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# Finite 2-groups with a non-Dedekind non-metacyclic norm of Abelian non-cyclic subgroups

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**Abstract.** The authors study finite 2-groups with non-Dedekind non-metacyclic norm $N_G^A$ of Abelian non-cyclic subgroups depending on the cyclicness or the non-cyclicness of the center of a group $G$. The norm $N_G^A$ is defined as the intersection of the normalizers of Abelian non-cyclic subgroups of $G$. It is found out that such 2-groups are cyclic extensions of their norms of Abelian non-cyclic subgroups. Their structure is described.

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## 1 Introduction

One of the main directions in group theory is the study of the impact of characteristic subgroups on the structure of the whole group. Such characteristic subgroups include different $\Sigma$-norms of a group. A $\Sigma$-norm is the intersection of the normalizers of all subgroups of a system $\Sigma$ (assuming that the system $\Sigma$ is non-empty). It is clear that when the $\Sigma$-norm coincides with a group, then all subgroups of the system $\Sigma$ are normal in the last one.

For the first time, R. Baer [1] considered the $\Sigma$-norm as a proper subgroup of a group in 1935 for the system of all subgroups of this group. He called it the norm of a group and denoted by $N(G)$. Narrowing the system of subgroups one can get different $\Sigma$-norms which can be considered as generalizations of the norm $N(G)$. Recently the interest in studying the $\Sigma$-norms does not decrease as evidenced by the series of works [2–4, 9, 11].

If $\Sigma$ is the system of all Abelian non-cyclic subgroups, then such a $\Sigma$-norm will be called the norm of Abelian non-cyclic subgroups and denoted by $N_G^A$. Thus *the norm* $N_G^A$ *of Abelian non-cyclic subgroups of a group* $G$ is the intersection of the normalizers of all Abelian non-cyclic subgroups of a group $G$, assuming that the system of such subgroups is non-empty.

Here we improve and extend some earlier results [8].

## 2 Preliminary Results

In a group $G$ which coincides with the norm $N_G^A$ all Abelian non-cyclic subgroups (assuming the existence of at least one such a subgroup) are normal. Non-Abelian groups with this property were called $\overline{HA}$-groups ($\overline{HA}_2$-groups in the case of 2-groups) [7].

**Proposition 1.** *[7] A non-Hamiltonian $\overline{HA}_2$-group does not contain an elementary Abelian subgroup of order 8.*

**Proposition 2.** *[7] Finite non-Hamiltonian $\overline{HA_2}$-groups are groups of the following types:*

*1) $G = (\langle a\rangle \times \langle b\rangle) \rtimes \langle c\rangle$, where $|a| = 2^n$, $n > 1$, $|b| = |c| = 2$, $[a, b] = [a, c] = 1$, $[b, c] = a^{2^{n-1}}$;*

*2) $G = \langle a\rangle \rtimes \langle b\rangle$, where $|a| = 2^n$, $|b| = 2^m$, $n \geq 2$, $m \geq 1$, $[a, b] = a^{2^{n-1}}$;*

*3) $G = (H \times \langle b\rangle) \rtimes \langle c\rangle$, where $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4$, $h_1^2 = h_2^2$, $|b| = |c| = 2$, $[h_1, h_2] = h_1^2$, $[H, \langle b\rangle] = [H, \langle c\rangle] = E$, $[b, c] = h_1^2$;*

*4) $G = (\langle a\rangle \times \langle b\rangle)\langle c\rangle$, where $|a| = |b| = |c| = 4$, $c^2 = a^2b^2$, $[c, b] = c^2$, $[c, a] = a^2$;*

*5) $G = (\langle a\rangle \times \langle b\rangle)\langle c\rangle\langle d\rangle$, where $|a| = |b| = |c| = |d| = 4$, $c^2 = d^2 = a^2b^2$, $[a, c] = [d, c] = a^2$, $[b, d] = b^2$, $[c, b] = [d, a] = c^2$;*

*6) $G = H \times \langle c\rangle$, where $H$ is the quaternion group, $|c| = 2^n$, $n \geq 2$;*

*7) $G = H \times Q$, where $H$ and $Q$ are the quaternion groups;*

*8) $G = (H \times \langle b\rangle)\langle c\rangle$, where $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = |b| = |c| = 4$, $[h_1, h_2] = h_1^2 = h_2^2$, $[H, \langle b\rangle] = [H, \langle c\rangle] = E$, $c^2 = b^2h_1^2$, $[b, c] = b^2$;*

*9) $G = (\langle h_2\rangle \times \langle c\rangle)\langle h_1\rangle$, where $|h_1| = |h_2| = 4$, $[h_1, h_2] = h_1^2 = h_2^2$, $|c| = 2^n > 2$, $[c, h_1] = c^{2^{n-1}}$;*

*10) $G = (H \times \langle b\rangle)\langle c\rangle$, where $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4$, $|b| = 2$, $|c| = 8$, $[b, c] = [h_1, h_2] = h_1^2 = h_2^2$, $c^2 = h_1$, $[h_2, c] = b$;*

*11) $G = \langle a\rangle\langle b\rangle$, where $|a| = 8$, $|b| = 2^n > 2$, $a^4 = b^{2^{n-1}}$, $a^{-1}ba = b^{-1}$.*

It is clear that the subgroup $N_G^A$ is characteristic and contains the center $Z(G)$ of the group $G$.

To reduce the presentation, a finite 2-group with non-Dedekind non-metacyclic norm $N_G^A$ of Abelian non-cyclic subgroups will be called *a group of type $\alpha$* if the center $Z(G)$ of the group $G$ is non-cyclic, and *a group of type $\beta$* if the center $Z(G)$ of the group $G$ is cyclic.

The following corollary immediately follows from Proposition 2.

**Corollary 1.** *If $G$ is a group of type $\alpha$ and $G = N_G^A$, then $N_G^A$ is a group of one of the types (4)-(9) of Proposition 2. If $G$ is a group of type $\beta$ and $G = N_G^A$, then $N_G^A$ is a group of one of the types (1), (3), (10) of Proposition 2.*

It turns out that there exist groups such that the center $Z(N_G^A)$ of the norm $N_G^A$ of the group $G$ is non-cyclic but the center $Z(G)$ of the group $G$ is cyclic. The following example shows it.

**Example 1.** $G = (\langle b \rangle \leftthreetimes H)\langle y \rangle$, where $|b| = 4$, $H = \langle h_1, h_2 \rangle$, $|h_1| = 4$, $[h_1, h_2] = h_1^2 = h_2^2$, $[b, h_2] = 1$, $y^2 = h_1$, $[y, h_2] = b^2 h_1^2$, $[y, b] = h_2$.

In this group all Abelian non-cyclic subgroups are contained in the group $\langle b \rangle \leftthreetimes H$ and are normal in it. So it is easy to verify that $N_G^A = \langle b \rangle \leftthreetimes H$ and $Z(N_G^A) = \langle b^2 \rangle \times \langle h_1^2 \rangle$ is non-cyclic. At the same time $Z(G) = \langle h_1^2 \rangle$ is cyclic.

**Lemma 1.** *If $Z$ is a central non-cyclic subgroup of a group $G$, then $\overline{N_G^A} \subseteq N(\overline{G})$ in the quotient-group $G/Z = \overline{G}$, where $N(\overline{G})$ is the norm of the group $\overline{G}$.*

*Proof.* It suffices to show that the group $\overline{N_G^A}$ normalizes every cyclic subgroup of the group $\overline{G} = G/Z$.

Let $\overline{x} \in \overline{G}$. Then the full preimage of the subgroup $\langle \overline{x} \rangle$ in the group $G$ is the Abelian non-cyclic subgroup $\langle x, Z \rangle$. Therefore, $N_G^A \subseteq N_G(\langle x, Z \rangle)$. In the quotient-group $\overline{G}$

$$[\overline{N_G^A} \subseteq \overline{N_G(\langle x, Z \rangle)} \subseteq N_{\overline{G}}(\langle \overline{x} \rangle)],$$

thus $\overline{N_G^A} \subseteq N(\overline{G})$. □

Let's denote the lower layer of a group $G$ by $\omega(G)$. It is the subgroup generated by all elements of prime order of the group $G$.

**Lemma 2.** *If the norm $N_G^A$ of Abelian non-cyclic subgroups of a finite 2-group $G$ is non-Dedekind non-metacyclic and its lower layer $\omega(N_G^A)$ is an elementary Abelian subgroup of order 4, then $N_G^A$ contains all involutions of the group $G$ and $\omega(N_G^A) = \omega(G)$.*

*Proof.* Let a group $G$ and its norm $N_G^A$ of Abelian non-cyclic subgroups satisfy the conditions of the lemma. Then $N_G^A$ is a group of one of types (4)-(10) of Proposition 2. Since $\omega(N_G^A) \lhd N_G^A$ and the subgroup $\omega(N_G^A)$ is characteristic in $N_G^A$, $\omega(N_G^A) \lhd G$. Therefore $\omega(N_G^A) \bigcap Z(G) \neq E$.

