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n-Torsion Regular Rings

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Abstract. As proper subclasses of the classes of unit-regular and strongly regular rings, respectively, the two new classes of n-torsion regular rings and strongly n-torsion regular rings are introduced and investigated for any natural number n. Their complete isomorphism classification is given as well. More concretely, although it has been recently shown by Nielsen-Šter (TAMS, 2018) that unit-regular rings need not be strongly clean, the rather curious fact that, for each positive odd integer n, the n-torsion regular rings are always strongly clean is proved.

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Introduction and Fundamentals

Everywhere in the text of the present article, let all rings be assumed associative with unity. Our standard notations and notions are in agreement with those from [18], [20] and [25]. For instance, for such a ring R, $U(R)$ denotes the group of all units, $Nil(R)$ the set of all nilpotents, $\mathrm{Id}(R)$ the set of all idempotents and $J(R)$ the Jacobson radical of R, respectively. Besides, the finite field with m elements will be denoted by $\mathbb{F}_m$; $m \in \mathbb{N}$ – the set of all naturals. For an element g of a group G, the letter $o(g)$ will denote the order of g. The symbol $LCM(n_1, \ldots, n_k)$ will be reserved for the *least common multiple* of $n_1, \ldots, n_k \in \mathbb{N}$; $k \in \mathbb{N}$.

About the more specific terminology, let us remember that a ring R is called *unit-regular* in [16] if, for every $r \in R$, there is $u \in U(R)$ with $r = rur$. If, however, $r = r^2u$, the ring R is called *strongly regular*. It is well known that strongly regular rings are exactly the reduced unit-regular rings or, in other words, the abelian unit-regular rings as being a subdirect product of division rings. In [8] it was also shown that strongly regular rings are precisely the pseudo uniquely unit-regular rings.

At the same time, a ring R is called *clean* in [21] provided that, for every $r \in R$, there are $u \in U(R)$ and $e \in \mathrm{Id(R)}$ such that $r = u + e$. It was established in [2] that unit-regular rings are always clean. If, in addition, $ue = eu$, the clean ring R is called *strongly clean*. Surprisingly, in [23] a mysterious matrix example o f a unit-regular ring which is *not* strongly clean was constructed.

Besides, a ring R is said to be *n-torsion clean* in [13] if, for each $r \in R$, there exist $u \in U(R)$ and $e \in \mathrm{Id(R)}$ such that $r = u + e$ and $u^n = 1$ with n being the smallest possible having this property. If, in addition, $ue = eu$, the n-torsion clean ring is said to be *strongly n-torsion clean.*

The aim of this article is to investigate in detail the following two newly defined proper subclasses of unit-regular rings and (strongly) n-torsion clean rings, respectively, by finding their close relationship.

Definition 1. A ring R is said to be *n-torsion regular* if there is $n \in \mathbb{N}$ and, for each element r of R, there exists $u \in U(R)$ such that $r = rur$ with $u^n = 1$ and n is the minimal possible having this property.

Without the restriction of minimality, the ring will just be called *almost n-torsion regular*.

Note that these rings form a proper subclass of the class of all unit-regular rings. It is clear that every unit-regular ring possessing unit group of bounded exponent n for some $n \in \mathbb{N}$, that is, $U^n(R) = \{1\}$, is (almost) n-torsion regular. Even something more: $o(w) \leq n$ for every $w \in U(R)$. In fact, one can write that $w^{-1} = w^{-1}uw^{-1}$ for some $u \in U(R)$ with $u^n = 1$ (and minimal n for all possible decompositions). Thus $w = u$ and $o(w)/n$, whence $o(w) \leq n$, as claimed.

Definition 2. A ring R is said to be *strongly n-torsion regular* if there is $n \in \mathbb{N}$ and, for each element r of R, there exists $u \in U(R)$ such that $r = r^2u$ with $u^n = 1$ and n is the minimal possible having this property.

Without the restriction of minimality, the ring will just be called *almost strongly n-torsion regular*.

