S_1, \ldots, S_{n+1}; U_{A',1} \ldots, U_{A',n}, \{x, 0, -x\}) -$$

$$\big(\widetilde{F}_n(S_1, \ldots, S_{n+1}; U_{B',1} \ldots, U_{B',n}) \backslash \{0\}\big)$$

of rational functions over the field R in variable x.

Since, according to Statement 5 of Proposition 2.4,

$$\widetilde{F}_{n+1}(S_1, \ldots, S_{n+1}; U_{A',1} \ldots, U_{A',n}, \{0\}) = \widetilde{F}_n(S_1, \ldots, S_{n+1}; U_{A',1}, \ldots, U_{A',n}),$$

then according to inductive assumption,

$$\widetilde{F}_n(S_1, \ldots, S_n; U_{A',1}, \ldots, U_{A',n}) \cap \big(F_n(S_1, \ldots, S_n; U_{B',1}, \ldots, U_{B',n}) \backslash \{0\}\big) = \emptyset.$$

If $\tilde{f}(x) \in \Omega_{(A',B')}$ for $A', B' \subseteq \{1, \ldots, n\}$, and $A' \cap B' = \emptyset$ then $\tilde{f}(x) = \tilde{f}_1(x) + r$ for any $\tilde{f}_1(x) \in \widetilde{F}_{n+1}(S_1, \ldots, S_{n+1}; U_{A',1} \ldots, U_{A',n}, \{x, 0, -x\}$ and $r \in \widetilde{F}_n(S_1, \ldots, S_n; U_{B',1}, \ldots, U_{B',n}$. As

$$\tilde{f}_1(0) \in \widetilde{F}_{n+1}(S_1, \ldots, S_{n+1}; U_{A',1} \ldots, U_{A',n}, \{0\}) = \widetilde{F}_n(S_1, \ldots, S_{n+1}; U_{A',1}, \ldots, U_{A',n})$$

then $\tilde{f}_1(0) \neq -r$, and hence, $\tilde{f}(0) \neq 0$.

Since the set $\{1, \ldots, n\}$ has a finite number of subsets, then the set

$$\Phi_n = \bigcup_{A', B' \subseteq \{1, \ldots, n\}, A' \bigcap B' = \emptyset} \Omega_{(A',B')}$$

is a finite set of rational functions $\tilde{f}(x)$ over the field R in variable x such that $\tilde{f}(0) \neq 0$.

Then, by Theorem 2.8, there exists a neighborhood W of zero in the topological field (R, τ_0) such that any element $r \in W$ is not a root of any rational function of the set Φ_n.

Then there exists a natural number k_{n+1} such that $k_{n+1} > k_n$ and $V_{k_{n+1}} \subseteq W$. We take h_{n+1} an arbitrary element of the set $V_{k_{n+1}} \backslash \{0\}$.

We prove that

$$\widetilde{F}_{n+1}(S_1, \ldots, S_{n+1}; U_{A,1}, \ldots, U_{A,n+1}) \cap \widetilde{F}_{n+1}(S_1, \ldots, S_{n+1}; U_{B,1}, \ldots, U_{B,n+1}) = \{0\}$$

for any subsets A and B of natural number such that $A \cap B = \emptyset$ (definition of sets $U_{C,k}$ see above).

Assume the contrary, and let

$$0 \neq r \in \widetilde{F}_{n+1}(S_1, \ldots, S_{n+1}; U_{A,1}, \ldots, U_{A,n+1}) \cap \widetilde{F}_{n+1}(S_1, \ldots, S_{n+1}; U_{B,1}, \ldots, U_{B,n+1}).$$

Since $A \cap B = \emptyset$ then from inductive assumption it follows that

$$\widetilde{F}_{n+1}(S_1, \ldots, S_{n+1}; U_{A,1}, \ldots, U_{A,n}, \{0\}) \cap \widetilde{F}_{n+1}(S_1, \ldots, S_{n+1}; U_{B,1}, \ldots, U_{B,n}, \{0\}) =$$

$$\widetilde{F}_n(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n})\cap\widetilde{F}_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n})=\{0\},$$

and hence $U_{A,n+1}=\{h_{n+1},0,-h_{n+1}\}$ or $U_{B,n+1}=\{h_{n+1},0,-h_{n+1}\}$ and as $A\cap B=\emptyset$ then from the definition of sets $U_{C,i}$ it follows that $U_{A,n+1}=\{0\}$ or $U_{B,n+1}=\{0\}$.