Let $\omega(N_G^A) = \langle a_1 \rangle \times \langle a_2 \rangle$, $|a_1| = |a_2| = 2$, $a_1 \in Z(G)$ for the definiteness. Suppose that $G$ contains an involution $x \notin N_G^A$. Then the subgroup $\langle a_1, x \rangle$ is Abelian and normal in the group $G_1 = \langle x \rangle N_G^A$. Since $[G_1 : C_{G_1}(\langle a_1, x \rangle)] \leq 2$, $[y^2, x] = 1$ for an arbitrary element $y \in N_G^A$. If $N_G^A$ is a group of one of types (4)-(9) of Proposition 2, then $[(N_G^A)^2, \langle x \rangle] = [\omega(N_G^A), \langle x \rangle] = E$. Therefore $\langle x \rangle \lhd G_1$ as the intersection of normal subgroups $\langle a_1, x \rangle$ and $\langle a_2, x \rangle$. Thus $G_1 = \langle x \rangle \times N_G^A$ is a non-Hamiltonian $\overline{HA}_2$-group which contains an elementary Abelian subgroup of order 8, which contradicts Proposition 1. So, in this case $\omega(N_G^A) = \omega(G)$.

Let $N_G^A$ be a group of type (10) from Proposition 2. Then $Z(N_G^A) = \langle h_1^2 \rangle$, where $h_1 \in H$, $|h_1| = 4$ and $h_1^2 = a_1 \in Z(G)$. By the proved above for the involution:

$$[\langle x \rangle, N_G^A] \subseteq \langle a_1 \rangle = \langle h_1^2 \rangle .$$

Therefore $[x, b^2] = [x, h_1] = 1$. If $[x, h_2] = 1$ then $\langle x, h_2 \rangle \bigcap N_G^A = \langle h_2 \rangle \lhd N_G^A$, which is impossible. Thus, $[x, h_2] = h_1^2$ and $|xh_2| = 2$. Since $xh_2 \notin N_G^A$, $[xh_2, b] \in \langle h_1^2 \rangle$, $[xh_2, b^2] = [xh_2, h_1] = 1$.

On the other hand, $[xh_2, h_1] = [h_2, h_1] = h_1^2 \neq 1$. The contradiction proves that $\omega(N_G^A) = \omega(G)$. □

**Corollary 2.** *If the norm $N_G^A$ of Abelian non-cyclic subgroups of a finite 2-group $G$ is non-Dedekind non-metacyclic and has the non-cyclic center $Z(N_G^A)$, then $\omega(N_G^A) = \omega(G)$.*

**Lemma 3.** *If the norm $N_G^A$ of Abelian non-cyclic subgroups of a finite 2-group $G$ is non-Dedekind, has the non-cyclic center and the non-central in $G$ lower layer $\omega(N_G^A)$, then $G = C\langle y\rangle$, where $C = C_G(\omega(N_G^A))$, $C \lhd G$, $|y| > 4$, $y^2 \in C$. In this case every Abelian non-cyclic subgroup of a finite 2-group $G$ is contained in $C$ and $N_G^A = N_C^A \subseteq C$.*

*Proof.* By the condition of the lemma the norm $N_G^A$ is a group of one of the types (4)-(9) of Proposition 2. In each of these cases $\omega(N_G^A)$ is an elementary Abelian subgroup of order 4 and $\omega(N_G^A) \not\subset Z(G)$ according to the condition of the lemma.

Let's denote $C = C_G\left(\omega(N_G^A)\right)$. Since $\omega(N_G^A) \lhd G, C \lhd G, [G : C] = 2$. Thus $G = C\langle y\rangle$, where $y^2 \in C$.

Since $\omega(N_G^A) \subseteq Z(N_G^A)$, $N_G^A \subseteq C$ and $y \notin N_G^A$. By Lemma 2 $\omega(N_G^A) = \omega(G)$, so $|y| > 2$. Let $|y| = 4$, then the subgroup $\langle y\rangle\omega(G)$ is a dihedral group of order 8. Since $\langle y\rangle\omega(G) = \langle y, b\rangle$, we have $|yb| = 2$. But $yb \in \omega(G)$ and $y \in \omega(G)$ by such conditions, which is impossible. Thus $|y| > 4$. Taking into account that every Abelian non-cyclic subgroup contains $\omega(N_G^A)$, we conclude that it is contained in $C$. Therefore $N_G^A = N_C^A \subseteq C$. □

**Lemma 4.** *Let $G$ be a group of type $\beta$ and the center $Z(N_G^A)$ is cyclic and contains an involution $a$. Then the element $a$ is contained in every cyclic subgroup of composite order of the group $G$.*

*Proof.* Let $x$ be an arbitrary element of the group $G$, $|x| = 2^k$, $k > 1$. Let $\langle x\rangle \cap \langle a\rangle = E$ and $a \in Z(N_G^A)$, $|a| = 2$. Then $[x, a] = 1$ and $\langle x, a\rangle \lhd G_1 = \langle x\rangle N_G^A$. Since $\left\langle x^2\right\rangle \lhd G_1$ and $\left\langle x^{2^{k-1}}\right\rangle \lhd G_1$, we have $x^{2^{k-1}} \in Z(G_1)$.

If $x^{2^{k-1}} \notin N_G^A$, then for an arbitrary element $y \in N_G^A$ $\langle y\rangle \times \left\langle x^{2^{k-1}}\right\rangle \lhd G_1$,

$$(\langle y\rangle \times \left\langle x^{2^{k-1}}\right\rangle) \cap N_G^A = \langle y\rangle \lhd N_G^A.$$

Thus the norm $N_G^A$ is Dedekind, which is impossible. Then $x^{2^{k-1}} \in N_G^A$, $x^{2^{k-1}} \in Z(N_G^A)$, $a \in Z(N_G^A)$ and $Z(N_G^A)$ is non-cyclic, which contradicts the condition. Thus, $\langle x\rangle \cap \langle a\rangle \neq E$ and $a \in \langle x\rangle$. □

## 3 Finite 2-groups with a non-cyclic center and a non-Dedekind non-metacyclic norm of Abelian non-cyclic subgroups (groups of type $\alpha$)

The norm $N_G^A$ of Abelian non-cyclic subgroups is closely related to the norm $N_G$ of non-cyclic subgroups. The last one is the intersection of the normalizers of all non-cyclic subgroups of a group $G$ and was studied in [5] for the case of finite

2-groups. If $G = N_G$, then all non-cyclic subgroups are normal in the group $G$. Such groups were studied in [6] and were called $\overline{H}$-groups.

In the general case $N_G \subseteq N_G^A$. However, if every non-cyclic subgroup is covered by Abelian non-cyclic subgroups, then $N_G = N_G^A$. In particular, we obtain the following.

**Theorem 1.** *If $G$ is a group of type $\alpha$ and does not contain the quaternion group, then $N_G^A = N_G$.*

*Proof.* Since the center of the group $G$ is non-cyclic, $\omega(G) = \omega(N_G^A)$ by Corollary 2. Taking into account that the group $G$ does not contain the quaternion group and has a non-cyclic center, every non-cyclic subgroup contains the lower layer $\omega(G)$. Therefore $\langle x, \omega(G)\rangle$ is an Abelian non-cyclic subgroup for any element $x$ of an arbitrary non-cyclic subgroup. Thus, every non-cyclic subgroup is covered by Abelian non-cyclic subgroups and $N_G^A = N_G$. □

**Lemma 5.** *Any group of type $\alpha$ of exponent 4 is an $\overline{HA_2}$-group.*

*Proof.* Let a group $G$ satisfy the conditions of the lemma. Then $\omega(N_G^A) = \omega(G)$ by Corollary 2 and $\omega(G)$ is a central elementary Abelian group of order 4.

The quotient-group $\overline{G} = G/\omega(G)$ is a group of exponent 2. Thus $\overline{G}$ is Abelian and $G' \subseteq \omega(G)$. Since every Abelian non-cyclic subgroup of a group $G$ contains $\omega(G)$, every such subgroup is normal in $G$ and $G$ is an $\overline{HA_2}$-group. □

**Corollary 3.** *Let $G$ be a group of type $\alpha$. If the group $G$ contains elements of order 4 which are not contained in the norm $N_G^A$, then $expG > 4$.*

**Lemma 6.** *Let $G$ be a group of type $\alpha$. If an element $x \in G \backslash N_G^A$, $|x| = 4$ exists, then the subgroup $G_1 = \langle x\rangle N_G^A$ is an $\overline{HA_2}$-group.*

*Proof.* Let $x \in G \backslash N_G^A$, $|x| = 4$. By Corollary 2 $\omega(N_G^A) = \omega(G) \subseteq Z(G)$. Therefore $\langle x\rangle\omega(G) \lhd G_1 = \langle x\rangle N_G^A$ and

$$G_1' \subseteq \langle x\rangle\omega(G) \cap N_G^A = \omega(G).$$

Since every Abelian non-cyclic subgroup of the group $G_1$ contans $\omega(G)$, it is normal in $G_1$. Thus $G_1$ is an $\overline{HA_2}$-group. □

Let's denote a subgroup which is generated by the elements of order not exceeding $2^m$ by $\omega_m(G)$. In particular, $\omega_1(G) = \omega(G)$ is the lower layer of the group $G$.