Notice that these rings form a proper subclass of the class of all strongly regular rings which, as aforementioned, are known to be a subdirect product of division rings. It is plainly seen that boolean rings are precisely the rings which are strongly 1-torsion regular. Thus the introduced above classes of rings can be treated as a natural generalization of boolean rings. Moreover, one sees that the (almost) strongly n-torsion regular rings are actually the commutative (almost) n-torsion regular ones. In fact, for any $r \in R$, one may have the sequence of equalities $r = r^2u = r^3u^2 = \cdots = r^{n+1}u^n = r^{n+1}$ and so the utilization of the famous classical Jacobson's Theorem substantiates our claim.

Let us notice that in [9] the classes of *invo-regular* and *strongly invo-regular* rings were investigated. In our terminology, (strongly) invo-regular rings are precisely rings which are either (strongly) 1-torsion regular or (strongly) 2-torsion regular or, in other words, they are almost 2-torsion regular removing the condition on minimality.

1 Preliminaries and Examples

Imitating [13], for some arbitrary but fixed $n \in \mathbb{N}$ we shall say that a ring R is *almost n-torsion clean* if, for every $r \in R$, there exists $u \in U(R)$ with $u^n = 1$ and there exists $e \in \mathrm{Id}(R)$ such that $r = u + e$. If, in addition, the elements u and e commute, R is said to be *almost strongly n-torsion clean.* The case $n = 2$ was settled in detail in [5] under the name *invo-clean* rings. Precisely, there was proved that invo-regular rings are, actually, invo-clean (see also [7] and [9]).

Proposition 1. *All almost n-torsion regular rings are almost m-torsion clean for some $m \leq n$.*

Proof. Suppose that R is almost n-torsion regular, then R is unit-regular and, as noted above, R has to be clean. But it is readily checked as above that in almost n-torsion regular rings all units are n-bounded. Therefore, R is m-torsion clean for some natural m less than or equal to n, as asserted. $\square$

Proposition 2. *All almost strongly n-torsion regular rings are almost strongly n-torsion clean for some $m \leq n$.*

Proof. Similar arguments as those from Proposition 1 work to get the pursued assertion. $\square$

As immediate commutative examples of n-torsion regular rings for a concrete $n \in \mathbb{N}$, a direct check shows that $\mathbb{F}_2$ is 1-torsion regular, $\mathbb{F}_3$ and $\mathbb{F}_3 \times \mathbb{F}_3$ are 2-torsion regular and $\mathbb{F}_5$ is 4-torsion regular. It is, however, very intriguing whether or *not* non-commutative examples do exist. As it will be hopefully showed in the sequel (compare with Theorem 1 stated and proved below), non-commutative examples do not exist in the case of odd number n, however.

2 Preliminaries and Results

We start here with the following somewhat surprising fact.

Proposition 3. *Let n be an odd natural. Then R is an almost n-torsion regular ring if and only if R is an almost strongly n-torsion regular ring.*

Proof. Since the right-to-left implication is self-evident, we will be concentrating on the left-to-right one. First, one sees that $2 = 0$. In fact, writing $(-1) = (-1)w(-1)$ for some w satisfying $w^n = 1$, we infer that $w = -1$ and thus $(-1)^n = 1$, i.e., $-1 = 1$ which amounts to $2 = 0$, as expected.

What we intend to show next is that R is reduced (that is, all its nilpotents are zero) and thus abelian (that is, all its idempotents are central). To that purpose, given $q \in Nil(R)$, we deduce that $1 + q \in 1 + Nil(R) \subseteq U(R)$. So, by what we have commented above, $(1 + q)^n = 1$ and expanding this by the Newton's binomial formula, we derive that

$$(1 + q)^n = \sum_{i=0}^{n} \binom{n}{i} q^i.$$

But $\binom{n}{0} = \binom{n}{n} = 1$ as well as $\binom{n}{1} = n = 2k + 1$, which gives that

$$q + k_2 q^2 + \cdots + k_{n-1} q^{n-1} + q^n = 0,$$

where we put $k_i = \binom{n}{i} \in \mathbb{N}$ whenever $i = 2, \cdots, n - 1$.

Finally, one detects that $q(1+k_2q+\cdots+k_{n-1}q^{n-2}+q^{n-1})=0$. Since the element in the brackets is obviously invertible, we conclude that $q=0$, as required. This substantiates our claim that $Nil(R)=\{0\}$.