Assume, for definiteness, that $U_{A,n+1}=\{0\}$ and $U_{B,n+1}=\{h_{n+1},0,-h_{n+1}\}$.

Then

$$0\neq r\in\widetilde{F}_n(S_1,\dots,S_n;U_{A,1},\dots,U_{A,n})\bigcap\widetilde{F}_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n+1}),$$

and hence, $r=\tilde{f}(h_{n+1})$ for some rational function

$$\tilde{f}(x)\in\widetilde{F}_{n+1}(S_1,\dots,S_{n+1};U_{B,1},\dots,U_{B,n},\{x,0,-x\}).$$

As $U_{C,i}=U_{C\bigcap\{1,\dots,n\},i}$ for any natural number $1\le i\le n$ and any set C of natural numbers, then $\tilde{f}(h_{n+1})-r\in\widetilde{F}_{n+1}(S_1,\dots,S_{n+1};U_{A',1},\dots,U_{A',n},\{h_{n+1},0,-h_{n+1}\})-(\widetilde{F}_n(S_1,\dots,S_{n+1};U_{B',1},\dots,U_{B',n})\setminus\{0\})$,

for $A'=A\bigcap\{1,\dots,n\}$ and $B'=B\bigcap\{1,\dots,n\}$.

We have contradiction with the definition of the element h_{n+1}. Therefore

$$\widetilde{F}_{n+1}(S_1,\dots,S_{n+1};U_{A,1},\dots,U_{A,n+1})\cap\widetilde{F}_{n+1}(S_1,\dots,S_{n+1};U_{B,1}\dots,U_{B,n+1})=\{0\}.$$

So, we defined the sequence $k_1,k_2,\dots$ of natural numbers such that $k_i\ge i$ for any number i and the sequence $h_1,h_2,\dots$ of nonzero elements of the field R such that $\{-h_i,h_i\}\subseteq V_{k_i}$ for any natural number i and

$$\widetilde{F}_n(S_1,\dots,S_k;U_{A,1},\dots,U_{A,n})\cap\widetilde{F}_n\ (S_1,\dots,S_k;U_{B,1},\dots,U_{B,n})=\{0\}$$

for any natural number n and any sets A and B of natural numbers such that $A\cap B=\emptyset$.

Step II. For any pair (i,j) of natural numbers we consider the set

$$U_{(i,j),A}=\widetilde{F}_j(U_{i+1,A},\dots,U_{i+j,A};S_{i+1},\dots,S_{i+j}),$$

where $U_{i,A}=\{0\}$ if $i\notin A$ and $U_{i,A}=\{0,h_i,-h_i\}$ if $i\in A$.

We show that for the sets $U_{(i,j),A}$ the following inclusions are true:

1. From Statement 3 of Proposition 2.4 it follows that

$$0\in\widetilde{F}_n(S_{i+1},\dots,S_{i+n};U_{i+1,A},\dots,U_{i+n,A})\in U_{(i,j),A}$$

for any natural numbers i,j and

$$U_{(i,n),A}=\widetilde{F}_n(S_{i+1},\dots,S_{i+n};U_{i+1,A},\dots,U_{i+n,A})\subseteq$$

$$\widetilde{F}_n(S_{i+1},\dots,S_{i+n};V_{i+1},\dots,V_{i+n})\subseteq V_i$$

for any natural numbers i, n and any set A of natural numbers.

2. From Statements 4 and 5 of Proposition 2.4 it follows that $U_{(k,j),A} \subseteq U_{(k,n),A}$ for any natural numbers $j \le n$.

3. From Statement 8 of Proposition 2.4 it follows that $U_{(i,j),A} \subseteq U_{(k,j),A}$ for any natural numbers $k \le i$ and j.

4. From Statement 2 of Proposition 2.4 it follows that $U_{(i,j),A}$ is a symmetric set, i.e. $-U_{(i,j),A} = U_{(i,j),A}$ for any natural numbers i, j.