**Corollary 4.** *Let $G$ be a group of type $\alpha$. If the norm $N_G^A$ is a group of types (5), (7), (8), (6) ($n > 2$) and (9) ($n > 2$) of Proposition 2, then $\omega_2(N_G^A) = \omega_2(G)$ and $\omega_2(N_G^A)$ is a group of exponent 4.*

*Proof.* Suppose that the conditions of the corollary are satisfied and an element $x \in G \backslash N_G^A$, $|x| = 4$ exists. Then $G_1 = \langle x\rangle N_G^A$ is an $\overline{HA_2}$-group by Lemma 5. Taking into account the structure of the norm $N_G^A$ and the description of $\overline{HA_2}$-groups we get a contradiction. Thus $\omega_2(N_G^A) = \omega_2(G)$. □

**Lemma 7.** *If $\omega_2(N_G^A) = \omega_2(G)$ in a group $G$ of type $\alpha$, then the group $G$ does not contain a generalized quaternion group of order greater than 8. If in this case the group $G$ contains the quaternion group $H$, then $H \subset N_G^A$. Moreover $N_G = N_{N_G^A}$.*

**Corollary 5.** *Let $G$ be a group of type $\alpha$ and its norm $N_G^A$ does not contain the quaternion group. If $\omega_2(N_G^A) = \omega_2(G)$, then the group $G$ does not contain the quaternion group and $N_G^A = N_G$.*

**Theorem 2.** *$G$ is a group of type $\alpha$ if and only if it is a group of one of the following types:*

*1) $G$ is a non-metacyclic non-Dedekind $\overline{HA_2}$-group with a non-cyclic center, $G = N_G^A$;*

*2) $G = H \cdot Q$, where $H$ is the quaternion group of order 8, $Q$ is a generalized quaternion group, $H = \langle h_1, h_2 \rangle$, $|h_1| = |h_2| = 4$, $[h_1, h_2] = h_1^2 = h_2^2$, $Q = \langle y, x \rangle$, $|y| = 2^n$, $n \geq 3$, $|x| = 4$, $y^{2^{n-1}} = x^2$, $x^{-1}yx = y^{-1}$, $H \cap Q = E$, $[Q, H] \subseteq \langle x^2, h_1^2 \rangle$, $N_G^A = H \times \langle y^{2^{n-2}} \rangle$.*

*Proof.* The sufficiency of the conditions of the theorem is easy to verify directly. Let's prove the necessity of the conditions of the theorem.

Since the center $Z(G)$ is non-cyclic, $\omega(N_G^A) \subseteq Z(G)$ and $\omega(N_G^A) = \omega(G)$ by Lemma 2. By the condition of the theorem and Corollary 1 the norm $N_G^A$ is a group of one of the types (4)-(9) of Proposition 2.

Let's continue the proof of the theorem depending on the structure of the norm $N_G^A$.

**Lemma 8.** *Let $G$ be a group of type $\alpha$ and let its norm $N_G^A$ be a group of one of types (4), (5), (7), (8) and (9) ($n = 2$) of Proposition 2. Then $G = N_G^A$.*

*Proof.* Suppose that $G \neq N_G^A$. Let's prove that $G$ is a group of exponent 4. Let an element $x \in G$, $|x| = 8$, exist.

If in this case the norm $N_G^A$ is a group of one of types (5), (7), (8) of Proposition 2, then $\omega_2(N_G^A) = \omega_2(G)$ by Corollary 4 and $x^2 \in N_G^A$.

Let the norm $N_G^A$ be a group of one of types (4) or (9) ($n = 2$) of Proposition 2. Suppose that $x^2 \notin N_G^A$. Since $\langle x \rangle \omega(G) \lhd G_1 = \langle x \rangle N_G^A$ and

$$[\langle x \rangle, N_G^A] \subseteq \langle x \rangle \omega(G) \cap N_G^A = \omega(G),$$

we have $\langle x^2 \rangle \lhd G_1$, $[\langle x^2 \rangle, N_G^A] = E$ and $x^2 \in Z(G_1)$. But in this case $\omega(G_1) \neq \omega(N_G^A)$, which is impossible by Lemma 2. Thus $x^2 \in N_G^A$,

$$[\langle x \rangle, \omega(G)] \subseteq \langle x \rangle \omega(G) \cap N_G^A = \langle x^2 \rangle \omega(G).$$

Let us consider the quotient-group $\overline{G_1} = G_1/\omega(G)$. By the proved above $\overline{G_1}' \subseteq \langle \overline{x}^2 \rangle$. If $\overline{G_1}' \neq \overline{E}$ and $\overline{x} \notin Z(\overline{G_1})$, $\langle \overline{x} \rangle \lhd \overline{G_1}$, then $[\overline{G_1} : C_{\overline{G_1}}(\langle \overline{x} \rangle)] = 2$. Thus $\overline{N_G^A}$ contains an element $\overline{y}$ of order 2 which is permutable with $\overline{x}$. Therefore $\langle \overline{x}, \overline{y} \rangle$ is a dihedral group of order 8 and $|\overline{xy}| = 2$. Since $\omega(G)$ is a central non-cyclic subgroup,

$\overline{N_G^A} \leq N(\overline{G})$ by Lemma 1. Therefore $\langle \overline{xy} \rangle \lhd \overline{G_1}$. Thus $\overline{G_1}$ is Abelian, which is impossible.

Therefore $\overline{G_1}' = E$, $G_1' \subseteq \omega(N_G^A)$ and $G_1$ is an $\overline{HA_2}$-group which contains a central cyclic subgroup of order 4, which contradicts the structure of the norm $N_G^A$. Thus $G$ is a group of exponent 4. $G$ is an $\overline{HA_2}$-group by Lemma 5. $\square$

**Lemma 9.** *Let $G$ be a group of type $\alpha$ and its norm $N_G^A$ is a direct or a semi-direct product of a normal cyclic group of order greater than 4 and the quaternion group. Then $G = N_G^A$.*

*Proof.* Let the norm $N_G^A$ satisfies the conditions of the lemma. It is a group of type (6) or (9) ($n > 2$). Suppose that $G \neq N_G^A$. Since the center of the group $G$ is non-cyclic, then $\omega(N_G^A) = \omega(G)$ by Lemma 2. Moreover, $\omega_2(N_G^A) = \omega_2(G)$ by Corollary 4.

If the norm $N_G^A$ is a group of type (6) ($n > 2$), then $N_G^A = N_G$ by Lemma 7. By Theorem 2 [5] $G$ is an $\overline{HA_2}$-group and it is a semi-direct product of a normal cyclic subgroup of order greater than 4 and the quaternion group. Thus $G = N_G^A$, which is impossible.

Let the norm $N_G^A$ be a group of type (9) ($n > 2$). Then $N_G^A$ contains all quaternion groups by Lemma 7 and the non-cyclic norm $N_G$ of a group $G$ coincides with the non-cyclic norm of the subgroup $N_G^A$, $N_G = N_{N_G^A} = \langle c^2 \rangle \times H$, $|c^2| \geq 4$. By Theorem 2 [5] $G$ is an $\overline{HA_2}$-group and $G = N_G^A$, which is impossible. $\square$

**Lemma 10.** *Let $G$ be a group of type $\alpha$ and let its norm $N_G^A$ be a group of the type $N_G^A = H \times \langle c \rangle$, where $H$ is the quaternion group, $|c| = 4$. Then either $G = N_G^A$, or $G$ is a group of type (2) of Theorem 2.*

*Proof.* By Lemma 2 $\omega(N_G^A) = \omega(G)$. If $N_G^A = \omega_2(G)$, then $N_G = N_{N_G^A} = N_G^A$ by Lemma 7. By Theorem 2 [5] $G$ is an $\overline{HA_2}$-group and $G = N_G^A$.

Let assume that $N_G^A \neq \omega_2(G)$ and an element $x \in G \backslash N_G^A$, $|x| = 4$ exists. By Lemma 6 $G_1 = \langle x \rangle N_G^A$ is an $\overline{HA_2}$-group of exponent 4. If $[x, c] = 1$, then $G_1$ contains a central cyclic subgroup $\langle c \rangle$ of order 4, which is impossible by Proposition 2, because $|\omega_2(G_1)| = 64$. Thus $c \notin Z(G)$.