Furthermore, since for any $r\in R$ it must be that $r=rur$ for some n-torsion unit u, and since ru is obviously an idempotent, whence by what we have already proved above it is a central one, it follows that $r=(ru)r=r(ru)=r^2u$, as required. $\square$

Concerning the even case for n, it was established in [9] that almost 2-torsion regular rings are almost strongly 2-torsion regular. We will now somewhat strengthen this affirmation in the case when $n=4$.

Proposition 4. *Suppose R is an almost 4-torsion regular ring. Then R is almost strongly 4-torsion regular.*

Proof. If $2=0$, we may adapt the idea from [9] to get the wanted claim. In fact, for any $u\in U(R)$, we have that $u^4=1$ and hence $u=1+(u-1)$ with $(u-1)^4=u^4-1=0$. Consequently, $U(R)=1+Nil(R)$ and the application of [3] or [12] assures that R must be boolean.

So, we shall assume that $2\neq 0$. Writing $2=2v2=4v$, for some v in R having $v^4=1$, and squaring, we obtain that $16=256$, that is, $240=2^4.3.5=0$. Since $(2^4,3,5)=1$, an easy trick ensures that $R\cong R_1\times R_2\times R_3$, where R_1,R_2,R_3 are either zero rings (*not* necessarily simultaneously) or almost 4-torsion regular rings of characteristic 2, 3 and 5, respectively. We shall now examine the three possible cases separately:

Case 1: $2=0$. By what we have just shown, R_1 is boolean (and so commutative).

Case 2: $3=0$. As above, choosing $q\in Nil(R_2)$, we have $(1+q)^4=1$, i.e., $q^4+q^3+q=0$. Therefore, $q(1+q^2+q^3)=0$ which is tantamount to $q=0$ as $1+q^2+q^3$ inverts in R_2. Thus R_2 being reduced is abelian, and hence it has to be almost strongly 4-torsion regular.

Case 3: $5=0$. Same as above, for $t\in Nil(R_3)$, we obtain $(1+t)^4=1$, i.e., $t^4-t^3+t^2-t=0$. Consequently, $t(-1+t-t^2+t^3)=0$ which is equivalent to $t=0$ as $-1+t-t^2+t^3$ inverts in R_3. Thus R_2 will be reduced and thus abelian, and hence it must be almost strongly 4-torsion regular.

Finally, after all procedures done, R is almost strongly 4-torsion regular, as promised. $\square$

An important question which immediately arises is of whether or *not* the last statement can be strengthened to an arbitrary natural n.

We continue with one more useful and applicable technicality.

Lemma 1. *Let $n\in\mathbb{N}$ and let R be a ring whose elements satisfy the identity $x^{n+1}=x$, while $x^{k+1}\neq x$ for some x, provided $k<n$ and $k\in\mathbb{N}$, that is, for every $k<n$ there exists x in R for which x^{k+1} is not equal to x. The next three items are then true:*

1. *R is reduced (i.e., $Nil(R) = \{0\}$);*

2. *R is semiprimitive (i.e., $J(R) = \{0\}$);*

3. *If R is primitive, then $n = p^m - 1$ for some $m \in \mathbb{N}$ and R is a field with p^m elements.*

Proof. Items (1) and (2) are rather obvious, which follow directly from the condition $x^{n+1} = x$, so we omit their verification. The third item is an immediate consequence of the fact that R is a PI-ring and of the significant classical Kaplansky's Theorem by using the method presented in details in [13]. □

The next statement sheds some additional light on the well-known characterization of rings whose elements satisfy the equation $x^{n+1} = x$ (compare also with [24] and [10] for a more account).

Corollary 1. *Suppose that $n \in \mathbb{N}$. Then, for a ring R, the following two conditions are equivalent:*

1. *R satisfies the equation $x^{n+1} = x$.*

2. *R is a subdirect product of finite fields $\mathbb{F}_{p_k^{m_k}}$ for some primes p_k and integers m_k, $k \in \mathbb{N}$, where $(p_k^{m_k} - 1)/n$ for each k.*

Proof. "(1) $\Rightarrow$ (2)". With Lemma 1 at hand, R is a subdirect product of finite fields F_i satisfying the equality $x^{n+1} = x$. Let us fix such a field F with p^m elements. It is then well known that $U(F)$ is a cyclic group of order $p^m - 1$ which satisfies the identity $x^n = 1$. Thus $p^m - 1$ divides n.

