BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(89), 2019, Pages 39–51
ISSN 1024–7696

On fully idempotent semimodules

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Abstract. Let S be a semiring and M an S-semimodule. Let N and L be subsemimodules of M. Set $N \star L := Hom_S(M, L)N = \sum\{\varphi(N) \mid \varphi \in Hom_S(M, L)\}$. Then N is called an idempotent subsemimodule of M, if $N = N \star N$. An S-semimodule M is called fully idempotent if every subsemimodule of M is idempotent. In this paper we study the concept of fully idempotent semimodules as a generalization of fully idempotent modules and investigate some properties of idempotent subsemimodules of multiplication semimodules.

Mathematics subject classification: 16Y60.
Keywords and phrases: semiring, fully idempotent semimodule, multiplication semimodule, regular semimodule.

1 Introduction

A semiring is a nonempty set S with two operations addition $(+)$ and multiplication $(\cdot)$ such that $(S, +)$ is a commutative monoid with identity element 0; $(S, .)$ is a monoid with identity element $1 \neq 0$; $0a = 0 = a0$ for all $a \in S$ and multiplication distributes over addition. The semiring S is commutative if the monoid $(S, .)$ is commutative. All semirings in this paper are commutative. An ideal I of a semiring S is a subset of S such that $a + b \in I$ and $sa \in I$ for all $a, b \in I$ and $s \in S$.

Let S be a semiring. An S-semimodule is an additive abelian monoid $(M, +, 0_M)$ with a scalar multiplication $S \times M \to M((s, m) \mapsto sm)$ such that $(s_1s_2)m = s_1(s_2m)$; $s(m_1+m_2) = sm_1+sm_2$; $(s_1+s_2)m = s_1m+s_2m$; $1m = m$ and $s0_M = 0_M = 0m$ for all $s, s_1, s_2 \in S$ and all $m, m_1, m_2 \in M$. A nonempty subset N of an S-semimodule M is a subsemimodule of M if N is closed under addition and scalar multiplication. A subsemimodule N of an S-semimodule M is subtractive if $m + n \in N$ and $m \in N$ imply that $n \in N$ for all $m, n \in M$. An S-semimodule M is called subtractive if every subsemimodule of M is subtractive.

Suppose that M and M' are S-semimodules. Then a map α from M to M' is an S-homomorphism if $\alpha(m + m') = \alpha(m) + \alpha(m')$ for all $m, m' \in M$ and $\alpha(sm) = s(\alpha(m))$ for all $m \in M$ and $s \in S$.

Let M be an S-semimodule and N a subsemimodule of M. Then N induces a congruence relation on M as follows: $m \equiv_N n$ if and only if $m + a = n + b$ for some $a, b \in N$. The set of equivalence classes is an S-semimodule and denoted by M/N. The equivalence class of $m \in M$ is denoted by m/N. The S-semimodule M/N is called the factor semimodule of M by $\equiv_N$. Factor semiring can be defined in a similar way. For more details on factor semimodules see [7] and [4].

Let R be a ring. A submodule N of an R-module M is called idempotent if $N = Hom_R(M,N)N = \sum\{\varphi(N) \mid \varphi \in Hom_R(M,N)\}$ [9]. Hence an ideal I of a ring R is an idempotent submodule of R if and only if $I = I^2$. An R-module M is called fully idempotent if every submodule of M is idempotent. A semiring S is called fully idempotent if any ideal of S is idempotent. Fully idempotent semirings were studied in [3]. In this paper we introduce fully idempotent semimodules. In Section 2, we generalize some results of [9] to semimodules. We investigate some properties of idempotent subsemimodules and we prove that over semirings such that every nonzero semimodule has an injective envelope, all semimodules are fully idempotent if and only if all cyclic semimodules are injective. In Section 3, we study idempotent subsemimodules of multiplication semimodules and generalize some results of [11] to semimodules. We prove that a pure subsemimodule of a multiplication semimodule M is a multiplication and idempotent subsemimodule of M and the converse is true for some class of semirings and semimodules.

2 Idempotent subsemimodules

In this section we introduce fully idempotent semimodules as a generalization of fully idempotent modules and give some properties of these semimodules.

Suppose that M is an S-semimodule. Let N and L be subsemimodules of M. Set $N \star L := Hom_S(M,L)N = \sum\{\varphi(N) \mid \varphi \in Hom_S(M,L)\}$. It is clear that $\varphi(N)$ is a subsemimodule of L for all $\varphi \in Hom_S(M,L)$ and hence $N \star L$ is too.

Definition 1. A subsemimodule N of an S-semimodule M is called an idempotent subsemimodule if $N = N \star N = \sum\{\varphi(N) \mid \varphi \in Hom_S(M,N)\}$.

An S-semimodule M is called fully idempotent if every subsemimodule of M is idempotent.