If $\langle c \rangle \lhd G$ and $\langle c \rangle$ is a non-central subgroup, then $[G : C] = 2$, where $C = C_G(\langle c \rangle)$. Let's show that under these conditions all elements of order greater than 4 are permutable with the element $c$. Let $y \in G \backslash N_G^A$, $|y| = 2^s$, $s > 2$. If $\langle y \rangle \cap N_G^A \subseteq \omega(G)$, then

$$[\langle y \rangle, N_G^A] \subseteq \langle y \rangle \omega(G) \cap N_G^A = \omega(G)$$

and $[\langle y^2 \rangle, N_G^A] = E$. But in this case $\omega(G) \neq \omega(N_G^A)$, which contradicts Lemma 2. Therefore, $\langle y \rangle \cap N_G^A = \langle y^{2^{s-2}} \rangle$.

Let $y_1 = y^{2^{s-3}}$, $y_1^2 = c^m h^k$, where $h \in H$. Let us consider $G_2 = \langle y_1 \rangle N_G^A$. Since

$$[\langle y_1 \rangle, N_G^A] \subseteq \langle y_1 \rangle \omega(G) \cap N_G^A = \langle y_1^2 \rangle \omega(G),$$

we have $\langle y_1^2\rangle \lhd G_2$. Thus either $m \equiv 0 \pmod 2$ and $(k,2)=1$, or $k \equiv 0 \pmod 2$ and $(m,2)=1$.

In the first case $y_1^2 = c^{2m_1}h^k$, $(k,2)=1$. Let consider the quotient-group $\overline{G} = G/\omega(G)$. By the proved above,

$$[\langle \overline{y}_1\rangle, \overline{N_G^A}] \subseteq \langle \overline{y}_1\rangle \cap \overline{N_G^A} = \langle \overline{y}_1^2\rangle = \langle \overline{h}\rangle.$$

Let $h_1$ be an element of the subgroup $H$ which is not permutable with $h$. Then $[\langle \overline{h}_1\rangle, \langle \overline{y}_1\rangle] = \langle \overline{y}_1^{2l}\rangle = \langle \overline{h}^{kl}\rangle$. If $(l,2)=1$, then $\langle \overline{y}_1, \overline{h}_1\rangle$ is a dihedral group and $|\overline{y}_1\overline{h}_1| = 2$. By Lemma 1 $\langle \overline{y}_1\overline{h}_1\rangle \lhd \overline{G_2}$ and therefore $\overline{G_2} = \overline{N_G^A} \times \langle \overline{y}_1\overline{h}_1\rangle$. Hence $[\overline{y}_1\overline{h}_1, \overline{h}_1] = [\overline{y}_1, \overline{h}_1] = 1$, which is impossible. Thus $(l,2) \neq 1$ and $[\overline{h}_1, \overline{y}_1] = 1$. But then $[h_1, y_1] \in \omega(N_G^A)$, $[h_1, y_1^2] = [h_1, h] = 1$, which contradicts the choice of $h_1$.

Thus $y_1^2 = c^m h^{2k_1}$, where $(m,2)=1$, and $[y,c]=1$. Hence the elements of order greater then 4 are contained in the centralizer $C$.

Let $x \notin C$. Then $|x| = 4$. Taking into account $[G:C]=2$, we conclude that $G = C\langle x\rangle$, where $x^2 \in \omega(G)$, $[\langle x\rangle, N_G^A] \subseteq \omega(G)$. By the proved above, the norm $N_G^A$ contains all elements of order 4 of the centralizer $C$, i.e. $N_G^A = \omega_2(C)$. If $\exp C = 4$, then $N_G^A = C$ and $G = N_G^A \cdot \langle x\rangle$. By Lemma 6 $G$ is an $\overline{HA_2}$-group which does not coincide with $N_G^A$, which is impossible. Thus $\exp C > 4$.

Since the norm $N_C^A$ of the subgroup $C$ contains $N_G^A$ and $c \in Z(C)$, the norm $N_C^A$ is a group of one of the types:

1) $N_C^A = \langle y\rangle \times H$, $|y| = 2^n$, $n \geq 3$, $y^{2^{n-2}} = c$;

2) $N_C^A = \langle y\rangle \rightthreetimes H$, $[\langle y\rangle, H] = \langle y^{2^{n-1}}\rangle$, $|y| = 2^n$, $n \geq 3$, $y^{2^{n-2}} = c$.

By Lemma 9, $N_C^A = C$. Let's consider each of these cases separately.

(1) Let $C = N_C^A = \langle y\rangle \times H$, then $G = (\langle y\rangle \times H)\langle x\rangle$, $x^2 \in C$. Let's consider the quotient group $\overline{G} = G/\omega(G) \cong (\langle \overline{y}\rangle \times \overline{H})\langle \overline{x}\rangle$. Since $\langle \overline{y}\rangle = \overline{Z(C)}$, the subgroup $\langle \overline{y}, \overline{x}\rangle$ contains a cyclic subgroup of index 2. Therefore the following relations are possible between $\overline{x}$ and $\overline{y}$.

If $[\overline{y}, \overline{x}] = 1$, then $G' \subseteq \omega(G)$ and $G$ is an $\overline{HA_2}$-group, which contradicts $G \neq N_G^A$.

If $\overline{x}^{-1}\overline{yx} = \overline{y}^{-1}\overline{y}^{2^{n-2}}$, $n \geq 4$, then turning to the preimages $x^{-1}yx = y^{-1}cz$, where $z \in \omega(G)$. Therefore $x^{-2}yx^2 = x^{-1}y^{-1}czx = yc^{-2}$, which contradicts $x^2 \in Z(G)$.

If $\overline{x}^{-1}\overline{yx} = \overline{yy}^{2^{n-2}}$, where $n \geq 4$, then $|y| \geq 16$, $x^{-1}yx = ycz$, where $z \in \omega(G)$, and $x^{-1}y^2x = y^2c^2$. Since $c \in \langle y\rangle$, $y^2 = c$ and $|y| = 8$, which is impossible.

Thus $G = H \cdot Q$ is a group of the type (2) of Theorem 2, where one of the groups $H$ or $Q$ is a generalized quaternion group of order greater than 8, and the other one is the quaternion group, $[H,Q] \subseteq \omega(G)$.

(2) Let $C = N_C^A = \langle y\rangle \rightthreetimes H$, $[\langle y\rangle, H] = \langle y^{2^{n-1}}\rangle$, $|y| = 2^n$, $n \geq 3$, $y^{2^{n-2}} = c$.

Let us consider the quotient-group

$$\overline{G} = G/\omega(G) \cong (\langle \overline{y}\rangle \rightthreetimes \overline{H})\langle \overline{x}\rangle,$$

where $[\overline{H}, \langle \overline{x}\rangle] = E$, $[\langle \overline{y}\rangle, \langle \overline{x}\rangle] \subseteq \langle \overline{y}, \overline{H}\rangle$. Let $\overline{x}^{-1}\overline{yx} = \overline{y}^{\alpha}\overline{h}^{\beta}$, where $\overline{h} \in \overline{H}$. Then by the condition $[\overline{x}^2, \overline{y}] = 1$, we have

$$\overline{x}^{-2}\overline{yx}^2 = (\overline{y}^{\alpha}\overline{h}^{\beta})^{\alpha}\overline{h}^{\beta} = \overline{y}^{\alpha^2}\overline{h}^{\beta(\alpha+1)} = \overline{y}.$$

If $\beta \equiv 1 \pmod 2$, then $\alpha^2 \equiv 1 \pmod{2^{n-1}}$ and $\alpha = \pm 1 + 2^{n-1}t$ or $\alpha = \pm 1 + 2^{n-2}t$. It is easy to verify that in each case $[h_1, (xy)^2] \neq 1$ for the element $h_1 \in H$ which is not permutable with $h$. On the other hand, $[h_1, x] \in \omega(G)$, $[h_1, y] \in \omega(G)$. Thus, $[h_1, xy] \in \omega(G)$ and $[h_1, (xy)^2] = 1$. We get a contradiction.

Thus $\beta \equiv 0 \pmod 2$ and $\langle \overline{y} \rangle \lhd \overline{G}$. Repeating the above proof we get that $\overline{x}^{-1}\overline{yx} = \overline{y}^{-1}$. Then $G = \langle y \rangle G_1$, where $G_1 = N_G^A \langle x \rangle$ is an $\overline{HA_2}$-group, which is a direct or a semi-direct product of two quaternion groups. Thus $G = H \cdot Q$ is a group of the type (2) of Theorem 2.