"(2) $\Rightarrow$ (1)". Letting R be a subdirect product of the required fields F_i, we then easily see that each such field satisfies $x^{n+1} = x$, whence R will also satisfy this identity. □

By the same token, we can derive the following consequence.

Corollary 2. *Suppose $n \in \mathbb{N}$. Then, for a ring R, the following two conditions are tantamount:*

1. *R satisfies the equation $x^{n+1} = x$ with n minimal possible.*

2. *R is a subdirect product of finite fields $\mathbb{F}_{p_k^{m_k}}$ for some primes p_k and integers m_k, $k \in \mathbb{N}$, where $(p_k^{m_k} - 1)/n$ for each k, and $n = LCM(p_k^{m_k} - 1 \mid k \in \mathbb{N})$ provided n is not a prime integer.*

Some more clarifications to the quoted above statements are too needed. We do that in the next statement.

Remark 1. For an arbitrary but fixed natural n, let $\mathrm{J}(n)$ be the class of rings which satisfy the identity $x^{n+1} = x$, let $\mathrm{TR}(n)$ be the class of almost strongly n-torsion regular rings (i.e., as in Definition 2, every element can be presented in the form $r = r^2u = ur^2$ with $u^n = 1$), and let $\mathrm{TC}(n)$ be the class of strongly clean rings such that every element can be presented as $e + u$ with $e^2 = e$ and $u^n = 1$, where e and u commute (see [13] for a more precise information). *Then the relations $J(n) = TR(n) \subseteq TC(n)$ are valid.* In fact, the inclusion $\mathrm{TR}(n) \subseteq \mathrm{J}(n)$ is clear and also it was already obtained above, whereas the converse containment is guaranteed with the aid of Corollary 1. Thus $\mathrm{J}(n) = \mathrm{TR}(n)$ holds, indeed. The same corollary also shows that $\mathrm{TR}(n) \subseteq \mathrm{TC}(n)$ (as well as, without any difficulty, Corollary 2 gives that the same relations will also be fulfilled additionally assuming that n is minimal).

On the other hand, in order to confirm the above relationships by using an element-wise presentation, for any element x the equality $x = x^{n+1}$ forces that $x = x^2y = yx^2$, where $y = x^n + x^{n-1} - 1$ and possibly $n \geq 2$. It is not too hard to check that $y^2 = x^{2n-2} - x^n + 1$ and so $y^2 = 1$ when $n = 2$. However, $y^3 = 2x^3 - 1 \neq 1$ when $n = 3$, provided $x^3 \neq 1$ and $2 \neq 0$, etc., which unambiguously demonstrates the complication of the general situation. Nevertheless, the utilization of [22] is a guarantor that y could be chosen to be a unit and so $y^n = 1$ because $y^{n+1} = y$. About the minimality, as observed above, $x = x^2y$ with $y^m = 1$ for some $m < n$ will imply by iteration that $x = x^{m+1}y^m = x^{m+1}$ which contradicts that $x = x^{m+1}$ is *not* always an identity, that is, there exists some $z \in R$ such that $z \neq z^{m+1}$.

We are now ready to proceed to proving our main structural result, which is the following criterion:

Theorem 1. *Suppose $n \in \mathbb{N}$ is odd and R is a ring. Then the next two items are equivalent:*

(i) R is n-torsion regular.

(ii) R is strongly n-torsion regular.