Definition 2. [1] Let M and N be two S-semimodules. Then N is called M-generated if there exists a surjective S-homomorphism $\varphi : M^{(I)} \to N$ for some set I. Note that N is M-generated if and only if for each $x \in N$, there exist a positive integer k, S-homomorphisms $\alpha_i : M \to N$ $(1 \leq i \leq k)$ and elements $x_i \in M$ $(1 \leq i \leq k)$ such that $x = \sum_{i=1}^{k} \alpha_i(x_i)$. An S-semimodule M is called a self-generator semimodule if every subsemimodule of M is M-generated.

From above definition, it is obvious that any idempotent subsemimodule of an S-semimodule M is M-generated.

Definition 3. A subsemimodule N of an S-semimodule M is called a fully invariant subsemimodule of M if for every $\varphi \in Hom_S(M,M)$, $\varphi(N) \subseteq N$.

Theorem 1. *Let M be an S-semimodule.*

1. *Suppose that N is a subsemimodule of M. If L is an idempotent subsemimodule of M such that $L \subseteq N$, then L is an idempotent subsemimodule of N.*

2. *If N is a direct summand of M and L an idempotent subsemimodule of N, then L is an idempotent subsemimodule of M.*

3. *Let M be a subtractive semimodule. Suppose that K is a fully invariant subsemimodule of M, and let L be an idempotent subsemimodule of M such that $K \subseteq L$. Then L/K is an idempotent subsemimodule of M/K.*

Proof. The proofs of (1) and (2) are similar to [9, Lemma 2.2]. Since the definition of M/K for semimodules is different from the one for modules, we only prove part (3). Let $x \in L$. Then there are a positive integer k, S-homomorphisms $\varphi_i : M \to L$ $(1 \leq i \leq k)$ and elements $x_i \in L$ $(1 \leq i \leq k)$ such that $x = \sum_{i=1}^{k} \varphi_i(x_i)$. For each i $(1 \leq i \leq k)$, we define an S-homomorphism $\overline{\varphi_i} : M/K \to L/K$ by $\overline{\varphi_i}(m/K) = \varphi_i(m)/K$. If $m/K = n/K$ for some $m, n \in M$, then $m+t = n+s$ for some $s, t \in K$. Thus $\varphi_i(m)+\varphi_i(t) = \varphi_i(n)+\varphi_i(s)$. Since K is a fully invariant subsemimodule of M, $\varphi_i(s), \varphi_i(t) \in K$. Hence $\varphi_i(m)/K = \varphi_i(n)/K$. Therefore $\overline{\varphi_i}$ is well-defined. Then $x/K = \sum_{i=1}^{k} \overline{\varphi_i}(x_i/K)$. Hence L/K is an idempotent subsemimodule of M/K. □

Remark 1. Suppose that S is a subtractive semiring. Let I be an idempotent ideal of S and J an ideal of S such that $I \subseteq J$. If J/I is an idempotent ideal of the semiring S/I, then J is an idempotent ideal of S. If $x \in J$, then $x/I = (\sum_{i=1}^{k} y_i z_i)/I$ where $y_i, z_i \in J$ for all i, $1 \leq i \leq k$. Thus $x+t = \sum_{i=1}^{k} y_i z_i + s$ for some $t, s \in I$. Since I is an idempotent ideal of S, $t = \sum_{i=1}^{n} a_i b_i$ and $s = \sum_{j=1}^{m} c_j d_j$ where $a_i, b_i, c_j, d_j \in I \subseteq J$ for all i, $1 \leq i \leq n$ and all j, $1 \leq j \leq m$. Since J^2 is a subtractive ideal of S, we conclude that $x \in J^2$.

Definition 4. Let M be an S-semimodule. An S-semimodule U is called M-projective, if for each surjective S-homomorphism $f : M \to N$ and each S-homomorphism $g : U \to N$, there exists an S-homomorphism $h : U \to M$ such that $fh = g$.

Theorem 2. *Suppose that M is a subtractive semimodule. Let K be an idempotent subsemimodule of M and let N be a subsemimodule of M such that $K \subseteq N$ and N/K is an idempotent subsemimodule of M/K. If M is N-projective, then N is an idempotent subsemimodule of M.*

Proof. Suppose that $x \in N$. Then there exist a positive integer k, S-homomorphisms $\varphi_i : M/K \to N/K$ $(1 \leq i \leq k)$ and elements $x_i \in N$ $(1 \leq i \leq k)$ such that $x/K = \sum_{i=1}^{k} \varphi_i(x_i/K)$. Let $1 \leq i \leq k$. Put $\psi_i = \varphi_i \pi$ where $\pi : M \to M/K$ is defined by $\pi(m) = m/K$. Then $\psi_i \in Hom_S(M, N/K)$. Since M is N-projective, there exists an S-homomorphism $\theta_i : M \to N$ such that $\psi_i = \pi' \theta_i$ where $\pi' : N \to N/K$ is defined by $\pi'(n) = n/K$.