Suppose that $\langle c \rangle \ntriangleleft G$. Hence $[\langle c \rangle, G] \subseteq \omega(G)$.

Let $x$ be an element of $G$, $|x| \geq 8$. If $\langle x \rangle \bigcap N_G^A \subseteq \omega(G)$, then

$$[\langle x \rangle, N_G^A] \subseteq \langle x \rangle \omega(G) \bigcap N_G^A = \omega(G)$$

and $[\langle x^2 \rangle, N_G^A] = E$. Hence $G_1 = \langle x^2 \rangle N_G^A$ is an $\overline{HA_2}$-group which has two central cyclic subgroups $\langle x \rangle$ and $\langle c \rangle$ of order 4, which contradicts the description of $\overline{HA_2}$-groups. Thus, $x^{2^k} = c^\alpha h^\beta$ (where either $\alpha$ or $\beta$ is not divisible by 2) and

$$[\langle x \rangle, N_G^A] \subseteq \langle x^{2^k} \rangle \omega(G).$$

Since $\langle x^2 \rangle \lhd G_1$, either $\alpha \equiv 0 \pmod 2$ and $\beta \equiv 1 \pmod 2$, or $\alpha \equiv 1 \pmod 2$ and $\beta \equiv 0 \pmod 2$.

If $\alpha \equiv 0 \pmod 2$ and $\beta \equiv 1 \pmod 2$, then $[\langle x \rangle, N_G^A] \subseteq \omega(G)$ and $[x^2, h_1] = 1$. On the other hand, $[x^2, h_1] = [c^{2\alpha} h^\beta, h_1] = [h^\beta, h_1] \neq 1$. We get a contradiction. Thus $x^{2^k} = c^\alpha h^{2\beta}$, where $(\alpha, 2) = 1$. Hence $[x, c] = 1$ and $\langle x \rangle \bigcap N_G^A = \langle c h^{2\beta} \rangle$, where $\beta \in \{0, 1\}$.

Let denote $N = N_G(\langle c \rangle)$. It is clear that $N \supseteq N_G^A$ and for any element $y \in G$ $|y| \geq 8$, $y \in N$. If $N \neq G$, then an element $a \in G \backslash N$ exists, $|a| = 4$, $a^2 \in \omega(G)$, $[\langle a \rangle, N_G^A] \subseteq \omega(G)$.

Let $a, b \notin N$. Then $[a, c] = c^{2r} h^2$, $[b, c] = c^{2s} h^2$. Hence $[ab, c] \in \langle c \rangle$ and $ab \in N$. It is easy to verify that $a^{-1}N = aN = bN$. Hence $[G : N] = 2$ and $N \lhd G$, $G = N\langle a \rangle$, $a^2 \in \omega(N_G^A)$.

By the proved above, the subgroup $N$ is a product of the quaternion group of order 8 and a generalized quaternion group of order equal or greater than 16: $N = H \cdot Q$, $|H| = 8$, $|Q| \geq 16$, $H = \langle h_1, h_2 \rangle$, $Q = \langle y, x \rangle$, $|y| = 2^n > 4$, $y^{2^{n-2}} = c$, $[H, Q] \subseteq \omega(G)$.

If $|y| > 8$, then $N' = \langle y^2 \rangle \times \langle h^2 \rangle \lhd G$ and $\langle y^4 \rangle \lhd G$, $\langle c \rangle \lhd G$, which contradicts the assumption. Thus, $|y| = 8$.

Let us consider the quotient-group

$$G/N_G^A \cong (\langle \overline{y} \rangle \times \langle \overline{x} \rangle)\langle \overline{a} \rangle,$$

$|\overline{y}| = |\overline{x}| = |\overline{a}| = 2$. If $G/N_G^A$ is non-Abelian, then it is a dihedral group and contains an element $\langle \overline{a}\overline{t} \rangle$ of order 4, where $\overline{t} \in \langle \overline{y}, \overline{x} \rangle$. It is clear that $|at| > 4$. Hence $at \in N$ and $a \in N$, which is impossible. Thus the quotient-group $G/N_G^A$ is Abelian,

$[\overline{N}, \langle \overline{a} \rangle] = 1$ and $[y, a] = c^k h^m$. If $m \equiv 0 \pmod 2$, then $[y^2, a] = c^{2k} \in \langle c \rangle$, which is impossible because $a \in N$. Thus $m = 1$ and $[y, a] = c^k h$. Hence

$$(ya)^2 = ya^2 y c^k h = c^{1+k} h z,$$

$z \in \omega(G)$. On the other hand, since $|ya| > 4$, $\langle ya \rangle \cap N_G^A \subseteq \langle c \rangle \omega(G)$ by the proved above. We get a contradiction. □

The theorem is proved. □

**Corollary 6.** *A group $G$ of type $\alpha$ does not contain a quaternion subgroup if and only if the norm $N_G^A$ does not contain such a subgroup.*

## 4 Finite 2-groups with cyclic center and a non-Dedekind non-metacyclic norm of Abelian non-cyclic subgroups (groups of type $\beta$)

**Lemma 11.** *Let $G$ be a finite 2-group with a non-Dedekind norm $N_G^A$ of Abelian non-cyclic subgroups which is a group of one of the types (4)-(8) of Proposition 2. Then the center $Z(G)$ of the group $G$ is non-cyclic.*

*Proof.* Let $N_G^A$ be a group of one of the types which have been noted in the condition of the lemma. Then the center $Z(N_G^A)$ of the norm $N_G^A$ is non-cyclic. If the norm $N_G^A$ is a group of type (6) of Proposition 2, then $\omega(N_G^A) \subseteq Z(G)$ and the group $G$ has the non-cyclic center.

So we will assume that $N_G^A$ is a group of one of types (4)-(5) or (7)-(8). In each of these cases $\omega(N_G^A)$ is an elementary Abelian subgroup of order 4. Since $\omega(N_G^A) \subseteq Z(N_G^A)$, we have $\omega(N_G^A) = \omega(G)$ by Lemma 2.

Suppose $\omega(N_G^A) \not\subset Z(G)$, contrary to the conditions of the lemma. Then

$$\omega(N_G^A) \cap Z(G) \neq E$$

by the condition $\omega(N_G^A) \lhd G$. Let $\omega(N_G^A) = \langle a_1 \rangle \times \langle a_2 \rangle$, $|a_1| = |a_2| = 2$, where $a_1 \in Z(G)$ and $a_2 \notin Z(G)$.

Let's denote $C = C_G\left(\omega(N_G^A)\right)$. Then $G = C \cdot \langle y \rangle$, $|y| > 4$, $y^2 \in C$ by Lemma 3. Since $N_C^A \subset C$, $N_G^A \subseteq N_C^A$ and $C$ contains all Abelian non-cyclic subgroups of $G$, $N_G^A = N_C^A$. Since the norm $N_C^A$ is non-metacyclic and $Z(C)$ is non-cyclic, $C$ is either a non-metacyclic non-Dedekind $\overline{HA}_2$-group by Theorem 2 and $C = N_C^A = N_G^A$, or $C = H \cdot Q$ is a product of the quaternion group $H = \langle h_1, h_2 \rangle$ of order 8 and a generalized quaternion group $Q = \langle t, q \rangle$, $|t| = 2^k > 8$, $t^{2^{k-1}} = q^2$, $q^{-1} t q = t^{-1}$, $[H, Q] \subseteq \omega(C)$ and $N_C^A = N_G^A = \left\langle t^{2^{k-2}} \right\rangle \times H$.

In the previous case $N_G^A$ is a group of type (6) of Proposition 2, which contradicts the proved above.

Thus we will assume that $C = N_G^A$ and $G = N_G^A \cdot \langle y \rangle$, where $y^2 \in N_G^A$. In this case $N_G^A$ is a non-Dedekind $\overline{HA}_2$-group of exponent 4. So $|y| = 8$, $y^4 = a_1 \in Z(G)$

by Lemma 4. It is also easy to prove that the norm $N_G^A$ contains all elements of order 4 of the group $G$.

Let's consider the quotient-group

$$\overline{G} = G/\omega(G) \cong \overline{N_G^A} \cdot \langle \overline{y} \rangle, \overline{y}^2 \in \overline{N_G^A},$$

where $|\overline{y}| = 4$. Since $\omega\left(\overline{G}\right) = \overline{N_G^A} \lhd \overline{G}$, $\left|\overline{N_G^A}\right| \geq 8$ and $\overline{y}$ induces an automorphism of order 2 on $\omega(\overline{G})$, there is an involution $\overline{z}$ such that $\langle \overline{y} \rangle \cap \langle \overline{z} \rangle = \overline{E}$ and $[\overline{z}, \overline{y}] = 1$ in $\omega(\overline{G})$. Turning to the preimages, we have $[z, y] = a$, where $a \in \omega(G)$. Since $\left[z^2, y\right] = 1$, we conclude that $z^2 = a_1$. Let $a \in \langle a_1 \rangle$, then $\left[z, y^2\right] = 1$ and $\left|y^2 z\right| = 2$. But in this case $y^2 \in \langle z \rangle \omega(G)$ and the intersection $\langle \overline{y} \rangle \cap \langle \overline{z} \rangle$ is non-identity in the quotient-group $\overline{G}$. It is a contradiction. Thus, $a \notin \langle a_1 \rangle$ and we can assume without loss of generality that $a = a_2$. Then $y^{-1} z y = z a_2$, $\left[z, y^2\right] = z^2 = a_1$, and $\left\langle y^2, z \right\rangle$ is the quaternion group, which is impossible if the norm $N_G^A$ is a group of type (4) or (5) of Proposition 2.

