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A note on almost contact metric 2- and 3-hypersurfaces in W_4-manifolds

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Abstract. It is proved that 2-hypersurfaces and 3-hypersurfaces of W_4-manifolds admit identical almost contact metric structures.

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1 Introduction

The class of W_4-manifolds is one of so-called small Gray–Hervella classes [7] of almost Hermitian manifolds. Some specialists identify this class with the class of locally conformal Kahlerian (LCK-) manifolds that is not quite correct. In fact, the W_4 class contains all locally conformal Kählerian manifolds, but it coincides with the class of LCK-manifolds only for dimension at least six [8]. W_4-manifolds were studied in detail from different points of view by such outstanding mathematicians as Alfred Gray (USA), Vadim Feodorovich Kirichenko (Russian Federation) and Izu Vaisman (Israel).

As it is known, almost contact metric structures are induced on oriented hypersurfaces of an almost Hermitian manifold. In [5] it has been proved that if the almost Hermitian manifold belongs to the class W_4 and the type number of its hypersurface is equal to one, then the almost contact metric structure on such a hypersurface is identical to the structure on a totally geodesic hypersurface. The similar results were obtained for 0- and 1-hypersurfaces of W_1- and W_3-manifolds [2, 4]. Namely, it has been proved that the almost contact metric structures on 0- and 1-hypersurfaces in W_1-manifolds and in W_3-manifolds (i.e. in nearly Kählerian manifolds and in special Hermitian manifolds) are identical. We also distinguish some results on 2-hypersurfaces of Kählerian and nearly Kählerian manifolds [1, 3].

In this paper we consider some hypersurfaces with type number 2 and 3 in W_4-manifolds. The main result of the present note is the following:

Theorem 1. *3-hypersurfaces of W_4-manifolds admit almost contact metric structures that are identical to the structures induced on 2-hypersurfaces of such manifolds.*

2 Preliminaries

We remind that the almost almost contact metric structure on an odd-dimensional manifold N is defined by the system of tensor fields $\{\Phi,\ \xi,\ \eta,\ g\}$ on this manifold, where ξ is a vector field, η is a covector field, Φ is a tensor of the type (1, 1) and $g = \langle\cdot,\ \cdot\rangle$ is the Riemannian metric [8]. Moreover, the following conditions are fulfilled:

$$\eta(\xi) = 1,\ \Phi(\xi) = 0,\ \eta \circ \Phi = 0,\ \Phi^2 = -id + \xi \otimes \eta,$$

$$\langle \Phi X, \Phi Y\rangle = \langle X, Y\rangle - \eta(X)\,\eta(Y)\,,\ X, Y \in \aleph(N),$$

where $\aleph(N)$ is the module of smooth vector fields on N. As the most important examples of almost contact metric structures we can mark out the cosymplectic structure, the nearly cosymplectic structure (or the Endo structure), the Sasakian structure and the Kenmotsu structure.

We remind also that an almost Hermitian manifold is an even-dimensional manifold M^{2n} equipped with a Riemannian metric $g = \langle\cdot,\ \cdot\rangle$ and an almost complex structure J. These objects must satisfy the following condition:

$$\langle JX,\ JY\rangle = \langle X,\ Y\rangle\,, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n} [7],[8]. The fundamental form of an almost Hermitian manifold is determined by the relation

$$F(X,\ Y) = \langle X,\ JY\rangle\,, \quad X, Y \in \aleph(M^{2n}).$$

An almost Hermitian structure belongs to the W_4 class, if

$$\nabla_X\ (F)\ (Y, Z) \ = \ -\frac{1}{2\,(n-1)}\ \{\langle X, Y\rangle\ \delta\, F\,(Z) - \langle X, Z\rangle\ \delta\, F\,(Y) -$$

$$- \langle X, JY\rangle\ \delta\, F\,(JZ) + \langle X, JZ\rangle\ \delta\, F\,(JY)\}\,, \quad X, Y, Z \in \aleph(M^{2n}),$$

where δ is the codifferentiation operator and ∇ is the Riemannian connection of the metric $g = \langle\cdot,\ \cdot\rangle$ [7].