Thus $x/K = \sum_{i=1}^{k} \theta_i(x_i)/K$. Then $x + a = \sum_{i=1}^{k} \theta_i(x_i) + b$ for some $a, b \in K$. Since K is an idempotent subsemimodule of M, there exist positive integers r_1, r_2, elements $s_i, t_j \in K$ $(1 \leq i \leq r_1, 1 \leq j \leq r_2)$ and S-homomorphisms $\alpha_i : M \to K$ $(1 \leq i \leq r_1)$, $\beta_j : M \to K$ $(1 \leq j \leq r_2)$ such that $a = \sum_{i=1}^{r_1} \alpha_i(s_i)$ and $b = \sum_{j=1}^{r_2} \beta_j(t_j)$. Then $x + \sum_{i=1}^{r_1} \alpha_i(s_i) = \sum_{i=1}^{k} \theta_i(x_i) + \sum_{j=1}^{r_2} \beta_j(t_j)$. Moreover $\sum_{i=1}^{r_1} \alpha_i(s_i)$,

$\sum_{i=1}^{k} \theta_i(x_i)$, $\sum_{j=1}^{r_2} \beta_j(t_j) \in N \star N$. Since $N \star N$ is a subtractive subsemimodule of M, we conclude that $x \in N \star N$. Therefore $N \star N = N$ and hence N is an idempotent subsemimodule of M. □

The above theorem is an extension of Proposition 2.3 in [9] to subtractive semimodules.

Lemma 1. *Let N be a subsemimodule of an S-semimodule M. Suppose that $N = \sum_{(i \in I)} Sx_i$ for some index set I. Then N is an idempotent subsemimodule of M if and only if for each $i \in I$ there exist a positive integer $k(i)$, S-homomorphisms $\varphi_{it} : M \to N$ $(1 \le t \le k(i))$ and elements $x_{it} \in N$ $(1 \le t \le k(i))$ such that $x_i = \sum_{t=1}^{k(i)} \varphi_{it}(x_{it})$.*

Proof. The proof is similar to [9, Lemma 2.4]. □

If $\{N_i | i \in I\}$ is a collection of idempotent subsemimodules of an S-semimodule M, then by Lemma 1, $N = \sum_{i \in I} N_i$ is an idempotent subsemimodule of M.

Let $\{M_i | i \in I\}$ be a family of S-semimodules. Then $\oplus_{i \in I} M_i = \oplus_{i \in I} \lambda_i(M_i)$ such that $\lambda_i : M_i \to \oplus_{i \in I} M_i$ is defined by $\lambda_i(m) = (n_j)$ where $n_j = 0$ if $j \neq i$ and $n_j = m$ if $j = i$ (see [12, Theorem 3.4]).

Now we obtain the following result similar to [9, Corollary 2.6].

Theorem 3. *Let $\{M_i | i \in I\}$ be a family of S-semimodules and $M = \oplus_{i \in I} M_i$. If for each $i \in I$, N_i is an idempotent subsemimodule of the S-semimodule M_i, then $\oplus_{i \in I} N_i$ is an idempotent subsemimodule of M.*

Proof. It follows from Theorem 1 (2). □

Theorem 4. *Let L be an idempotent subsemimodule of an S-semimodule M and N an L-generated subsemimodule of M. Then $N + L$ is an idempotent subsemimodule of M.*

Proof. It is similar to the proof of [9, Theorem 2.7]. □

Now we extend [9, Corollary 2.8] to semimodules as follows:

Corollary 1. *Let S be a semiring and M an S-semimodule. Let m be an element of M such that Sm is an idempotent subsemimodule of M and $Sm \cong S$. Then every subsemimodule N of M such that $m \in N$ is an idempotent subsemimodule of M.*

Proof. By [7, Proposition 17.11], there exist a set I and a surjective S-homomorphism $\varphi : S^{(I)} \to N$. Since $S \cong Sm$, N is Sm-generated. By Theorem 4, N is an idempotent subsemimodule of M. □

Theorem 5. *Let S be a Boolean algebra and M an S-semimodule. Suppose that $m \in M$. If $ann(m) = 0$, then $Sm \cong S$.*

Proof. Let $\varphi : S \to Sm$ be a map defined by $\varphi(s) = sm$. Then φ is a surjective S-homomorphism. Suppose that $\varphi(s) = \varphi(t)$ for some $s, t \in S$. Then $sm = tm$. Since S is a Boolean algebra, there exist $s', t' \in S$ such that $s + s' = t + t' = 1$ and $ss' = tt' = 0$. Then $s'tm = s'sm = 0$. Therefore $s't \in ann(m)$ and hence $s't = 0$. Moreover $ts + ts' = t$. It follows that $st = t$. A similar argument shows that $ts = s$. Therefore $s = t$ and hence φ is injective. □

Corollary 2. *Let S be a Boolean algebra and M an S-semimodule. Let m be an element of M such that Sm is an idempotent subsemimodule of M and $ann(m) = 0$. Then every subsemimodule N of M such that $m \in N$ is an idempotent subsemimodule of M.*

With a similar proof for [11, Theorem 2.8], the following theorem can be obtained.