Let $N_G^A$ contain the quaternion group, i.e. $N_G^A$ is a group of type (7) or (8) of Proposition 2. Then $N_G^A = H \cdot Q$ is a direct or a semidirect product of two quaternion groups $H$ and $Q$, $[H, Q] \subseteq Q^2$.

Then in the group $G = N_G^A \cdot \langle y \rangle$ the subgroup $\left\langle y^2, a_2 \right\rangle$ is Abelian non-cyclic by the inclusion $\omega(N_G^A) \subseteq Z(N_G^A)$ and therefore $\left\langle y^2, a_2 \right\rangle$ is a normal subgroup in $G$. The subgroup $\widetilde{N_G^A}$ is elementary Abelian of order 8 in the quotient-group

$$\widetilde{G} = G/\left\langle y^2, a_2 \right\rangle \cong \widetilde{N_G^A} \leftthreetimes \langle \tilde{y} \rangle.$$

Since $\tilde{y}$ induces an automorphism of order 2 on $\widetilde{N_G^A}$, it is always possible to point out involutions $\widetilde{z_1}$, $\widetilde{z_2} \in \widetilde{N_G^A}$ which are permutable with $\tilde{y}$. Turning to preimages we get that $[z_i, y] = y^{2m_i} a^{s_i}$, $i = 1, 2$.

If $s_1 = s_2 = 1$, then $[z_1 z_2, y] = y^{2t}$. If $(t, 2) = 1$, then $|y z_1 z_2| \leq 4$ and $y \in N_G^A$ by the proved, which is impossible. Thus $t = 2t_1$ and $[z_1 z_2, y] = y^{4t_1} \in Z(G)$. But

$$\left[z_1 z_2, y^2\right] = \left[(z_1 z_2)^2, y\right] = 1$$

by such conditions. From the second part of the equality we have $(z_1 z_2)^2 = a_1 = y^4$ and $\left|z_1 z_2 y^2\right| = 2$, which contradicts the structure of the norm $N_G^A$.

Thus we can assume that at least one of numbers $s_i = 0$. But then $[z_i, y] = y^{2m_i}$ and we again get a contradiction repeating the above argument. In this case $G = C$ and $\omega\left(N_G^A\right) \subseteq Z(G)$. □

**Theorem 3.** *$G$ is a group of type $\beta$ if and only if it is a group of one of the following types:*

*1) $G$ is a non-metacyclic non-Hamiltonian $\overline{HA_2}$-group with a cyclic center, $G = N_G^A$;*

*2) $G = (\langle x \rangle \leftthreetimes \langle b \rangle) \leftthreetimes \langle c \rangle$, $|x| = 2^n, n > 3$, $|b| = |c| = 2$, $[x, c] = x^{\pm 2^{n-2}} b$, $[b, c] = [x, b] = x^{2^{n-1}}$, $N_G^A = \left(\left\langle x^2 \right\rangle \times \langle b \rangle\right) \leftthreetimes \langle c \rangle$;*

*3)* $G = (\langle x\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle \leftthreetimes \langle d\rangle$, $|x| = 2^n, n > 2$, $|b| = |c| = |d| = 2, [x, c] = [x, b] = 1$, $[b, c] = [c, d] = [b, d] = x^{2^{n-1}}$, $d^{-1}xd = x^{-1}$, $N_G^A = \left(\left\langle x^{2^{n-2}}\right\rangle \times \langle b\rangle\right) \leftthreetimes \langle c\rangle$*;*

*4)* $G = (\langle c\rangle \leftthreetimes H)\langle y\rangle, H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4, h_1^2 = h_2^2 = [h_1, h_2], |c| = 4$, $[c, h_1] = c^2$, $[c, h_2] = 1$, $y^2 = h_1, [y, h_2] = c^2h_1^2$, $[y, c] = h_2^{\pm 1}, N_G^A = \langle c\rangle \leftthreetimes H$*.*

*Proof.* Let a group $G$ and its norm of Abelian non-cyclic subgroups satisfy the conditions of the theorem. Let's continue the proof of the theorem in the following lemmas.

**Lemma 12.** *Let $G$ be a finite 2-group and its norm $N_G^A$ of Abelian non-cyclic subgroups be a group of type (10) of Proposition 2. Then all Abelian non-cyclic subgroups are normal in $G$ and $G = N_G^A$.*

*Proof.* Let $N_G^A$ be a group of type (10) of Proposition 2, i.e.

$$N_G^A = (H \times \langle a\rangle)\langle b\rangle,$$

where $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4$, $|a| = 2$, $|b| = 8$, $b^2 = h_1$, $[h_2, b\ ] = a$, $[a, b] = [h_1,\ h_2] = h_1^2 = h_2^2$. In particular, $\omega\left(N_G^A\right) = \left\langle h_1^2, a\right\rangle$ and $Z\left(N_G^A\right) = \left\langle h_1^2\right\rangle \subset Z(G)$.

$N_G^A$ contains all elements of order 2 of the group $G$ by Lemma 3 and $\omega\left(N_G^A\right) = \omega(G)$. Let's denote $C = C_G(\omega(G))$. Then $[G : C] = 2$ and $G = C\langle b\rangle$, $b^2 \in C$. By the proved above, the lower layer $\omega\left(N_G^A\right)$ contains all involutions of the centralizer $C$, so the quotient-group $\overline{C} = C/\langle a\rangle$ contains only one involution by Lemma 4. Since $\overline{C}$ is non-Abelian, $\overline{C}$ is a quaternion 2-group:

$$\overline{C} \cong \overline{Q} = \langle \overline{x}, \overline{y}\rangle,$$

$|\overline{x}| = 2^n \geq 4$, $|\overline{y}| = 4$, $\overline{x}^{2^{n-1}} = \overline{y}^2$, $\overline{y}^{-1}\overline{xy} = \overline{x}^{-1}$.

Turning to the preimages and taking into account Lemma 4, we have that $x^{2^{n-1}} = y^2 = h_1^2$, $y^{-1}xy = x^{-1}a^m$, $m \in \{0, 1\}$. If $m = 1$, then $y^{-1}xy = x^{-1}a$ and $(xy)^2 = h_1^2a \notin \left\langle h_1^2\right\rangle$, which is impossible. Therefore $m = 0$, $y^{-1}xy = x^{-1}$ and

$$C = Q \times \langle a\rangle.$$

We can assume, without loss of generality, that $H \subseteq Q$, $h_1 \in \langle x\rangle, \langle h_2\rangle = \langle y\rangle$. If $|Q| > 8$, then $h_2 \notin N_G(\langle a, xh_2\rangle)$, which is impossible, because $h_2 \in N_G^A$. Thus $Q = H$, $C = H \times \langle a\rangle \subset N_G^A$ and

$$G = C\langle b\rangle = N_G^A.$$

□

**Lemma 13.** *If a finite 2-group $G$ has the norm $N_G^A$ of Abelian non-cyclic subgroups which is a group of type (3) of Proposition 2, then $G = N_G^A$.*

*Proof.* Let a group $G$ and its norm $N_G^A$ satisfy the conditions of the lemma,

$$N_G^A = (H \times \langle b\rangle) \leftthreetimes \langle c\rangle,$$

where $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4$, $[h_1,\ h_2] = h_1^2 = h_2^2$, $|b| = |c| = 2$, $[H, \langle b\rangle] = [H, \langle c\rangle] = E$, $[b, c] = h_1^2$.

Suppose that $G \neq N_G^A$ and let's prove that $N_G^A$ contains all involutions of the group $G$. Indeed, otherwise we have $\langle z, h_1^2\rangle \lhd G_1 = \langle z\rangle N_G^A$ for any involution $z \in G\backslash N_G^A$. Therefore $\left[G_1 : C_{G_1}\left(\langle z, h_1^2\rangle\right)\right] \leq 2$ and $G_1 \backslash \left\langle h_1^2\right\rangle$ contains an involution $y \neq h_1^2$ which is permutable with $z$. So,

$$\langle y, z\rangle \cap N_G^A = \langle y\rangle \lhd N_G^A,$$

which is impossible. Hence all involutions of a group $G$ are contained in $N_G^A$.

