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Examples of bipartite graphs which are not cyclically-interval colorable

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Abstract. A proper edge t-coloring of an undirected, simple, finite, connected graph G is a coloring of its edges with colors $1, 2, ..., t$ such that all colors are used, and no two adjacent edges receive the same color. A cyclically-interval t-coloring of a graph G is a proper edge t-coloring of G such that for each its vertex x at least one of the following two conditions holds: a) the set of colors used on edges incident to x is an interval of integers, b) the set of colors not used on edges incident to x is an interval of integers. For any positive integer t, let $\mathfrak{M}_t$ be the set of graphs for which there exists a cyclically-interval t-coloring. Examples of bipartite graphs that do not belong to the class $\bigcup\limits_{t\geq 1} \mathfrak{M}_t$ are constructed.

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We consider undirected, simple, finite and connected graphs. For a graph G we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. For a graph G, we denote by $\Delta(G)$ and $\chi'(G)$ the maximum degree of a vertex of G and the chromatic index [1] of G, respectively. The terms and concepts which are not defined can be found in [2].

For an arbitrary finite set A, we denote by $|A|$ the number of elements of A. The set of positive integers is denoted by $\mathbb{N}$. An arbitrary nonempty finite subset of consecutive integers is called an interval. An interval with the minimum element p and the maximum element q is denoted by $[p, q]$.

For any $t \in \mathbb{N}$ and arbitrary integers i_1, i_2 satisfying the conditions $i_1 \in [1, t]$, $i_2 \in [1, t]$, we define [3] the sets $intcyc_1[(i_1, i_2), t]$, $intcyc_1((i_1, i_2), t)$, $intcyc_2((i_1, i_2), t)$, $intcyc_2[(i_1, i_2), t]$ as follows:

$$intcyc_1[(i_1, i_2), t] \equiv [\min\{i_1, i_2\}, \max\{i_1, i_2\}],$$

$$intcyc_1((i_1, i_2), t) \equiv intcyc_1[(i_1, i_2), t] \backslash (\{i_1\} \cup \{i_2\}),$$

$$intcyc_2((i_1, i_2), t) \equiv [1, t] \backslash intcyc_1[(i_1, i_2), t],$$

$$intcyc_2[(i_1, i_2), t] \equiv [1, t] \backslash intcyc_1((i_1, i_2), t).$$

If $t \in \mathbb{N}$ and Q is a non-empty subset of the set $\mathbb{N}$, then Q is called a t-cyclic interval if there exist integers i_1, i_2, j_0 satisfying the conditions $i_1 \in [1, t]$, $i_2 \in [1, t]$, $j_0 \in \{1, 2\}$, $Q = intcyc_{j_0}[(i_1, i_2), t]$.

A function $\varphi : E(G) \to [1,t]$ is called a proper edge t-coloring of a graph G if adjacent edges are colored differently and each of t colors is used.

For a graph G and any integer t, where $t \in [\chi'(G), |E(G)|]$, we denote by $\alpha(G,t)$ the set of all proper edge t-colorings of G. Let us set

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)}^{|E(G)|} \alpha(G,t).$$

If G is a graph, $\varphi \in \alpha(G)$, and $x \in V(G)$, then the set $\{\varphi(e)/e \in E(G), e \text{ is incident to } x\}$ is denoted by $S_G(x,\varphi)$.

A proper edge t-coloring of a graph G is called a cyclically-interval t-coloring if for any $x \in V(G)$ at least one of the following two conditions holds: a) $S_G(x,\varphi)$ is an interval, b) $\{1,...,t\} \backslash S_G(x,\varphi)$ is an interval.

For any $t \in \mathbb{N}$, let $\mathfrak{M}_t$ be the set of graphs for which there exists a cyclically-interval t-coloring. Let

$$\mathfrak{M} \equiv \bigcup_{t \geq 1} \mathfrak{M}_t.$$

The concept of cyclically-interval edge coloring of graphs was introduced by de Werra and Solot [18] in 1991 and motivated by scheduling problems arising in flexible manufacturing systems. In [18], it was proved that all outerplanar bipartite graphs have cyclically-interval edge colorings. Kubale and Nadolski [14] proved that the problem of determining whether a given bipartite graph admits a cyclically-interval edge coloring is NP-complete. Nadolski [15] showed that all subcubic graphs are cyclically-interval colorable. He also constructed first examples of graphs that are not cyclically-interval colorable. In [3–5], all possible values of t for which simple cycles and trees have a cyclically-interval t-coloring eve determined. Petrosyan and Mkhitaryan [21] showed that all complete tripartite graphs are cyclically-interval colorable and conjectured that the same holds for all complete multipartite graphs. Recently, this conjecture was confirmed by Asratian, Casselgren and Petrosyan [23]. Some results on cyclically-interval edge colorings of bipartite graphs were obtained in [20–23]. Some other interesting results on this and related topics were obtained in [6–11, 13–24].