Theorem 6. *Let M be an S-semimodule and N an idempotent subsemimodule of M. Suppose that I is an idempotent ideal of S. Then IN is an idempotent subsemimodule of M.*

Corollary 3. *Let M be an S-semimodule and I an idempotent ideal of S. Then IM is an idempotent subsemimodule of M.*

Lemma 2. *Suppose that M is a fully idempotent semimodule. Let N be a subsemimodule of M and I an ideal of S. Then $N \cap IM = IN$.*

Proof. It is similar to the proof of [11, Lemma 2.13]. □

Definition 5. Let S be a semiring and M an S-semimodule. A subsemimodule N is called pure in M if $N \cap IM = IN$ for all ideals I of S.

Definition 6. Let M be an S-semimodule and N a subsemimodule of M. We say that N has property (A), if there exists an S-homomorphism $\phi : M \to N$ such that $\phi\imath = id_N$ where $\imath : N \to M$ is the inclusion map. Note that every direct summand of M and every injective subsemimodule of M has property (A).

In the following example we give a subsemimodule of a semimodule which has property (A), but is not a direct summand.

Example 1. Let $\mathbf{B}_3 = \{0, 1, 2\}$. Define the addition and multiplication on $\mathbf{B}_3$ by $a + b = max\{a, b\}$ and $a \cdot b = 0$ if $a = 0$ or $b = 0$ and $a \cdot b = max\{a, b\}$ otherwise [2, Example 3.7]. Then $(\mathbf{B}_3, +, \cdot)$ is a semiring. By [2, Fact 4.11], $\{0, 2\}$ is an injective $\mathbf{B}_3$-semimodule and hence it has property (A). But $\{0, 2\}$ is not a direct summand of $\mathbf{B}_3$.

Definition 7. Let M be an S-semimodule. We say that M has property (B), if every cyclic subsemimodule of M has property (A).

It is clear that if every cyclic subsemimodule of an S-semimodule M is a direct summand of M, then M has property (B). Moreover if S is a semiring such that all cyclic S-semimodules are injective, which is called CI-semiring [2], then every S-semimodule has property (B).

In [14], an S-semimodule M is called regular if for each $m \in M$, there exists an S-homomorphism $g \in Hom_S(M, S)$ such that $m = g(m)m$. If S is a semiring such that it is a regular S-semimodule, then S is a multiplicatively regular semiring [14, Definition 3.3.1].

It is known that an R-module M is a regular module if and only if every cyclic submodule of M is a direct summand of M and projective [16]. Similar to [16, Theorem 2.2], we have the following result.

Theorem 7. *Let M be an S-semimodule. Then M is a regular S-semimodule if and only if M has property (B) and every cyclic subsemimodule of M is projective.*

Proof. $(\rightarrow)$: Let $m \in M$. Then there exists an S-homomorphism $g \in Hom_S(M, S)$ such that $m = g(m)m$. Define a map $f : M \to Sm$ by $f(n) = g(n)m$. Then f is an S-homomorphism and $f(m) = g(m)m = m$. Hence $f\imath = id_{Sm}$ where $\imath : Sm \to M$ is the inclusion map. Therefore Sm has property (A). Moreover by [14, Proposition 3.3.4] every cyclic subsemimodule of M is projective.

$(\leftarrow)$: Now assume that M has property (B) and $m \in M$. Then since Sm is projective, by [14, Theorem 3.4.12], there exist a positive integer n and $\{m_i\}_{1 \le i \le n} \subseteq Sm$ and $\{f_i\}_{1 \le i \le n} \subseteq Hom_S(Sm, S)$ such that $m = \sum_{i=1}^{n} f_i(m)m_i$. For each i $(1 \le i \le n)$, there exists $s_i \in S$ such that $m_i = s_i m$. Then $m = \sum_{i=1}^{n} f_i(m)s_i m$. Let $g = \sum_{i=1}^{n} f_i s_i$. Since M has property (B), there exists an S-homomorphism $\phi : M \to Sm$ such that $\phi(m) = m$. Put $h = g\phi$. Then $h \in Hom_S(M, S)$ and $m = h(m)m$. Therefore M is a regular S-semimodule. $\square$

Now we can restate [14, Corollary 3.3.5] as follows:

Corollary 4. *A semiring S is a multiplicatively regular semiring if and only if S has property (B) and every cyclic subsemimodule of S is projective.*

Theorem 8. *Let N be a subsemimodule of M with property (A) and let L be an idempotent subsemimodule of N. Then L is an idempotent subsemimodule of M.*

Proof. Let $x \in L$. Then $x = \sum_{i=1}^{k} \alpha_i(x_i)$ where k is a positive integer, $\alpha_i \in Hom_S(N, L)$ and $x_i \in L$ for all i, $1 \le i \le k$. There exists an S-homomorphism $\phi : M \to N$ such that $\phi\imath = id_N$ since N has property (A). Let $\beta_i = \alpha_i\phi$. Then $x = \sum_{i=1}^{k} \beta_i(x_i)$ and hence L is an idempotent subsemimodule of M. $\square$

Corollary 5. *Let N be a subsemimodule of M with property (A). Then N is an idempotent subsemimodule of M.*

Now with the use of above definition we extend some more results of [9] to semimodules.