Suppose that an element $x$ of order 4 exists in $G\backslash N_G^A$. By Lemma 4 $x^2 = h_1^2$. Thus any element $a$ of order 4 of the norm $N_G^A$ is not permutable with $x$, otherwise $|ax|$=2 and $x \in N_G^A$ by the proved above. Let's denote $G_2 = \langle x\rangle N_G^A$ and consider the quotient-group $\overline{G_2} = G_2/\left\langle h_1^2\right\rangle$. Since $\overline{N_G^A}$ is an elementary Abelian group of order 16, normal in $\overline{G_2}$ and $\overline{x}$ induces an automorphism of order 2 on $\overline{N_G^A}$, there exist involutions $\overline{y_1}, \overline{y_2} \in \overline{N_G^A}$, $\langle\overline{y_1}\rangle \cap \langle\overline{y_2}\rangle = \overline{E}$, which are permutable with $\overline{x}$. Turning to the preimages we will have $[x, y_i] \in \left\langle h_1^2\right\rangle$, $i = 1, 2$. It is easy to prove that the group $\langle y_1, y_2\rangle$ contains an involution $y \neq h_1^2$ which is permutable with $x$. Then $\langle x, y\rangle \lhd G_2$ as an Abelian non-cyclic subgroup and

$$G_2' \subseteq \langle x, y\rangle \cap N_G^A = \left\langle y, h_1^2\right\rangle.$$

Let $t$ be an arbitrary non-central involution of $N_G^A$ which differs from $y$. Let's put

$$[x, t] = y^m h_1^{2k}, m, k \in \{0, 1\}.$$

Then $\left[x, t^2\right] = h_1^{2m}$. On the other hand, $\left[x, t^2\right] = 1$, therefore $m = 0$ and $\left[\langle x\rangle, N_G^A\right] \subseteq \left\langle h_1^2\right\rangle$. However in this case the group $G_2$ will contain an involution which does not belong to $N_G^A$, that contradicts the proved above. Therefore $N_G^A$ contains all elements of order 4 of the group $G$.

According to the assumption $G \neq N_G^A$, we conclude that there is an element $x \in G\backslash N_G^A$, $|x| = 8$. Since $x^2 \in N_G^A$, $\left|x^2\right| = 4$ and all cyclic subgroups of order 4 are normal in $N_G^A$, we have

$$\left\langle x^2\right\rangle \lhd G_3 = \langle x\rangle N_G^A.$$

Let's consider the quotient-group $\overline{G_3} = G_2/\left\langle x^2\right\rangle$. Since $\overline{N_G^A}$ is a normal elementary Abelian group of order 8 and $\overline{x}$ induces an automorphism of order 2 on it, there exist involutions $\overline{y_1},\ \overline{y_2} \in \overline{N_G^A}$, $\langle\overline{y_1}\rangle \cap \langle\overline{y_2}\rangle = \overline{E}$, which are permutable with $\overline{x}$. Turning to the preimages we get $[x, y_i] \in \left\langle x^2\right\rangle$, $i = 1, 2$. It is easy to check that $[x, y_i] \in \left\langle h_1^2\right\rangle$ and the group $\left\langle x^2,\ y_1,\ y_2\right\rangle$ contains an involution $y$ which is permutable with $x$. Then $\langle x, y\rangle \lhd G_3$ as an Abelian non-cyclic subgroup and

$$G_3' \subseteq \langle x, y\rangle \cap N_G^A = \langle y, x^2\rangle.$$

Let $[x,t]=x^{2m}y^k$, where $t$ is an arbitrary non-central involution of $N_G^A$ which differs from $y$. Since $N_G^A$ contains all elements of order 4, $[x,t]\in\langle h_1^2\rangle$ by the condition $\left[x,t^2\right]=1$. But then $\left[x^2,t\right]=1$ and $x^2\in Z(G_3)$, which is impossible, because the norm $N_G^A$ does not contain non-central elements of order 4. This contradiction proves that $G=N_G^A$. □

**Lemma 14.** *If a finite 2-group $G$ has a non-Dedekind norm $N_G^A\neq G$ which is a group of type (1) of Proposition 2, then $G$ is a group of one of types (2) or (3) of Theorem 3.*

*Proof.* Let $G\neq N_G^A$ and
$$N_G^A=(\langle a\rangle\times\langle b\rangle)\leftthreetimes\langle c\rangle,$$
where $|a|=2^n$, $n\geq 2$, $|b|=|c|=2$, $[a,c]=[a,b]=1,[b,c]=a^{2^{n-1}}$. Since $N_G^A\lhd G$, the intersection $\overline{N_G^A}\cap Z(\overline{G})\neq\overline{E}$ in the quotient-group $\overline{G}=G/\langle a\rangle$. We can assume without loss of generality that $\overline{b}\in Z(\overline{G})$. Then $\langle a,b\rangle\lhd G$, $\omega\left(\langle a,b\rangle\right)=\left\langle a^{2^{n-1}},b\right\rangle\lhd G$.

Let's denote $C=C_G\left(\left\langle a^{2^{n-1}},b\right\rangle\right)$. Then $C\lhd G$, $[G:C]=2$ and $G=C\leftthreetimes\langle c\rangle$, where $c\in N_G^A$, $|c|=2$. By Lemma 4 the quotient-group $\overline{C}=C/\langle b\rangle$ has only one involution and $\overline{C}$ is a cyclic group or a generalized quaternion group.

Let $\overline{C}$ be cyclic, then its full preimage $C=\langle x\rangle\times\langle b\rangle$ is Abelian and
$$[x,c]\in C\cap N_G^A=\langle a,b\rangle.$$
Let's put $[x,c]=a^mb^k$. If $|[x,c]|=2$, then $G'\subset\left\langle a^2\right\rangle$ and $G$ is an $\overline{HA_2}$-group, contrary to the assumption. Thus $|[x,c]|>2$. If $|a|=4$, then $[x,c]=a^{\pm 1}b$ by the condition $\left[x,c^2\right]=1$, so $(xc)^2\in Z(G)$ and $|x|\leq 8$. So $x^2=a^{\pm 1}b$. However, $c\notin N_G\left(\left\langle a^2\right\rangle\times\langle xbc\rangle\right)$ by such conditions, i.e. $c\notin N_G^A$, which is impossible.

Let $|a|>4$, then $m=2^{n-2}m_1$, where $(m_1,2)=1$, $(k,2)=1$. Thus $[x,c]=a^{\pm 2^{n-2}}b$, $(xc)^2=x^2a^{\pm 2^{n-2}}b$ and $(xc)^2\in Z(G)$. Since $Z\left(G\right)=\langle a\rangle$ and $|x|>|a|$ by the previous reasoning, $(xc)^2=a$. Let's denote $xc=y$. Then $|y|=2^{n+1}$, $[y,b]=y^{2^n}$, $[y,c]=y^{\pm 2^{n-1}}b$ and
$$G=(\langle y\rangle\leftthreetimes\langle b\rangle)\leftthreetimes\langle c\rangle$$
is a group of type (2) of Theorem 3.

Let $\overline{C}$ be a generalized quaternion group $\overline{C}=\left\langle\overline{h_1},\overline{h_2}\right\rangle$, where $\left|\overline{h_1}\right|=2^n$, $n\geq 2$, $\left|\overline{h_2}\right|=4$, $\overline{h_1}^{2^{n-1}}=\overline{h_2}^2$, $\overline{h_2}^{-1}\overline{h_1h_2}=\overline{h_1}^{-1}$. Let $h_1$ and $h_2$ denote the preimages of elements $\overline{h_1}$ and $\overline{h_2}$, respectively. Since the center $Z(G)$ is cyclic, $h_1^{2^{n-1}}=h_2^2=a^{2^{n-1}}$, $h_2^{-1}h_1h_2=h_1^{-1}b^m$, $m\in\{0,1\}$, by Lemma 4. If $m\neq 0$, then
$$(h_1h_2)^2=h_2^2b=a^{2^{n-1}}b,$$
which contradicts Lemma 4. Thus $m=0$, $C=H\times\langle b\rangle$, $H=\langle h_1,h_2\rangle$ is a generalized quaternion group. We also note that $\langle a\rangle\subseteq\langle h_1\rangle$ by the condition $\langle a\rangle\lhd G$.

Since

$$[h_2, c] \in \langle h_2, b\rangle \cap \langle b, c\rangle = \left\langle a^{2^{n-1}}, b\right\rangle$$

and $\left[h_2, c^2\right] = 1$, we conclude that $[h_2, c] \in \left\langle a^{2^{n-1}}\right\rangle$. Then one of the elements $h_2c$ or $h_2bc$ is of order 2, and hence one of the subgroups $\left\langle h_2c, a^{2^{n-1}}\right\rangle$ or $\left\langle h_2bc, a^{2^{n-1}}\right\rangle$ is elementary Abelian. Since $\langle a\rangle \subseteq N_G^A$, the element $a$ has to normalize these subgroups, which is possible only if $|a| = 4$.