Proof. The implication (i) $\Leftarrow$ (ii) is straightforward. To show the reverse part (i) $\Rightarrow$ (ii), we first will take into account the established above in Proposition 3 crucial fact that almost n-torsion regular rings are, actually, almost strongly n-torsion regular. That the minimality condition on n is trivially satisfied, now follows elementarily by a direct verification. □

For an arbitrary positive integer n, we can say slightly more in the "strongly" case. Specifically, the following is true:

Theorem 2. *Suppose that $n \in \mathbb{N}$. Then a ring R is strongly n-torsion regular $\iff$ R is a subdirect product of finite fields $\mathbb{F}_{p_k^{m_k}}$ for some primes p_k and integers m_k, $k \in \mathbb{N}$, where $(p_k^{m_k} - 1)/n$ for each k, and $n = LCM(p_k^{m_k} - 1 \mid k \in \mathbb{N})$ provided n is not a prime integer.*

Proof. It follows directly from Corollary 2 (2) in view of the comments in Remark 1 quoted above. □

As an immediate important consequence, in sharp contrast to [23], one yields the following:

Corollary 3. *For any odd $n \in \mathbb{N}$, all (almost) n-torsion regular rings are strongly regular, and thus they are strongly clean.*

3 Concluding Discussion and Open Problems

In conclusion, let us give some more detailed comments and pose a few problems of certain interest and importance:

In [3] the next class of rings was considered – for any $r \in R$, there exist $q \in Nil(R)$ and $e \in \mathrm{Id}(R)$ such that $r = r(q+e)r$. Besides, it was asked whether or *not* these rings are *nil-clean* in the sense of Diesl ([15]), saying that every element is the sum of a nilpotent and an idempotent.

In this direction, we shall now put into consideration the following generalization of invo-regular rings from [9]: Let, for each $r \in R$, there be $v \in U(R)$ with $o(v) \leq 2$ and $e \in \mathrm{Id}(R)$ such that $r = r(v+e)r$. We assert that such a ring R can be decomposed as the direct product of a ring of characteristic 2 and of a ring of characteristic 3. Indeed, writing $2 = 4(v+e) = 4v+4e$, and so squaring $2-4v = 4e$, we deduce that $12 = 0$. Since $4 = 0 \iff 2 = 0$ (in fact, $4 = 2^2 = 0$ yields that $2 \in J(R) = \{0\}$ as being a central nilpotent and taking into account that R is von Neumann regular). Now, the Chinese Remainder Theorem gives the desired decomposition, say $R \cong P \times L$. Moreover, it is clear that all units are of the type $v + e$, so that if t is an arbitrary nilpotent, then $1 + t$ is a unit and it must be that $v = t + (1 - e)$. The *Lemma on Involutions* (see, e.g., [4] and [5]) applies now to get that $1 - e = 1$, i.e., $e = 0$. Consequently, $t = v - 1$ and it follows that $t^2 = 0$ when $2 = 0$ and $t = 0$ when $3 = 0$; to see the latter, one has that $t^3 = (v-1)^3 = v^3 - 1 = v - 1 = t$ whence $t(t^2-1) = 0$ insures that $t = 0$ because $t^2 - 1$ is obviously a unit. Therefore, to look at the direct factor P having characteristic 2, we can present $v+e = (v+1)+(1+e) \in Nil(P)+\mathrm{Id}(P)$, where $(v+1)^2 = v^2+1 = 0$. So, in parallel to the above commentary, pertaining to [3], one reasonably may ask whether it is nil-clean. As for the direct factor L having characteristic 3, it is necessarily abelian being reduced and so any its element x satisfies the equation $x = x^2(v+e) = (v+e)x^2$ since $x(v+e)$ and $(v+e)x$ are both idempotents in L. That is why, L is a strongly regular ring and hence it is a subdirect product of division rings. One may suspect that it could be embedded in the direct product $\prod_\mu \mathbb{Z}_3$ for some ordinal μ. Furthermore, if in addition we generally have that $ve = ev$, then we may obtain a complete characterization like this: For any unit u of the ring P we have that $u = v+e$ will imply $u^2 = 1+e \in U(P) \cap \mathrm{Id}(P) = \{1\}$. Therefore, $(u-1)^2 = 0$, so that $u \in 1 + Nil(R)$ whence $U(R) = 1 + Nil(R)$, and thus, bearing in mind that P is exchange (as it is von Neumann regular), we can successfully employ [3] or [12] (see also [6] or [11]) to get the desired description of P. As for L, one observes for any $u = v+e \in L$ that $u^3 = (v+e)^3 = v^3+e^3 = v+e = u$, so that $u^2 = 1$ holds again and we can process as in [6] or [11] taking into account that L is also exchange being von Neumann regular. In addition, $U^2R = \{1\}$ and R is von Neumann regular (and hence exchange).