Theorem 9. *Let F be a nonzero free S-semimodule with basis $\{f_i\}_{(i\in I)}$ for some index set I. Let L be a subsemimodule of F such that $F = L + \sum_{(j\in J)} Sf_j$ for some proper subset J of I. Then L is an idempotent subsemimodule of F.*

Proof. Let $i \in I \backslash J$. Then there exist a finite subset J' of J and elements $s_j \in S$ ($j \in J'$) and $x \in L$ such that $f_i = \sum_{(j\in J')} s_j f_j + x$. Let $y \in F$. Then there exist a finite subset K of I and elements $u_t \in S$ ($t \in K$) such that $y = \sum_{(t\in K)} u_t f_t$. If $i \in K$, then $y = \sum_{t\neq i} u_t f_t + u_i f_i$. Thus $y = \sum_{t\neq i} u_t f_t + u_i \sum_{(j\in J')} s_j f_j + u_i x$. If $i \notin K$, we put $u_i = 0$. We can define a map $\varphi : F \to Sx$ by $\varphi(y) = u_i x$. Then φ is a well-defined S-homomorphism. Let $x = \sum_{(t\in I')} u_t f_t$ for some finite subset I' of I. If $i \notin I'$, then $f_i = \sum_{(j\in J')} s_j f_j + \sum_{(t\in I')} u_t f_t$ which is a contradiction. Thus $i \in I'$. Therefore $f_i = \sum_{(j\in J')} s_j f_j + \sum_{(t\in I', t\neq i)} u_t f_t + u_i f_i$. We conclude that $u_i = 1$. Hence $\varphi(x) = x$. Therefore Sx has property (A) and hence it is an idempotent subsemimodule of F.

Now, we prove that $Sx \cong S$. Define a map $\alpha : S \to Sx$ by $\alpha(s) = sx$. Then α is a surjective S-homomorphism. Let $sx = s'x$ for some $s, s' \in S$. Since $x = \sum_{(t\in I', t\neq i)} s_t f_t + f_i$, we conclude that $s = s'$. Therefore α is injective and $S \cong Sx$. Hence by Corollary 1, L is an idempotent subsemimodule of F. □

In [9, Theorem 3.1] it is shown that if R is a commutative ring, then every cyclic idempotent submodule of an R-module M, is a direct summand of M. For semimodules we have the following result.

Theorem 10. *Let N be a cyclic idempotent subsemimodule of an S-semimodule M. Then N has property (A).*

Proof. Suppose that $N = Sx$. Then $x = \sum_{i=1}^{k} f_i(s_i x)$ where k is a positive integer, $f_i \in Hom_S(M, N)$ and $s_i \in S$ for all i, $1 \leq i \leq k$. Let $g = \sum_{i=1}^{k} s_i f_i$. Then $g(x) = x$ and hence $g\imath = id_N$ where $\imath : N \to M$ is the inclusion map. Therefore N has property (A). □

Similar to [9, Lemma 2.15], we have the following result.

Lemma 3. *Let M be an S-semimodule. Then the following statements are equivalent:*

1. *M is fully idempotent.*
2. *Sm is an idempotent subsemimodule of M for every $m \in M$.*

Let R be a ring. In [9], an R-module M is called regular if every cyclic submodule of M is a direct summand and it is shown that for a commutative ring R, an R-module M is fully idempotent if and only if it is regular [9, Proposition 3.4]. Similarly we have the following result for semimodules.

Theorem 11. *Let S be a semiring. Then an S-semimodule M is fully idempotent if and only if it has property (B).*

Proof. Suppose that M is fully idempotent and N a cyclic subsemimodule of M. Then by Theorem 10, N has property (A). Now assume that M has property (B). Then by Corollary 5, every cyclic subsemimodule of M is an idempotent subsemimodule of M. Thus by Lemma 3, M is fully idempotent. □

Assume that R is a ring. Then every R-module can be embedded in an injective module. But this is not true for arbitrary semirings (see [7, Proposition 17.21]).

An S-monomorphism $\alpha : M \to N$ of S-semimodules is essential if and only if, for any S-homomorphism $\beta : N \to N'$, the map $\beta\alpha$ is an S-monomorphism only when β is an S-monomorphism. Suppose that M is an S-semimodule. Let E be an injective S-semimodule such that $M \subseteq E$ and the inclusion map $\imath : M \to E$ is an essential S-homomorphism. Then E is called an injective envelope of M. For more details on injective envelopes of semimodules see [7] and [8].

Theorem 12. *Let S be a semiring such that every S-semimodule has an injective envelope. Then every S-semimodule is fully idempotent if and only if every injective S-semimodule is fully idempotent.*

Proof. $(\to)$: Obvious.

$(\leftarrow)$: Let M be an S-semimodule and N a subsemimodule of M. Suppose that $E(M)$ is the injective envelope of M. Then N is an idempotent subsemimodule of $E(M)$ and hence by Theorem 1(1), N is an idempotent subsemimodule of M. □

Similar to [9, Theorem 3.7], we have the following result.