5. For any natural numbers i, j and $j > 1$ and any set A of natural numbers we prove by induction on the number j the following inclusions:

$$U_{(i+1,j),A} \cdot U_{(i+1,j),A} \subseteq U_{(i,j),A};$$

$$U_{(i+1,j),A} + U_{(i+1,j),A} \subseteq U_{(i,j),A};$$

$$\frac{U_{(i+1,j),A}}{e + U_{(i+1,j),A}} \subseteq U_{(i,j),A}.$$

In fact, if $j = 2$, then, from the definition of sets $U_{(i,j),A}$ and Statement 1 of Proposition 2.4 it follow:

$$U_{(i+1,2),A} \cdot U_{(i+1,2),A} = \widetilde{F}_1(S_{i+2}; U_{i+2,A}) \cdot \widetilde{F}_1(S_{i+2}; U_{i+2,A}) \subseteq$$

$$\widetilde{F}_1\big(S_{i+1}; \widetilde{F}_1(S_{i+2}; U_{i+2,A})\big) \subseteq \widetilde{F}_1(S_{i+1}; U_{i+1,A} \cup \widetilde{F}_1(S_{i+2}; U_{i+2,A})) =$$

$$\widetilde{F}_2(S_{i+1}, S_{i+2}; U_{i+1,A}, U_{i+2,A}) = U_{(i,2),A};$$

$$U_{(i+1,2),A} + U_{(i+1,2),A} = \widetilde{F}_1(S_{i+2}; U_{i+2,A}) + \widetilde{F}_1(S_{i+2}; U_{i+2,A}) \subseteq$$

$$\widetilde{F}_1\big(S_{i+1}; U_{i+1,A} \cup \widetilde{F}_1(S_{i+2}; U_{i+2,A})\big) = \widetilde{F}_1\big(S_{i+1}; \widetilde{F}_1(S_{i+2}; U_{i+2,A})\big) \subseteq$$

$$\widetilde{F}_2(S_{i+1}, S_{i+2}; U_{i+1,A}, U_{i+2,A}) = U_{(i,2),A};$$

$$\frac{U_{(i+1,2),A}}{e + U_{(i+1,2),A}} = \frac{\widetilde{F}_1(S_{i+2}; U_{i+2,A})}{e + \widetilde{F}_1(S_{i+2}; U_{i+2,A})} \subseteq$$

$$\widetilde{F}_1\big(S_{i+1}; \widetilde{F}_1(S_{i+2}; U_{i+2,A})\big) \subseteq \widetilde{F}_1(S_{i+1}; U_{i+1,A} \cup \widetilde{F}_1(S_{i+2}; U_{i+2,A})) =$$

$$\widetilde{F}_2(S_{i+1}, S_{i+2}; U_{i+1,A}, U_{i+2,A}) = U_{(i,2),A}$$

for any natural number i and any set A of natural numbers.

Assume that the required inclusions are proved for natural number $j = n \ge 2$ and any natural number i. Then:

$$U_{(i+1,n+1),A} \cdot U_{(i+1,n+1),A} = \widetilde{F}_n(S_{i+2}, \dots, S_{i+n+1};$$

$$U_{i+2,A}, \dots, U_{i+n+1,A}) \cdot \widetilde{F}_n(S_{i+2}, \dots, S_{i+n+1}; U_{i+2,A}, \dots, U_{i+n+1,A}) \subseteq$$

$$\widetilde{F}_1(S_{i+1}; \widetilde{F}_n(S_{i+2}, \dots, S_{i+n+1}; U_{i+2,A}, \dots, U_{i+n+1,A})) \subseteq$$

$$\widetilde{F}_1\big(S_{i+1}; U_{i+1,A} \cup \widetilde{F}_n(S_{i+2} \dots, S_{i+n+1}; U_{i+2,A}, \dots, U_{i+n+1,A})\big) =$$

$$\widetilde{F}_{n+1}(S_{i+1} \dots, S_{i+n+1}; U_{i+1,A}, \dots, U_{i+n+1,A}) = U_{(i,n+1),A};$$