In this paper, the examples of bipartite graphs that do not belong to the class $\mathfrak{M}$ are constructed.

The result was announced in [12].

For any integer $m \geq 2$, set:

$$V_{0,m} \equiv \{x_0\}, V_{1,m} \equiv \{x_{i,j}/1 \leq i < j \leq m\},$$

$$V_{2,m} \equiv \{y_{p,q}/1 \leq p \leq m, 1 \leq q \leq m\},$$

$$E'_m \equiv \{(x_0, y_{p,q})/1 \leq p \leq m, 1 \leq q \leq m\}.$$

For any integers i, j, m satisfying the inequalities $m \geq 2$, $1 \leq i < j \leq m$, set:

$$E''_{i,j,m} \equiv \{(x_{i,j}, y_{i,q})/1 \leq q \leq m\} \cup \{(x_{i,j}, y_{j,q})/1 \leq q \leq m\}.$$

For any integer $m \geq 2$, let us define a graph $G(m)$ by the following way:

$$G(m) \equiv \left(\bigcup_{k=0}^{2} V_{k,m}, E'_m \cup \left(\bigcup_{1 \leq i < j \leq m} E''_{i,j,m} \right) \right).$$

It is not difficult to see that for any integer $m \geq 2$, $G(m)$ is a bipartite graph with $\Delta(G(m)) = \chi'(G(m)) = m^2$, $|V(G(m))| = \frac{3m^2-m}{2} + 1$, $|E(G(m))| = m^3$.

Theorem 1. *For any integer $m \geq 8$, $G(m) \notin \mathfrak{M}$.*

Proof. Assume the contrary. It means that there exist integers m_0, t_0, k_0 satisfying the conditions $m_0 \geq 8$, $m_0^2 \leq t_0 \leq m_0^3$, $t_0 = m_0^2 + k_0$, $0 \leq k_0 \leq m_0^3 - m_0^2$, $G(m_0) \in \mathfrak{M}_{t_0}$.

Let φ_0 be a cyclically-interval t_0-coloring of the graph $G(m_0)$. Without loss of generality, we can suppose that $S_{G(m_0)}(x_0, \varphi_0) = [1, m_0^2]$. Let us consider the edges e' and e'' of the graph $G(m_0)$, which are incident to the vertex x_0 and satisfy the equalities $\varphi_0(e') = 1$, $\varphi_0(e'') = \lfloor \frac{m_0^2}{2} \rfloor$.

Suppose that $e' = (x_0, y')$, $e'' = (x_0, y'')$. Clearly, there exists a vertex $\tilde{x} \in V_{1,m_0}$ in the graph $G(m_0)$ which is adjacent to the vertices y' and y''. Lemma 1 from [3] implies that $S_{G(m_0)}(y', \varphi_0) \cup S_{G(m_0)}(\tilde{x}, \varphi_0) \cup S_{G(m_0)}(y'', \varphi_0)$ is a t_0-cyclic interval.

Clearly, the inequalities $m_0^2 + k_0 - 4m_0 + 4 > 4m_0 - 2$ and $4m_0 - 1 \leq \lfloor \frac{m_0^2}{2} \rfloor \leq m_0^2 + k_0 - 4m_0 + 3$ are true. Consequently,

$$\left\lfloor \frac{m_0^2}{2} \right\rfloor \in intcyc_1((4m_0 - 2, m_0^2 + k_0 - 4m_0 + 4), m_0^2 + k_0).$$

But it is incompatible with the evident relations $\lfloor \frac{m_0^2}{2} \rfloor \in S_{G(m_0)}(y'', \varphi_0)$ and

$$S_{G(m_0)}(y', \varphi_0) \cup S_{G(m_0)}(\tilde{x}, \varphi_0) \cup S_{G(m_0)}(y'', \varphi_0) \subseteq$$
$$\subseteq intcyc_2[(4m_0 - 2, m_0^2 + k_0 - 4m_0 + 4), m_0^2 + k_0].$$

The proof of the theorem is complete. □

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