On the other vein, in [10] and [24] were classified those rings R whose elements x satisfy the equation $x^n = x$ for some arbitrary fixed $n \in \mathbb{N}$ with $n > 1$. This was considerably extended in [10] to rings satisfying the more general polynomial identity $x^n = \pm x$. As these things are closely related to our considerations above, we will now consider those rings R for which $x^n - x \in Nil(R)$. According to [19, Theorem A.1], $Nil(R) \lhd R$ whenever $n \not\equiv 1(mod\ 3)$ and $n \not\equiv 1(mod\ 8)$ – note that by the paramount Dirichlet's Theorem there exists an infinite number of primes having that property. Consequently, in the factor-ring $R/Nil(R)$ the equation $x^n = x$ is true for all its elements, so by what we have noted above we can characterize such rings R. Similarly, one can treat the rings R for which $x^n \pm x \in Nil(R)$. Certainly, of some interest and importance is the description of these rings when n does not possess these two limitations, but this will be the theme of some other research work where a new approach is in use.

We close the work with several challenging questions.

In regard to Corollary 3, one states the following:

Problem 1. Are (almost) n-torsion regular rings strongly clean for any $n \in \mathbb{N}$?

Observing that Proposition 3 makes sense in that way when n is odd, so the question remains left-open when n is even. However, in [9] this was answered in the affirmative when $n = 2$ and in Proposition 4 when $n = 4$.

Problem 2. Does it follow that a ring is unit-regular if and only if it is both von Neumann regular and clean?

It was proved in [16] that semiprimitive Artinian rings are always unit-regular. Also, it is principally known that a ring is semiprimitive Artinian exactly when it is simultaneously von Neumann regular and Noetherian. So, in that aspect, one may ask the following:

Problem 3. Does it follow that a semiprimitive ring is Artinian if and only if it is both Noetherian and clean?

Recall once again that the definition of an n-torsion clean ring is given in [13].

Problem 4. Suppose n is an arbitrary natural. Is it true that a ring is n-torsion regular if and only if it is n-torsion clean and von Neumann regular (in particular, unit-regular)?

In order to expand the considered above generalized version of invo-regular rings, one can state:

Problem 5. Describe the following generalization of (almost) n-torsion regular rings: $\forall\ r \in R$, $\exists\ u \in U(R), u^n = 1, n \in \mathbb{N}$ and $\exists\ e \in \mathrm{Id}(R)$ such that $r = r(u + e)r$.

Since we are still in the class of von Neumann regular rings, a question that arises is: are these rings unit-regular and/or clean? As we already have indicated above, by [2] all unit-regular rings are always clean and this implication is not reversible in general.

On the other vein, let us recall that a ring R is called π-boolean if, for each $r \in R$, there exists $i \in \mathbb{N}$ such that $r^{2i} = r^i$. In view of the related problems posed in [3] and [12], one may ask the following:

Problem 6. Are π-boolean rings with nil-clean units also nil-clean?

Note that in such rings all units must be torsion.

Problem 7. Are von Neumann regular (in particular, unit-regular) rings with nil-clean inner (inverse) elements or even with nil-clean units also nil-clean? Likewise, is it true that von Neumann regular (in particular, unit-regular) rings with a finite number of inner (inverse) elements are (strongly) clean?

In that direction, the interested reader can see and consult [8] as well.

In regard to the aforementioned Diesl's *nil-clean* rings from [15], which were somewhat discussed also above, we finish off with the next expansion.

Problem 8. Describe the structure of those rings whose elements are sums or differences of a nilpotent of order ≤ 2 and an idempotent.

One observes that in such rings (in the general situation of a nilpotent they are known in the existing literature as *weakly nil clean rings* – see, for consultation, [14] and [1]), the index of nilpotence is at most 2. In fact, if $t = q \pm e$ is an arbitrary nilpotent, where $q^2 = 0$ and $e^2 = e$, then it follows from [17, Proposition 2] that $e = 0$ and so $t = q$ has exponent not exceeding 2, as claimed.

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