Theorem 13. *Let S be a semiring such that every nonzero S-semimodule has an injective envelope. Then every S-semimodule is fully idempotent if and only if S is a CI-semiring.*

Proof. $(\to)$: Suppose that every S-semimodule is fully idempotent. Let Sm be a cyclic S-semimodule and $E = E(Sm)$ an injective envelope of Sm. By Theorem 10, Sm has property (A). Hence Sm is an injective S-semimodule since it is a retract of E.

$(\leftarrow)$: Suppose that every cyclic S-semimodule is injective. Let M be an S-semimodule and $m \in M$. Since Sm is an injective S-semimodule, it has property (A). Thus by Theorem 11, M is fully idempotent. □

Let S be semiring and F a free S-semimodule with a basis $B = \{f_i\}_{(i\in I)}$ for some index set I. Suppose that $x \in F$. Then there are a finite subset K of I and elements $s_t \in S$ $(t \in K)$ such that $x = \sum_{(t\in K)} s_t f_t$. Let $\beta_B(x)$ be the ideal generated by coefficients of x. Let $B' = \{e_j\}_{(j\in J)}$ be another basis of F. For each $t \in K$, there exist a finite subset J_t of J and elements $s_{tj} \in S(j \in J_t)$ such that $f_t = \sum_{(j\in J_t)} s_{tj} e_j$. Hence $x = \sum_{(t\in K)} \sum_{(j\in J_t)} s_t s_{tj} e_j$. Therefore $\beta_{B'}(x) \subseteq \beta_B(x)$. A similar argument shows that $\beta_B(x) \subseteq \beta_{B'}(x)$. Hence $\beta_B(x)$ is independent of the basis B. Let N be a subsemimodule of the free S-semimodule F. Then we define $\beta(N) = \sum_{x\in N} \beta(x)$. Clearly $\beta(N)$ is an ideal of S (cf. [9, §4]).

The following theorem characterizes the idempotent subsemimodules of a free semimodule.

Theorem 14. *Let N be a subsemimodule of a free S-semimodule F. Then $\beta(N)N = Hom_S(F, N)N$.*

Proof. Let $\varphi : F \to N$ be an S-homomorphism and $B = \{f_i\}_{(i \in I)}$ a basis of F. Suppose that $x \in N$. Then there exist a finite subset J of I and elements $s_i \in S$ $(i \in J)$ such that $x = \sum_{(i \in J)} s_i f_i$. Thus $\varphi(x) = \sum_{(i \in J)} s_i \varphi(f_i) \in \beta(N)N$.

Now let $x, y \in N$ and $s \in S$ such that $s \in \beta(x)$. Suppose that $x = \sum_{i=1}^{n} b_i f_i$. Thus there exist $a_i \in S$ $(1 \leq i \leq n)$ such that $s = \sum_{i=1}^{n} a_i b_i$. For each i $(1 \leq i \leq n)$, define a map $\alpha_i : F \to N$ by $\alpha_i(f_i) = y$ and $\alpha_i(f_j) = 0$ for all $j \neq i$. Then α_i is a well defined S-homomorphism and $\alpha_i(x) = b_i y$. Let $\alpha_x = \sum_{i=1}^{n} a_i \alpha_i$. Then $\alpha_x : F \to N$ is an S-homomorphism and $\alpha_x(x) = \sum_{i=1}^{n} a_i \alpha_i(x) = \sum_{i=1}^{n} a_i b_i y = sy$. Now let $r \in \beta(N)$. Then $r = \sum_{i=1}^{m} s_i$ where for each i $(1 \leq i \leq m)$ there exists $x_i \in N$ such that $s_i \in \beta(x_i)$. Then $ry = \sum_{i=1}^{m} s_i y = \sum_{i=1}^{m} \alpha_{x_i}(x_i)$. Hence $ry \in Hom_S(F, N)N$. □

As immediate consequences of Theorem 14 we have the following corollaries.

Corollary 6. *A subsemimodule N of a free S-semimodule F is an idempotent subsemimodule of F if and only if $N = \beta(N)N$.*

Corollary 7. *Let N be a subsemimodule of a free S-semimodule F such that $\beta(N) = S$. Then N is an idempotent subsemimodule of F.*

A semiring S is called ideal-simple if it has no nontrivial ideals [6].

Corollary 8. *Let S be an ideal-simple semiring. Then every free S-semimodule is fully idempotent.*

3 Idempotent subsemimodules of multiplication semimodules

In this section we investigate idempotent subsemimoules of multiplication semimodules and generalize some results of [11] to semimodules.

If N and L are subsemimodules of an S-semimodule M, we set $(N : L) = \{s \in S \mid sL \subseteq N\}$. Note that $(N : L)$ is an ideal of S.

Definition 8. [5] Let S be a semiring and M an S-semimodule. Then M is called a multiplication S-semimodule if for each subsemimodule N of M there exists an ideal I of S such that $N = IM$. In this case it is easy to see that $N = (N : M)M$.