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the well-known unitary group $U(n)$ [6],[8]. Its elements are the so-called A-frames, i.e. the frames adapted to the structure. The construction of these frames is as follows:

$$(p,\ \varepsilon_1,\ \ldots, \varepsilon_n,\ \varepsilon_{\hat{1}},\ \ldots,\ \varepsilon_{\hat{n}}),$$

where ε_a are the eigenvectors corresponding to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\hat{a}}$ are the eigenvectors corresponding to the eigenvalue $-i$. Here the index a ranges from 1 to n, and $\hat{a} = a + n$.

By direct computation it is easy to obtain that the matrices of the operator of the almost complex structure, of the Riemannian metric and of the fundamental form written in an A-frame look as follows, respectively [6]:

$$\left(J_j^k\right)=\left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right);\ \ (g_{kj})=\left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right);$$

$$(F_{kj})=\left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right),$$

where I_n is the identity matrix; $k, j = 1, \dots, 2n$.

The first group of the Cartan structural equations of an almost Hermitian manifold written in an A-frame looks as follows [8]:

$$d\,\omega^a = \omega^a_b \wedge \omega^b + B^{ab}{}_c\, \omega^c \wedge \omega_b + B^{abc}\, \omega_b \wedge \omega_c\ ;$$

$$d\,\omega_a = -\omega^b_a \wedge \omega_b + B_{ab}{}^c\, \omega_c \wedge \omega^b + B_{abc}\, \omega^b \wedge \omega^c\ ,$$

where

$$B^{ab}{}_c = -\frac{i}{2}\, J^a_{\hat{b},\, c};\ B_{ab}{}^c = \frac{i}{2}\, J^{\hat{a}}_{b,\, \hat{c}};$$

$$B^{abc} = \frac{i}{2}\, J^a_{[\hat{b},\, \hat{c}]};\ B_{abc} = -\frac{i}{2}\, J^{\hat{a}}_{[b,\, c]}.$$

The systems of functions $\{\ B^{ab}{}_c\ \}$, $\{\ B_{ab}{}^c\ \}$, $\{\ B^{abc}\ \}$, $\{\ B_{abc}\ \}$ are components of the Kirichenko tensors of the almost Hermitian manifold M^{2n} [6], $\{J^j_{k,m}\}$ are components of ∇J; $a, b, c = 1, ..., n$; , $\hat{a} = a + n$.

At the end of this section, note that all considered manifolds, tensor fields and similar objects are assumed to be of the class C^∞.

3 Proof of the theorem

Let us consider the Cartan structural equations of the almost contact metric structure on an oriented hypersurface N^{2n-1} of a W_4-manifold M^{2n} [5]:

$$\begin{aligned} d\omega^\alpha &= \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}{}_\gamma \omega^\gamma \wedge \omega_\beta + \left(\sqrt{2} B^{\alpha n}{}_\beta + i\sigma^\alpha_\beta\right) \omega^\beta \wedge \omega + \\ &+ \left(-\frac{1}{\sqrt{2}} B^{\alpha\beta}{}_n + i\sigma^{\alpha\beta}\right) \omega_\beta \wedge \omega; \\ d\omega_\alpha &= -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta}{}^\gamma \omega_\gamma \wedge \omega^\beta + \left(\sqrt{2} B_{\alpha n}{}^\beta - i\sigma^\beta_\alpha\right) \omega_\beta \wedge \omega + \\ &+ \left(-\frac{1}{\sqrt{2}} B_{\alpha\beta}{}^n - i\sigma_{\alpha\beta}\right) \omega^\beta \wedge \omega; \qquad (1) \\ d\omega &= \left(\sqrt{2} B^{n\alpha}{}_\beta - \sqrt{2} B_{n\beta}{}^\alpha - 2i\sigma^\alpha_\beta\right) \omega^\beta \wedge \omega_\alpha + \left(B_{n\beta}{}^n + i\sigma_{n\beta}\right) \omega \wedge \omega^\beta + \\ &+ \left(B^{n\beta}{}_n - i\sigma^\beta_n\right) \omega \wedge \omega_\beta, \end{aligned}$$

where σ is the second fundamental form of the immersion of N^{2n-1} into M^{2n}.