$$U_{(i+1,n+1),A}+U_{(i+1,n+1),A}=\widetilde{F}_n(S_{i+2},\dots,S_{i+n+1};U_{i+2,A},\dots,U_{i+n+1,A})+$$
$$\widetilde{F}_n(S_{i+2},\dots,S_{i+n+1};U_{i+2,A},\dots,U_{i+n+1,A})\subseteq$$
$$\widetilde{F}_1(S_{i+1};\widetilde{F}_n(S_{i+2},\dots,S_{i+n+1};U_{i+2,A},\dots,U_{i+n+1,A}))\subseteq$$
$$\widetilde{F}_1(S_{i+1};U_{i+1,A}\cup\widetilde{F}_n(S_{i+2},\dots,S_{i+n+1};U_{i+2,A},\dots,U_{i+n+1,A}))=$$
$$\widetilde{F}_{n+1}(S_{i+1}\dots,S_{i+n+1};U_{i+1,A},\dots,U_{i+n+1,A})=U_{(i,n+1),A};$$
$$\frac{U_{(i+1,n+1),A}}{e+U_{(i+1,n+1),A}}=$$
$$\frac{\widetilde{F}_n(S_{i+2},\dots,S_{i+n+1};U_{i+2,A},\dots,U_{i+n+1,A})}{e+\widetilde{F}_n(S_{i+2},\dots,S_{i+n+1};U_{i+2,A},\dots,U_{i+n+1,A})}\subseteq$$
$$\widetilde{F}_1(S_{i+1};\widetilde{F}_n(S_{i+2},\dots,S_{i+n+1};U_{i+2,A},\dots,U_{i+n+1,A}))\subseteq$$
$$\widetilde{F}_1(S_{i+1};U_{i+1}\cup\widetilde{F}_n(S_{i+2},\dots,S_{i+n+1};U_{i+2,A},\dots,U_{i+n+1,A}))=$$
$$\widetilde{F}_{n+1}(S_{i+1}\dots,S_{i+n+1};U_{i+1,A},\dots,U_{i+n+1,A})=U_{(i,n+1),A},$$

and hence all inclusions specified in 5 are proved.

6. For any natural numbers i,j,k and $j>1$ and any set A of natural numbers we prove that $r_k\cdot U_{(i+k,j),A}\subseteq U_{(i,j),A}$.

In fact,

$$r_k\cdot U_{(i+k,j),A}\subseteq S_{i+k}\cdot\widetilde{F}_{k+i+j}(S_{k+i+1},\dots,S_{k+i+j};U_{k+i+1,A},\dots,U_{k+i+j,A})\subseteq$$
$$\widetilde{F}_1(S_{k+i};U_{k+i,A}\cup\widetilde{F}_{k+i+j}(S_{k+i+1},\dots,S_{k+i+j};U_{k+i+1,A},\dots,U_{k+i+j,A}))=$$

$U_{(i+k-1,j),A}\subseteq U_{(i,j),A}$, and hence inclusion 6 is proved.

Step III. For every infinite set A of natural numbers and any natural number i we take $\hat{U}_i(A)=\bigcup\limits_{j=1}^{\infty}U_{(i,j),A}$ and show that the set $\{\hat{U}_i(A)|i\in\mathbb{N}\}$ satisfies the conditions of Theorem 2.1, and hence, this set is a basis of the filter of neighborhoods of zero for a field topology $\tau(A)$ on the field R.

In fact, since

$$U_{(i,n+1),A}=\widetilde{F}_{n+1}(S_{i+1},\dots,S_{i+n+1};U_{i+1,A},\dots,U_{i+n+1,A})\subseteq$$
$$\widetilde{F}_{n+1}(S_{i+1}\dots,S_{i+n+1};V_{i+1},\dots,V_{i+n+1})\subseteq V_i$$

for any natural numbers i and n, then $\hat{U}_i(A)=\bigcup\limits_{j=1}^{\infty}U_{(i,j),A}\subseteq V_i$. Then

$\{0\}\subseteq\bigcap\limits_{i=1}^{\infty}\hat{U}_i(A)\subseteq\bigcap\limits_{i=1}^{\infty}V_i=\{0\}$, and hence, the condition 1 of Theorem 2.1 is satisfied.