From Corollary 3, we conclude that every multiplication semimodule over a Boolean algebra is a fully idempotent semimodule.

Lemma 4. *Assume that M is a subtractive multiplication semimodule and $K \subseteq N \subseteq M$. If N is an idempotent subsemimodule of M, then N/K is an idempotent subsemimodule of M/K.*

Proof. Similar to the proof of [15, Lemma 3] we can show that K is a fully invariant subsemimodule of M. Therefore, since M is a multiplication semimodule, there is an ideal I of S such that $K = IM$. If $f \in End(M)$, then $f(K) = f(IM) = If(M) \subseteq IM = K$. Thus by Theorem 1(3), N/K is idempotent. □

Now with the use of above theorem, we can show that for a fully idempotent subtractive multiplication semimodule M and a subsemimodule N of M, M/N will be fully idempotent too.

A semiring S is yoked if for all $a, b \in S$, there exists an element t of S such that $a + t = b$ or $b + t = a$.

An element m of an S-semimodule M is cancellable if $m + m' = m + m''$ implies that $m' = m''$. An S-semimodule M is cancellative if every element of M is cancellable.

Let M be an S-semimodule and $\mathfrak{p}$ a maximal ideal of S. We say that M is $\mathfrak{p}$-cyclic if there exist $m \in M$, $t \in S$ and $q \in \mathfrak{p}$ such that $t + q = 1$ and $tM \subseteq Sm$.

Theorem 15. *Let S be a yoked semiring such that every maximal ideal of S is subtractive and let M be a finitely generated faithful multiplication cancellative S-semimodule. Then a subsemimodule N of M is an idempotent subsemimodule of M if and only if $I = (N : M)$ is an idempotent ideal of S.*

Proof. ($\rightarrow$): Let N be an idempotent subsemimodule of M. Assume that $I^2 \neq I$ and $s \in I \backslash I^2$. Put $J = \{t \in S \mid ts \in I^2\}$. Then there exists a maximal ideal $\mathfrak{p}$ such that $J \subseteq \mathfrak{p}$. By [13, Theorem 25], $M \neq M\mathfrak{p}$ and hence M is $\mathfrak{p}$-cyclic from [13, Theorem 6]. Therefore there exist $m \in M$, $t \in S$ and $q \in \mathfrak{p}$ such that $t+q = 1$ and $tM \subseteq Sm$. We prove that $t^2N \subseteq Im$. Let $x \in N$. Then $tx = um$ for some $u \in S$. Then $tuM \subseteq (u)m = (t)x \subseteq N$. Thus $tu \in I$ and hence $t^2x = tum \in Im$. Moreover, since $s \in I$, $sm \in N$. Thus $sm = \sum_{i=1}^{k} \varphi_i(x_i)$, where k is a positive integer, $\varphi_i \in Hom_S(M, N)$ and $x_i \in N$ for all i, $1 \leq i \leq k$. Now let $\varphi : M \rightarrow N$ and $x \in N$. Then $t^2x \in Im$ and hence $t^2\varphi(x) \in I\varphi(m)$. Since $\varphi(m) \in N$, $t^2\varphi(m) \in Im$. Thus $t^4\varphi(x) \in t^2I\varphi(m) \subseteq I^2m$. Now define an S-homomorphism $\alpha : M \rightarrow N$ by $\alpha(m) = sm$. Thus $t^4sm = mb$ for some $b \in I^2$. Since S is yoked, there exists $c \in S$ such that $t^4s+c = b$ or $b+c = t^4s$. Suppose that $t^4s+c = b$. Then $t^4sm+cm = bm$. Thus $cm = 0$ and hence $tcM \subseteq cm = 0$. Since M is a faithful S-semimodule, $tc = 0$. Then $t^5s = bt \in I^2$. Thus $t^5 \in J \subseteq \mathfrak{p}$ which is a contradiction. Therefore $J = S$ and hence $I^2 = I$. Now suppose that $b+c = t^4s$. A similar argument shows that $I^2 = I$.

($\leftarrow$): It follows from Corollary 3. □

The above theorem is a generalization of [9, Theorem 3.9].

Lemma 5. *Let S be a semiring and M a multiplication S-semimodule. Then M is a self-generator S-semimodule.*

Proof. Let N be a subsemimodule of M and $x \in N$. Then there exists an ideal I of S such that $N = IM$. Thus $x = \sum_{i=1}^{n} a_ix_i$ where $a_i \in I$ and $x_i \in M$ for all i

$(1 \leq i \leq n)$. For each i, we define a map $\theta_i : M \to IM = N$ by $\theta_i(m) = a_i m$. Then θ_i is an S-homomorphism and $x = \sum_{i=1}^{n} \theta_i(x_i)$. Therefore N is M-generated. $\square$

In the next result we state [10, Corollary 3.2] for semimodules.