Now we consider the case [6], in which the matrix of the second fundamental form of the hypersurface N^{2n-1} in a W_4-manifold M^{2n} looks as follows:

$$(\sigma_{ps}) = \left(\begin{array}{c|c|c} (\sigma_{\alpha\beta}) & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & 0 \\ \hline 0\dots 0 & \sigma_{nn} & 0 \dots 0 \\ \hline 0 & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & (\sigma_{\hat{\alpha}\hat{\beta}}) \end{array}\right), \quad p, s = 1,\, 2,\, 3, \dots, n-1,$$

and moreover

$$rank\,(\sigma_{\alpha\beta}) \ = rank\ (\sigma_{\hat{\alpha}\hat{\beta}}) = 1.$$

That is why the rank of the matrix (σ_{ps}) is equal to 2 if and only if $\sigma_{nn} = 0$; otherwise $rank\,(\sigma_{ps}) = 3$. We also mark out the important fact that the Cartan structural equations (1) of the almost contact metric structure do not contain the component σ_{nn}. Therefore if this component vanishes, then it does not affect the Cartan structural equations (1). For almost contact metric structures on such hypersurfaces of $rank\,(\sigma_{ps}) = 2$ or of $rank\,(\sigma_{ps}) = 3$, i.e. on hypersurfaces with type number 2 or 3, respectively, the Cartan structural equations (1) are absolutely identical. Namely:

$$\begin{aligned} d\omega^{\alpha} &= \omega^{\alpha}_{\beta} \wedge \omega^{\beta} + B^{\alpha\beta}{}_{\gamma}\omega^{\gamma} \wedge \omega_{\beta} + \sqrt{2}B^{\alpha n}{}_{\beta}\, \omega^{\beta} \wedge \omega + \\ &+ \left(-\frac{1}{\sqrt{2}}B^{\alpha\beta}{}_{n} + i\sigma^{\alpha\beta}\right)\omega_{\beta} \wedge \omega; \\ d\omega_{\alpha} &= -\omega^{\beta}_{\alpha} \wedge \omega_{\beta} + B_{\alpha\beta}{}^{\gamma}\omega_{\gamma} \wedge \omega^{\beta} + \sqrt{2}B_{\alpha n}{}^{\beta}\, \omega_{\beta} \wedge \omega + \\ &+ \left(-\frac{1}{\sqrt{2}}B_{\alpha\beta}{}^{n} - i\sigma_{\alpha\beta}\right)\omega^{\beta} \wedge \omega; \\ d\omega &= \left(\sqrt{2}B^{n\alpha}{}_{\beta} - \sqrt{2}B_{n\beta}{}^{\alpha}\right)\, \omega^{\beta} \wedge \omega_{\alpha} + B_{n\beta}{}^{n}\, \omega \wedge \omega^{\beta} + B^{n\beta}{}_{n}\, \omega \wedge \omega_{\beta}. \end{aligned} \tag{2}$$

Taking into account that if the Cartan structural equations are completely identical for such hypersurfaces of a W_4-manifold, then the almost contact metric structures induced on such hypersurfaces are also completely identical, Q.E.D.

4 Some comments

Using the above mentioned fact that the class of W_4-manifolds contains all LCK-manifolds, we get the following additional result:

Corollary 1. 3-hypersurfaces of locally conformal Kählerian manifolds admit almost contact metric structures that are identical to the structures induced on 2-hypersurfaces of such manifolds.

We remark that the structure induced on the considered 2- and 3-hypersurfaces of W_4-manifolds does not belong to any well-studied classes of almost contact metric structures (cosymplectic, Kenmotsu, Sasaki structures etc).

Indeed, let us compare (2) with the well-known Cartan structural equations of the cosymplectic structure [6],[8]:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta,$$

$$d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta,$$

$$d\omega = 0,$$

or with the Cartan structural equations of the Kenmotsu structure [6],[8]:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + \omega \wedge \omega^\alpha,$$

$$d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + \omega \wedge \omega_\alpha,$$

$$d\omega = 0,$$

or with the Cartan structural equations of the Sasakian structure [6],[8]:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta - i\,\omega \wedge \omega^\alpha,$$

$$d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + i\,\omega \wedge \omega_\alpha,$$

$$d\omega = -2i\omega^\alpha \wedge \omega_\alpha.$$

It is easy to conclude that equations (2) do not correspond to these relations.

Corollary 2. 2-hypersurfaces and 3-hypersurfaces of W_4-manifolds admit almost contact metric structures that are non-cosymplectic, non-Kenmotsu and non-Sasakian.

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