Based on the fact that $\langle h_1h_2\rangle \times \langle b\rangle$ is an Abelian non-cyclic subgroup, we have

$$[h_1h_2, c] \in (\langle h_1h_2\rangle \times \langle b\rangle) \cap N_G^A = \left\langle a^2, b\right\rangle.$$

It is easy to prove that $[h_1h_2, c] \in \left\langle a^2\right\rangle$ by Lemma 4. It also follows that $[h_1, c] \in \left\langle a^2\right\rangle$. Thus $\left[H,\ N_G^A\right] = \left\langle a^2\right\rangle$.

Let's denote $B = \langle b, c\rangle$. Since $B$ is a 2-generated non-Abelian subgroup and the commutant $[B, G] \subseteq \left\langle a^2\right\rangle$ is of order 2, we have $G = BC_G(B)$ by [10]. We can assume without loss of generality that $H = C_G(B)$. If $|H| = 8$, then $G$ is an $\overline{HA_2}$-group, which contradicts the assumption. So $|H| > 8$ and $G$ is a group of type (3) of Theorem 3. □

**Lemma 15.** *If a finite 2-group $G$ has the norm $N_G^A \neq G$ which is a group of type (9) of Proposition 2, then $G$ is a group of type (4) of Theorem 3.*

*Proof.* Let $N_G^A$ be a group of type (9) of Proposition 2: $N_G^A = \langle c\rangle \leftthreetimes H$, where $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4, h_1^2 = h_2^2 = [h_1, h_2]$, $|c| = 2^n > 2$, $[c, h_1] = c^{2^{n-1}}$, $[c, h_2] = 1$.

Suppose that $N_G^A \neq G$. Since $\omega\left(N_G^A\right) \subset Z\left(N_G^A\right)$ and $\omega\left(N_G^A\right) \not\subset Z(G)$, we have $\omega(G) = \omega\left(N_G^A\right)$ by Lemma 2 and $G = C\langle y\rangle$, where $C = C_G\left(\omega\left(N_G^A\right)\right) \lhd G$, $y^2 \in C$, $|y| > 4$ by Lemma 4. The group $C$ contains all Abelian non-cyclic subgroups of the group $G$, so

$$N_G^A \subseteq N_C^A \subseteq C.$$

Thus $C$ is a 2-group which has the norm of Abelian non-cyclic subgroups of type (9) of Proposition 2 and the non-cyclic center. We conclude that $C$ is an $\overline{HA_2}$-group and

$$C = N_G^A = \langle c\rangle \leftthreetimes H$$

by Theorem 2. Thus

$$G = C\langle y\rangle = (\langle c\rangle \leftthreetimes H)\langle y\rangle, |y| > 4, y^2 \in C.$$

Let $|y| = 2^k$. Since $y \notin C, \omega(G) \cap \langle y\rangle \subseteq Z(G)$. Let's denote $\langle a_1\rangle = \omega(G) \cap \langle y\rangle$ and consider the quotient-group

$$\overline{G} = G/\omega(G) \cong \overline{C}\langle\overline{y}\rangle.$$

Since the lower layer $\omega\left(\overline{C}\right)$ is an elementary Abelian subgroup of order 8 and $\omega\left(\overline{C}\right) \lhd \overline{G}$, we conclude that $\omega\left(\overline{C}\right)$ contains an involution $\overline{z}$ such that $[\overline{z}, \overline{y}] = \overline{1}$,

$\langle\overline{z}\rangle \cap \langle\overline{y}\rangle = \overline{E}$. Turning to the preimages we put $[z, y] = a$, where $|a| = 2$, $a \in \omega(G)$. Then $\left[z^2, y\right] = 1$ and $z^2 = a_1 \in Z(G)$. If $a \in Z(G)$, then $\left[z, y^2\right] = 1$, $\left|y^{2^{k-2}}z\right| = 2$, which is impossible, because the elements of the order 4 of $N_G^A$ do not have such property. Thus $a \notin Z(G)$ and $\left[z, y^2\right] = a_1$. It follows that $\left\langle z, y^2\right\rangle$ is the quaternion group and $|y| = 8$.

If $|c| > 4$, then $a_1 = c^{2^{n-1}} \in Z(G)$ and $c^{2^{n-1}} \in \left\langle z, y^2\right\rangle$. But any quaternion group in $N_G^A$ does not contain $c^{2^{n-1}}$. This means that $|c| = 4$, $c^2 \notin Z(G)$ and $a_1 = h_1^2 \in \left\langle z, y^2\right\rangle$. Taking into account the structure of the quaternion subgroups in $N_G^A$, we have $\left\langle z, y^2\right\rangle = \left\langle h_2 c^{2m}, h_1 h_2^l c^s\right\rangle$.

Suppose that $\left\langle y^2\right\rangle \lhd G$. Then we can assume that $y^2 = h_2 c^{2m}$, $z = h_1 h_2^l c^s$. Let's consider the quotient-group

$$\widetilde{G} = G/\left\langle y^2\right\rangle \cong \left(\langle\widetilde{c}\rangle \leftthreetimes \left\langle\widetilde{h_1}\right\rangle\right) \leftthreetimes \langle\widetilde{y}\rangle .$$

Since $\langle\widetilde{c}\rangle$ is a characteristic subgroup in $\widetilde{N_G^A}$, $\langle\widetilde{c}\rangle \lhd \widetilde{G}$ and $[\widetilde{c}, \widetilde{y}] \in \left\langle\widetilde{c}^2\right\rangle$. Turning to the preimages we have $[c, y] = c^{2r} y^{2i}$. So $\left[c^2, y\right] = h_2^{2i} \neq 1$ and $i \equiv 1 \pmod 2$. It is easy to verify that in this case $|cy| \le 4$, which contradicts the proved.

Thus $\left\langle y^2\right\rangle \ntriangleleft G$. Then we can assume that $y^2 = h_1 h_2^l c^s$ and $z = h_2 c^{2m}$, respectively. Let's consider the quotient-group

$$\overline{G} = G/\omega(G) \cong \left(\langle\overline{c}\rangle \times \left\langle\overline{h_1}\right\rangle \times \left\langle\overline{h_2}\right\rangle\right)\langle\overline{y}\rangle .$$

Without loss of generality, $\langle\overline{y}\rangle \cap \overline{N}_G^A = \left\langle\overline{h_1}\right\rangle$ and $\overline{z} = \overline{h_2}$. Then $[\overline{y}, \overline{z}] = \left[\overline{y}, \overline{h_2}\right] = \overline{1}$ according to the choice of $\overline{z}$. We get

$$\left[\langle\overline{y}\rangle, \overline{N}_G^A\right] \subseteq \overline{N}_G^A \cap \left\langle\overline{y}, \overline{h_2}\right\rangle = \left\langle\overline{y}^2, \overline{h_2}\right\rangle = \overline{H}$$

by the condition $\left\langle\overline{y}, \overline{h_2}\right\rangle \lhd \overline{G}$. Thus $[y, h_2] = c^{2l} h_1^{2s}$ and $[y, c] = c^{2l_1} h_1^m h_2^r$. We have $l \not\equiv 0 \pmod 2$ by the first equality and the condition $\left[y, c^2\right] \neq 1$. We have $m \equiv 0 \pmod 2$ and $r \not\equiv 0 \pmod 2$ by the second equality and the condition $\left[y, c^2\right] \neq 1$. Thus $[y, h_2] = c^2 h_1^{2s}$ and $[y, c] = c^{2l} h_2^{\pm 1}$. Further $l_1 \equiv s \pmod 2$, because $\left[y^2, c\right] = c^2$.

We can assume without loss of generality that

$$G = C\langle y\rangle = (\langle c\rangle \leftthreetimes H)\langle y\rangle ,$$

where $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4$, $h_1^2 = h_2^2 = [h_1, h_2]$, $|c| = 4$, $[c, h_1] = c^2$, $[c, h_2] = 1$, $y^2 = h_1$, $[y, h_2] = c^2 h_1^2$, $[y, c] = h_2^{\pm 1}$. In this group all Abelian non-cyclic subgroups are contained in $\langle c\rangle \leftthreetimes H$ and are normalized by this subgroup. At the same time $y \notin N_G^A$, because $y \notin N_G\left(\left\langle c, h_1^2\right\rangle\right)$. $\square$

Theorem is proved. $\square$

**Corollary 7.** *Any group $G$ of type $\beta$ is a cyclic or metacyclic extension of the norm $N_G^A$.*

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