From inclusions 2 and 3 (see Step II), it follows

$$\hat{U}_i(A)\bigcap \hat{U}_k(A) = (\bigcup_{j=1}^{\infty}(U_{(i,j),A})\bigcap(\bigcup_{l=1}^{\infty} U_{(k,l),A}) =$$

$$\bigcup_{j=1}^{\infty}\bigcup_{l=1}^{\infty}(U_{(i,j),A}\bigcap U_{(k,l),A}) = \bigcup_{j=1}^{\infty} U_{(t,j),A} = \hat{U}_t(A),$$

where $t = \max\{i,k\}$, and hence, the condition 2 of Theorem 2.1 is satisfied.

From inclusions 2 and 5 (see Step II) it follows

$$\hat{U}_i(A) + \hat{U}_k(A) = (\bigcup_{j=1}^{\infty} U_{(i,j),A}) + (\bigcup_{l=1}^{\infty} U_{(i,l),A}) =$$

$$\bigcup_{j=1}\bigcup_{j=1}(U_{(i,j),A} + U_{(i,l),A}) = \bigcup_{t=1}^{\infty} U_{(i-1,t),A} = \hat{U}_{i-1}(A)$$

and

$$\hat{U}_i(A)\cdot \hat{U}_k(A) = (\bigcup_{j=1}^{\infty} U_{(i,j),A})\cdot(\bigcup_{l=1}^{\infty} U_{(i,l),A}) =$$

$$\bigcup_{j=1}^{\infty}\bigcup_{j=1}^{\infty}(U_{(i,j),A}\cdot U_{(i,l),A}) = \bigcup_{t=1}^{\infty} U_{(i-1,t),A} = \hat{U}_{i-1}(A)$$

for any natural number $i > 1$, and hence, conditions 3 and 6 of Theorem 2.1 are satisfied.

From inclusion 3 (see Step II) it follows

$$-\hat{U}_i(A) = -(\bigcup_{j=1}^{\infty} U_{(i,j),A}) = \bigcup_{j=1}^{\infty}\bigcup_{l=1}^{\infty}(-U_{(i,j),A}) = \bigcup_{j=1}^{\infty} U_{j,A} = \hat{U}_i(A)$$

for any natural number i, and hence, the condition 4 of Theorem 2.1 is satisfied.

Let now $r \in R$.

If $r = 0$, then $r\cdot\hat{U}_i(A) = \{0\} \subseteq \hat{U}_i(A)$ and $\hat{U}_i(A)\cdot r = \{0\} \subseteq \hat{U}_i(A)$ for any natural number i and any set A of natural numbers.

If $r \neq 0$, then $r = r_n$ or $r = -r_n$ for some natural number n. Then, from the inclusion of 6 (see Step II), it follows $r_n\cdot\hat{U}_{i+n}(A) \subseteq \hat{U}_i(A)$ for any natural number i, and hence, the condition 5 of Theorem 2.1 is satisfied.

If now $a,b \in \hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(i,j),A}$ then from inclusion 3 (see Step II) it follows that there exists a natural number n such that $a,b \in U_{(i,n),A}$. Then (see inclusion 5 of Step II) $\frac{a}{e+b} \in \frac{U_{(i,n),A}}{e+U_{(i,n),A}} \subseteq U_{(i-1,n),A}$ for any natural number $i > 1$, and from the arbitrariness of elements a and b it follows $\frac{\hat{U}_i(A)}{e+\hat{U}_i(A)} \subseteq \hat{U}_{i-1}(A)$ for any natural number $i > 1$, and hence condition 7 of Theorem 2.1 is satisfied.

Thus, we have shown that the set $\{\hat{U}_i(A) | i \in \mathbb{N}\}$ satisfies conditions $1-7$ of Theorem 2.1, and hence, this set is a basis of the filter of neighborhoods of zero for a field topology $\tau(A)$ of the field R.

Since $\hat{U}_i(A) = \bigcup\limits_{j=1}^{\infty} U_{(i,j),A} \subseteq V_i$ for any natural number i, then $\tau_0 \leq \tau(A)$.

Thus Statement 3.1.1 is proved.

Proovs of Statements 2 - 5 can be obteined if we repeat the proof word for word of the corresponding statements 3.12 – 3.1.5 in [6].

The theorem is proved. $\square$

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