Theorem 16. *Suppose that M is a multiplication semimodule. Let N and L be subsemimodules of M. Then $N \star L = IJM$ where I and J are ideals of S such that $N = IM$ and $L = JM$.*

Proof. Since L is M-generated, $\sum\{\varphi(M) \mid \varphi \in Hom_S(M, L)\} = L$. Thus $N \star L = \sum\{\varphi(N) \mid \varphi \in Hom_S(M, L)\} = \sum\{\varphi(IM) \mid \varphi \in Hom_S(M, L)\} = \sum\{I\varphi(M) \mid \varphi \in Hom_S(M, L)\} = IL = IJM$. $\square$

Now with the use of Theorem 16, we generalize [11, Theorem 2.6] to multiplication semimodules.

Theorem 17. *Suppose that M is a multiplication S-semimodule. Let K be a multiplication idempotent subsemimodule of M and N an idempotent subsemimodule of K. Then N is an idempotent subsemimodule of M.*

Proof. Since M is a multiplication semimodule, $K = (K : M)M$. Then $K = K \star K = (K : M)M \star (K : M)M = (K : M)^2 M = (K : M)K$ by Theorem 16. Since K is a multiplication semimodule, $N = (N : K)K$. Then $N = N \star N = (N : K)K \star (N : K)K = (N : K)^2 K = N(N : K)$. It follows that $N = (N : K)K = (N : K)(K : M)K \subseteq ((N : K)K : M)K = (N : M)K \subseteq (N : K)K$. Therefore $N = (N : M)K$. Moreover by Theorem 16, $N \star N = (N : M)M \star (N : M)M = (N : M)^2 M = (N : M)N$. Since K is a multiplication semimodule, $(N : M)N \cap K = (N : M)N = ((N : M)N : K)K \supseteq (N : K)(N : M)K = (N : K)N = N$. $\square$

Theorem 18. *Let M be a multiplication semimodule and let M_1 and M_2 be fully idempotent subsemimodules of M such that $M = M_1 \oplus M_2$. Then M is also fully idempotent.*

Proof. Since M is a multiplication semimodule, every subsemimodule of M is fully invariant. Let $\pi_i : M \to M_i$ be the canonical projection of M onto M_i for $i = 1, 2$. Suppose that N is a subsemimodule of M. Then $\pi_i(N) \subseteq N \cap M_i$ for $i = 1, 2$. Let $x \in N$. Then there exist $x_1 \in M_1$ and $x_2 \in M_2$ such that $x = x_1 + x_2$. Hence $\pi_1(x) = x_1 \in N \cap M_1$ and $\pi_2(x) = x_2 \in N \cap M_2$. Thus $x \in N \cap M_1 + N \cap M_2$. Therefore $N = N \cap M_1 + N \cap M_2$. Since the representation $x = x_1 + x_2$ is unique, $N = N \cap M_1 \oplus N \cap M_2$. Since $N \cap M_1$ and $N \cap M_2$ are idempotent subsemimodules of M_1 and M_2 respectively, N is the idempotent subsemimodule of M by Theorem 3. $\square$

In the next theorem, we show that every pure subsemimodule of a multiplication semimodule is idempotent similar to [11, Theorem 3.1].

Theorem 19. *Let M be a multiplication S-semimodule. If N is a pure subsemimodule of M, then N is a multiplication idempotent subsemimodule of M.*

Proof. Let K be a subsemimodule of M. We prove that $K \cap N = (K : N)N$. Since M is a multiplication S-semimodule, $K = (K : M)M$. Since N is a pure subsemimodule, $(K : N)N = N \cap (K : N)M$. Then $(K : N)N = N \cap (K : N)M \supseteq N \cap (K : M)M = N \cap K \supseteq (K : N)N$. Hence $K \cap N = (K : N)N$. Therefore N is a multiplication subsemimodule of M. Now since N is a pure subsemimodule of M, $(N : M)N = N \cap (N : M)M = N$. Moreover by Theorem 16, $N \star N = (N : M)M \star (N : M)M = (N : M)^2M = (N : M)N$. Therefore $N \star N = N$ and hence N is an idempotent subsemimodule of M. □

In the following theorem we prove the converse of Theorem 19 for some class of semirings and semimodules.

Theorem 20. *Let S be a yoked semiring such that every maximal ideal of S is subtractive and let M be a finitely generated faithful multiplication cancellative S-semimodule. If N is a multiplication idempotent subsemimodule of M, then N is a pure subsemimodule of M.*

Proof. Since N is an idempotent subsemimodule of M, $(N : M)$ is an idempotent ideal by Theorem 15. Then $N = (N : M)M = (N : M)^2M = (N : M)(N : M)M = (N : M)N$. Suppose that K is a subsemimodule of M. Then $(K : N)N = (K : N)(N : M)N \subseteq (K : M)N \subseteq (K : N)N$. Therefore $(K : N)N = (K : M)N$. Now let I be an ideal of S. Since N is a multiplication S-semimodule, $N \cap IM = (IM : N)N$. Hence $N \cap IM = (IM : N)N = (IM : M)N = IN$. Note that $IM = (IM : M)M$ and hence $I = (IM : M)$ by [13, Theorem 25]. □

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Received May 26, 2018